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Original Paper

LN-Factorization, Classification of k-Tridiagonal Toeplitz Matrices, and Recursions

Hamide Dogan, William Dunlap, and Eugene Agyei-Kodie

Hamide Dogan: Mathematical Sciences, University of Texas at El Paso, USA;
William Dunlap: Computer Science, Texas Tech University, USA;
Eugene Agyei-Kodie: Mathematics, Michigan State University, USA

Abstract

In this paper, we introduce a new method, called LN-Factorization, for identifying the determinants of k-tridiagonal Toeplitz matrices in cases where $\mathrm{W}_p = 0$. Here, W_p denotes the determinant of the smallest singular tridiagonal Toeplitz T_n. Specifically, p is the smallest positive integer for which T_p is singular, that is, T_k is non-singular for all $1 \leq k < p$, while $\mathrm{W}_p = \det(\mathrm{T}_p) = 0$. The LN-factorization yields a classification of singular k-tridiagonal Toeplitz matrices. In addition, we establish recursive relations among the determinants of both singular and nonsingular k-tridiagonal Toeplitz matrices.

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1. Introduction

1.1. Significance and Motivation

The significance of matrices in engineering and computational sciences cannot be overstated. Recently, it was established that every $n \times n$ matrix can be represented as a product of $\lfloor \frac{n}{2} \rfloor + 1$ Toeplitz matrices [12]. Additionally, Ye et al. [12] demonstrated that any $n \times n$ matrix can be decomposed into a product of, at most, $2n + 5$ Toeplitz matrices. Their findings revealed that $\lfloor \frac{n}{2} \rfloor + 1$ is the minimal count of r-Toeplitz matrices required to express any generic $n \times n$ matrix.

In a related study, Mackey et al. [8] also explored the relationship between $n \times n$ matrices and Toeplitz matrices. They showed that any $n \times n$ matrix, where $n \leq 4$, can be transformed into a Toeplitz matrix through a similarity transformation.

These revelations position Toeplitz matrices at the forefront of Matrix Theory and its applications in engineering and computational sciences.They are instrumental in various fields such as spline function computation, parallel and distributed computing [2], signal and image processing, differential equation solutions [4], boundary value problems, interpolation, physics, and polynomial and power series computations. Notably, they are

Email addresses: hdogan@utep.edu; Wildunla@ttu.edu; agyeikod@msu.edu

crucial in analyzing the spatial distribution of zeros in eigen-polynomials concerning Hermitian Toeplitz matrices, a facet relevant to signal processing [10].Therefore, any research exploring properties of Toeplitz matrices is poised to enhance their applications across numerous STEM disciplines, thereby driving progress.

Concerning tridiagonal matrices, numerous studies have investigated their structural and spectral properties. Da Fonseca and Yılmaz [5] examined properties of tridiagonal matrices while Noschese, Pasquini, and Reichel [9] studied the properties and applications of tridiagonal Toeplitz matrices. Other researchers have focused on the inverses of tridiagonal Toeplitz matrices [7]. Yalciner [11] investigated LU-decomposition structures for tridiagonal Toeplitz matrices by applying the classical Doolittle factorization, in which, the lower triangular matrix has a unit diagonal and the diagonal entries of the upper triangular matrix is computed recursively [6]. Additional studies established determinant properties for both 1-tridiagonal [1] and 2-tridiagonal Toeplitz matrices [3].In the present work, we extend these results to k-tridiagonal Toeplitz matrices for all $k > 2$, and derive recursive determinant formulas.

A common feature of the last three studies is the assumption that all principal Toeplitz matrices generated by the same entries are nonsingular. Under this assumption, Gauss-Jordan elimination process produces an LU-decomposition whose diagonal entries of the upper-triangular matrix U satisfy a recursive pattern [11]. In particular, each diagonal entry can be expressed as the quotient of the determinants of two consecutive principal submatrices. Consequently, these methods require all Toeplitz matrices of orders 1, 2,. . . ,n, generated by the same entries, to have nonzero determinants.

To date, singular k-tridiagonal Toeplitz matrices have received limited attention in the literature, and no existing study appears to provide a systematic classification of such matrices. The present work addresses this gap by introducing a novel factorization, termed the LN-Factorization, which provides a framework for the analysis, characterization, and classification of singular k-tridiagonal Toeplitz matrices.

The significance of this work lies in the introduction of a unified framework for both recursive determinant computation and the classification of singular k-tridiagonal Toeplitz matrices. Unlike existing approaches, which typically assume nonsingularity and require a separate factorization for each value of k, the reported determinant recurrence is valid for arbitrary k. Consequently, it enables efficient recursive determinant evaluation for both singular and nonsingular matrices while simultaneously providing a systematic classification of singular k-tridiagonal Toeplitz matrices. Thus, determinant computation and singularity analysis are unified within a single framework, providing both computational efficiency and a systematic method for identifying singular k-tridiagonal Toeplitz matrices.

Taken together, the recursive formulas and classification results developed in this work may stimulate further discoveries and expand the applications of k-tridiagonal Toeplitz matrices in matrix analysis and related fields.

1.2. Notations and Definitions

Notations:

n : matrix size.

a, b, and c : nonzero complex numbers.

k : tridiagonality.

p : the size of the smallest singular matrix.

Definition 1.1. A Toeplitz matrix is a matrix T_n with

$$\mathrm{T}_{ij} = \mathrm{T}_{i+1,j+1}, \quad \forall i, j = 1, 2, \cdots, n.$$

Or $\mathrm{T}_n = [t_{i,j}]$ where $t_{i,j} = t_{i-j}$.

Definition 1.2. An $n \times n$ k-tridiagonal Toeplitz matrix is

$$\mathrm{T}_{ij}^{(k)} = \begin{cases} a & ; \quad i = j \\ b & ; \quad j - i = k \\ c & ; \quad i - j = k \\ 0 & \text{otherwise} \end{cases}$$

$\forall i, j = 1, \cdots, n$ and $a, b, c \in \mathbb{C}$.

Determinant of an $n \times n$ k-tridiagonal Toeplitz matrix $\mathrm{T}_n^{(k)}$ is denoted by $\mathrm{W}_n^{(k)}$. For, $k = 1$, we consider W_n and $\mathrm{W}_n^{(1)}$ interchangeably.

Among all tridiagonal Toeplitz matrices T_n with entries a, b, and c, let W_p denote the determinant of the smallest size singular matrix. Here, p is the smallest positive integer such that T_p is singular. In other words, T_m is non-singular for all $1 \leq m < p$, and $\mathrm{W}_p = det(\mathrm{T}_p) = 0$.

For example, the matrix T_5 is the smallest singular tridiagonal Toeplitz matrix of all tridiagonal Toeplitz matrices with entry values a=b=3 and c=1. That is, $\mathrm{W}_p = \mathrm{W}_5 = 0$ and T_m is non-singular for $1 \leq m < 5$.

2. Preliminaries

For nonsingular k-tridiagonal Toeplitz matrices, we derive an LU-decomposition by exploiting the recursive relationship among the determinants of successive Toeplitz submatrices sharing the same entries, as established by Yalciner [11]. That is, for $n = km + s$, $a, b, c \in \mathbb{C}$, $\mathrm{T}_n^k = \mathrm{LU}$ where L is a lower-triangular matrix with diagonals 1 and U is an upper-triangular matrix with diagonal entries $\frac{\mathrm{W}_i}{\mathrm{W}_{i-1}}$ (provided $\mathrm{W}_{i-1} \neq 0$ for all $i \geq 1$), repeated k times in each of the m groups of k-clusters. The last s-terms contain $\frac{\mathrm{W}_{m+1}}{\mathrm{W}_m}$. This structure is illustrated in the figure (2.1).

Building on this framework, we obtain recursive determinant formulas for any value of k. Consequently, determinants can be evaluated recursively, eliminating the need to derive an LU-decomposition for each value of k.

$$\mathrm{U} = \begin{bmatrix}
W_1 & \overbrace{\cdots}^{\text{(k-1)-many zeros}} & b & & & & & & & & & \\
 & \ddots & \cdots & \ddots & & & & & & & & \\
 & & W_1 & \cdots & \ddots & & & & & & & \\
 & & & \frac{W_2}{W_1} & \cdots & b & & & & & & 0 \\
 & & & & \ddots & \cdots & b & & & & & \\
 & & & & & \frac{W_2}{W_1} & \cdots & b & & & & \\
 & & & & & & \ddots & \cdots & \ddots & & & \\
 & & 0 & & & & & \frac{W_i}{W_{i-1}} & \cdots & \ddots & & \\
 & & & & & & & & \ddots & \cdots & \ddots & \\
 & & & & & & & & & \frac{W_i}{W_{i-1}} & \cdots & b \\
 & & & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & & & \frac{W_{m+1}}{W_m}
\end{bmatrix} \tag{2.1}$$

The recursive pattern in the diagonal entries of U in the LU-decomposition of [11] can be derived directly from the recursive relation given in the following lemma (See page 286 in [1])

Lemma 2.1. *[1] Let* $W_0^{(1)} = 1$. *For* $n \geq 1$,

$$W_{n+1} = a\,W_n - bc\,W_{n-1}.$$

2.1. Recursive Properties of Nonsingular k-Tridiagonal Toeplitz Matrices

A cluster analysis, in which like terms are grouped together, leads to the following theorem.

Theorem 2.2. *For n=km+s,*

$$W_n^{(k)} = (\,W_m)^{k-s}\,(\,W_{m+1})^s\,.$$

Proof.
For each $k-$cluster of the diagonal entries of U (seen in the figure (2.1)), regrouping the first s terms gives

$$\overbrace{\overbrace{\mathrm{W}_1^{(1)} \times \cdots \times \mathrm{W}_1^{(1)}}^{s-many} \times \overbrace{\frac{\mathrm{W}_2^{(1)}}{\mathrm{W}_1} \times \cdots \times \frac{\mathrm{W}_2^{(1)}}{\mathrm{W}_1}}^{s-many} \times \cdots \times \overbrace{\frac{\mathrm{W}_{m+1}^{(1)}}{\mathrm{W}_m^{(1)}} \times \cdots \times \frac{\mathrm{W}_{m+1}^{(1)}}{\mathrm{W}_m^{(1)}}}^{s-many}}^{(m+1)-many} \tag{2.2}$$

.
Next, regrouping the remaining $(k-s)$ terms in each $k-$cluster gives

$$\overbrace{\overbrace{\mathrm{W}_1^{(1)} \times \cdots \times \mathrm{W}_1^{(1)}}^{\{k-s\}-many} \times \overbrace{\frac{\mathrm{W}_2^{(1)}}{\mathrm{W}_1} \times \cdots \times \frac{\mathrm{W}_2^{(1)}}{\mathrm{W}_1}}^{(k-s)-many} \times \cdots \times \overbrace{\frac{\mathrm{W}_{m}^{(1)}}{\mathrm{W}_{m-1}^{(1)}} \times \cdots \times \frac{\mathrm{W}_{m}^{(1)}}{\mathrm{W}_{m-1}^{(1)}}}^{\{k-s\}-many}}^{m-many} \tag{2.3}$$

Simplifying the two expressions (2.2) and (2.3) results in

$$\mathrm{W}_n^{(k)} = (\mathrm{W}_m)^{k-s}\,(\mathrm{W}_{m+1})^s$$

□

The case $n = km$ yields the following corollary.

Corollary 2.3. *For* $n = km$,

$$W_n^{(k)} = (\,W_m)^k$$

A more general version of Theorem 2.2 is presented in the following theorem.

Theorem 2.4. *For* $n = km + s$, $1 < l < k$, $l \in \mathbb{N}$, *and* $l \leq s$,

$$W_n^{(k)} = W_{l(m+1)}^{(l)}\,W_{m(k-l)+s-l}^{(k-l)}$$

Proof.
The proof follows directly from the l-grouping analysis of each k-cluster. □

It is worth noting here that this theorem confirms the $k = 2$ case reported in [3].

Theorems 2.2 and 2.4, as well as Corollary 2.3, constitute our contributions. They are derived from the existing framework in the literature [11], namely, the recursive pattern on the diagonal entries of upper-triangular matrix U in LU-decomposition of [11]. In what follows, we introduce several new ideas that have not yet been explored in the literature.

3. LN-factorization and Singular k-Triadiagonal Toeplitz Matrices

3.1. LN- Factorization

LN-factorization is a modification of the recursive pattern, introduced by Yalciner [11], for cases $W_p = 0$ where $p < n$ is the smallest order of a singular tridiagonal Toeplitz matrix among all matrices with the same entries.

LN-factorization yields matrices L and N such that $LT_n^k = N$. Let us first discuss the procedure that generates elementary matrices E_i, $\forall i = 1, \cdots, n$.

Given T_n and $p < n$ with $W_p = 0$. Set $E_1 =$ I, Identity matrix.

For $i > 1$,

Case 1 : For $i \neq t(p+1)$ or $i \neq t(p+1)+1$, $t \in \mathbb{N}$,
E$_i$ is:

(a). The i$^{\text{th}}$row and the (i-1)$^{\text{th}}$column $= -c$

(b). The i^{th} diagonal entry = The $(i-1)^{\text{th}}$ diagonal entry of $\mathrm{E}_{i-1} \cdots \mathrm{E}_1 \mathrm{T}_n$.

(c). All other diagonal entries=1.

(d). Otherwise=0.

The following is an illustration of Case 1.

$$\mathrm{E}_i = \begin{bmatrix} 1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & & \ddots & & & & 0 & & \\ & & & & 1 & & & & & \\ \cdots & \cdots & 0 & \cdots & -c & b^l \mathrm{W}_t (\mathrm{W}_{p-1})^l & 0 & \cdots & & \\ & & & & & & 1 & & & \\ 0 & & & & & & & & \ddots & \\ & & & & & & & & & 1 \end{bmatrix} \longrightarrow \mathrm{i}^{\text{th}}\text{row}. \tag{3.1}$$

where $0 \leq t \leq p-1$ and $0 \leq l \leq (r+1)$ for $n = r(p+1)+s$.

Case 2 : $i = t(p+1)$

$\mathrm{E}_i = \mathrm{I}$, Identity matrix.

Case 3 : $i = t(p+1)+1$

E_i is:

(a). The i$^{\text{th}}$ row and $(\mathrm{i}-2)^{\text{th}}$ column $= -c$

(b). The i^{th} diagonal entry = The $(i-2) \times (i-1)$ entry of $E_{i-1} \cdots E_1 T_n$.

The left multiplication of E_i with T_n yields the matrix N.

$$E_n \cdots E_{i+1} I E_{i-1} \cdots E_1 T_n = N.$$

Here, L is the product of the matrices E_i. That is,

$$L = E_n \cdots E_{i+1} I E_{i-1} \cdots E_1.$$

Note that L is the product of nonsingular matrices. We now present an example to illustrate the steps of this process.

Example 3.1. For T_6 with nonzero entry values a,b,c, and $p = 3$.

$\mathrm{E}_1 = \mathrm{I}$.

$$\mathrm{E}_2 = \begin{bmatrix} 1 & & & & & 0 \\ -c & a & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ 0 & & & & & 1 \end{bmatrix}$$

$$\mathrm{E}_3 = \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & -c & \mathrm{W}_2 & & & \\ & & & 1 & & \\ & & & & 1 & \\ 0 & & & & & 1 \end{bmatrix}$$

$\mathrm{E}_4 = \mathrm{I}$.

$$\mathrm{E}_5 = \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & -c & 0 & bW_2 & \\ 0 & & & & & 1 \end{bmatrix}$$

$$\mathrm{E}_6 = \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & 0 & & & -c & abW_2 \end{bmatrix}$$

The structures of N, N_0 and N_i are illustrated below.

$$\mathrm{N} = \begin{bmatrix} \mathrm{N}_0 & & & & & & & & & & & & \\ c & a & b & & & & & & & & & & \\ & & \mathrm{N}_1 & & & & & & & & & & \\ & & c & a & b & & & & 0 & & & & \\ & & & & \ddots & & & & & & & & \\ & & & & & \ddots & & & & & & & \\ & & & & & & & N_i & & & & & \\ & & & & & & & c & a & b & & & \\ & & & & & & & & & \ddots & & & \\ & & & & & & & & & & \ddots & & \\ & & 0 & & & & & & & & & \ddots & \\ & & & & & & & & & & & c & a & b \\ & & & & & & & & & & & & & \mathrm{N}_{r+1} \end{bmatrix}$$

$$\mathrm{N}_0 = \begin{bmatrix} a & b & 0 & & & & \\ 0 & \mathrm{W}_2 & b\mathrm{W}_1 & & & & \\ & \ddots & & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & \mathrm{W}_{p-1} & b\mathrm{W}_{p-2} & 0 \\ & & & & 0 & 0 & b\mathrm{W}_{p-1} \end{bmatrix}$$

$$\mathrm{N}_i = \begin{bmatrix} b^i\mathrm{W}_1\left(\mathrm{W}_{p-1}\right)^i & b^{i+1}\left(\mathrm{W}_{p-1}\right)^i & 0 & & \\ 0 & b^i\mathrm{W}_2\left(\mathrm{W}_{p-1}\right)^i & b^{i+1}\mathrm{W}_1\left(\mathrm{W}_{p-1}\right)^i & & \\ & \ddots & \ddots & & \\ & & \ddots & & \\ & & & \ddots & \\ & & b^i\mathrm{W}_{p-1}\left(\mathrm{W}_{p-1}\right)^i & b^{i+1}\mathrm{W}_{p-2}\left(\mathrm{W}_{p-1}\right)^i & 0 \\ & & 0 & 0 & \left(b\mathrm{W}_{p-1}\right)^{i+1} \end{bmatrix}$$

Remark: 1. N_i is an $p \times (p+1)$ size matrix.
2. The first row, first column entry of each N_i is aligned with b on the same column.
3. The last row, last column entry of each N_i is aligned with a on the same column.
4. N is described in terms of N_is illustrated above.

An illustration of the second remark is given in the figure (3.2).

$$\begin{bmatrix} c & a & b & 0 & \\ 0 & 0 & b^i W_1 (W_{p-1})^i & b^{i+1}(W_{p-1})^i & \\ & & & & \end{bmatrix} \tag{3.2}$$

An illustration of the third remark is given in the figure (3.3).

$$\begin{bmatrix} 0 & 0 & (bW_{p-1})^{i+1} & 0 & \\ 0 & c & a & b & \\ & & & & \end{bmatrix} \tag{3.3}$$

3.2. Classification and Recursive Properties

The LN- factorization provides a framework for the classification of singular tridiagonal Toeplitz matrices, given in the following theorem.

Theorem 3.2. *Let $a, b, c \in \mathbb{C}$, and p be the smallest positive integer such that $W_p = 0$. Then for all $t \in \mathbb{N}$,*

$$W_{(t+1)p+t} = 0.$$

Consequently, $T_{(t+1)p+t}$ are singular.

Proof. The LN-factorization gives the $((t+1)p+t)^{th}$ diagonal of N to be

$$ab^t(W_{p-1})^{t+1} - cb^{t+1}W_{p-2}(W_{p-1})^t.$$

Thus,

$$\begin{aligned} ab^t(W_{p-1})^{t+1} - cb^{t+1}W_{p-2}(W_{p-1})^t &= b^t(W_{p-1})^t(aW_{p-1} - cbW_{p-2}) \\ &= b^t(W_{p-1})^t W_p = 0 \end{aligned}$$

By Lemma 1, $aW_{p-1} - cbW_{p-2} = W_p$. ☐

An immediate consequence of this theorem is that T_n is nonsingular whenever $n \neq (t+1)p+t$.

The following theorem presents recursive relations for the determinants of these matrices.

Theorem 3.3. *For $n = l(p+1)+s$, $W_p^{(1)} = 0$, $s < p$, $LT_n^{(1)} = N$, and $n \neq (t+1)p+t$, $\forall t \in \mathbb{N}$,*

$$\det(L) = \prod_{j=0}^{l-1}\prod_{i=0}^{p-1} W_i(bW_{p-1})^j \prod_{i=0}^{s-1}(bW_{p-1})^l W_i \tag{3.4}$$

$$\det(N) = (-1)^l c^l \prod_{i=1}^{l}(bW_{p-1})^i \prod_{j=0}^{l-1}\prod_{i=1}^{p-1} W_i(bW_{p-1})^j \prod_{i=1}^{s}(bW_{p-1})^l W_i \tag{3.5}$$

thus

$$W_n = \det(T_n) = (-1)^l c^l (bW_{p-1})^l W_s \tag{3.6}$$

Proof.
The proof of the expression (3.4) follows directly from taking the product of the determinants of the matrices E_i. In proving the expression (3.5), we delete the $((t+1)p+t)^{\text{th}}$ row and the $((t+1)p+(t+1))^{\text{th}}$ column of N per $t \in \mathbb{N}$ at a time. Note that these entries have all zeros with the only non-zero entry on the $((t+1)p+(t+1))^{\text{th}}$ column entry, which is $(bW_{p-1})^{t+1}$ (See the illustration of N_i above).

Repeating the process per $t \in \mathbb{N}$ yields an upper-triangular matrix whose diagonal entries are $b^t(W_i)(W_{p-1})^t$ for $i = 1, \cdots, p-1$. The rows corresponding to multiples of p have c on the diagonal entry and b on the column immediately following the diagonal entry. Consequently, the determinant of N is given by the expression (3.5). The proof of the expression (3.6) follows from

$$\det(L)\det(T_n^{(1)}) = \det(N).$$

Note that $det(L) \neq 0$ since $L = E_t \cdots E_2 E_1$ and E_i are all non-singular matrices.

Considering

$$\det(T_n) = \frac{\det N}{\det L}$$

gives the expression (3.6) □

Our next theorem extends Theorem 3.3 to $k-$tridiagonal Toeplitz matrices with $k > 1$ satisfying

$W_p = 0, \;\; p < m$ where $n = km + s$.

Theorem 3.4. *For $n = km + s, \;\; W_p = 0$ and $m = l(p+1) + s', \;\; s' < p-1$,*

$$W_n^{(k)} = (-1)^{lk} c^{lk} (bW_{p-1})^{lk} (W_{s'})^{k-s} (W_{s'+1})^s.$$

Proof. Proof is the direct result of Theorems 2.2 and 3.3. □

In addition, Theorems 3.2 and 3.4 give way to the classification of $k-$tridiagonal singular toeplitz matrices. They are presented in the following corollaries.

Corollary 3.5. *For $n = km + s$, $W_p = 0$, with p being the smallest singular size, and $m = p$ (or $s' = p$ in $m = l(p+1) + s'$),*

$$W_n^{(k)} = 0.$$

Corollary 3.6. *For $W_p = 0$, with p being the smallest singular size,*

$$W_{kp(t+1)+kt}^{(k)} = 0$$

for all integers $t \geq 0$, and $k \geq 1$. Consequently, $T_{kp(t+1)+kt}^{(k)}$ are singular.

Furthermore, the following theorem reports recursive relations among $k-$tridiagonal Toeplitz matrices extending the results presented in Theorems 2.4 and 3.3. Although some of the associated submatrices may be singular, this theorem considers only nonsingular k-tridiagonal Toeplitz matrices.

Theorem 3.7. *For $n = k(p+1)l + s$ and $t < k$, $t \in \mathbb{N}$,*

$$W_n^{(k)} = W_{t(p+1)l+s}^{(t)} W_{(k-t)(p+1)l}^{(k-t)}$$

We now present a couple of examples of singular tridiagonal Toeplitz matrices.

Example 3.8.

Given $a = c = 3$, $b = 1$ and $W_5 = 0$. Here, $p = 5$.
Then,

$$W_{(t+1)5+t} = 0 = W_5 = W_{11} = W_{17} = \cdots$$

Meaning, the following tridiagonal Toeplitz matrices are singular:

$$T_5, T_{11}, T_{17}, \cdots, T_{(t+1)5+t}$$

Example 3.9.

Given $a = b = 3$, $c = 1$, $W_5 = 0$. Here, $p = 5$ and $k = 2$.
Then:

$$W^{(2)}_{10(t+1)+2t} = 0 = W^{(2)}_{10} = W^{(2)}_{22} = W^{(2)}_{34} \cdots$$

Thus, the following k-tridiagonal Toeplitz matrices are singular:

$$T^{(2)}_{10}, T^{(2)}_{22}, T^{(2)}_{34} \cdots$$

In short, using the formula

$$W^{(k)}_{k(p(t+1)+t)}$$

we can generate all singular $k-$tridiagonal Toeplitz matrices with the same entries, provided that $W_p = 0$ for the smallest singular size p. Moreover, the same formula can be used to identify the nonsingular matrices.

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