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Bayesian and Frequentist Reliability Analysis of the Akash Distribution with Progressive First-Failure Censoring

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Abstract

This paper presents the results of a reliability analysis of the one-parameter Akash distribution under the progressive first-failure censoring technique, which is known to be one of the most economical life-testing plans in terms of time and cost for data acquisition. Due to the inferential and computational difficulties presented by censored samples, both frequentist and Bayesian methods of estimation for the unknown parameter and key reliability measures, such as reliability function, hazard rate, and mean residual life are outlined and assessed. The frequentist methods are primarily based on maximum likelihood estimation along with asymptotic and bootstrapped confidence intervals. The Bayesian methods, on the other hand, are based on the Tierney-Kadane approximation, importance sampling, and Metropolis-Hastings with squared error loss and non-informative and informative (gamma) priors. A complete Monte Carlo simulation study systematically examines all estimation methods with respect to bias, mean squared error, and the coverage probability of confidence intervals for different censoring levels. The results of the study demonstrate that Bayesian analysis, and in particular the Tierney-Kadane approximation, with informative priors, yield more accurate and efficient estimates.

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1. Introduction

Through experimenting the lifetime of industrial products, manufacturers are trying to determine the potential failure behaviors of the products in order to supply consumers with quality, long-lasting items. Some of the models are designed with persistent or continuously increasing or decreasing failure patterns. The classical works such as (9) and (15) used the exponential, gamma, and Weibull distributions. Since they are simple to understand and easy to use, the models are still popular.

But in reality, many lifetime datasets obtained from experiments do not fit these conventional models well. Hazard rates are typically quite complex and show nonlinear behavior, which are hard to capture using simple models. This has spurred the development of more flexible, copula-based lifetime models that are designed to flexibly fit a larger range of hazard shapes and age dynamics. The importance of flexible distributions arises in actuarial science, medical sciences, and engineering, where the study involves the analysis of lifetime data.

Although these generalized distributions are more flexible for modeling, they can also pose problems in computation and inference. The existence of various parameters, which are crucial for capturing distributional nuances, makes the parameter estimation process more challenging. Such censoring situations occur in life-testing experiments with limited time horizons; traditional methods, e.g., maximum likelihood estimation (MLE), may be poor in adherence, especially with limited sample sizes. In those situations, the MLE estimates can become inconsistent, and the values provided do not give a full representation of the uncertainty about the parameters.

It is common in reliability studies and life-testing plans to come upon something referred to as censoring, in which we only see part of an object's real lifetime. Censoring occurs when items are either taken out of experiments intentionally or by accident. This concept is crucial in fields like medication, survival evaluation, medical trials, and engineering. Researchers mainly focus on two kinds of censoring. Type-I censoring happens while they have a look at the ends at a fixed time, and any surviving objects are censored. Type-II censoring concludes after a particular number of failures, leaving the ultimate items censored.

In actual-existence situations, lengthy-lasting products could make Type-I and Type-II censoring much less effective. To address this, (1) introduced the first-failure censoring plan, in which groups of items are tested until the first failure occurs in each group. This method saves money and time but doesn't permit for items to be eliminated during the check. To introduce some flexibility, (3) developed the modern censoring scheme, which allows units to be removed at distinct times at some point of the check. Later, (17) mixed those thoughts into the innovative first-failure censoring scheme, creating a more efficient and realistic existence-checking-out plan. In this method, groups of objects are tested simultaneously, and every time a failure takes place, the failed group and a few extra groups are step by step removed. This approach helps researchers save time, reduce prices, and create more adaptable censoring structures.

Bayesian strategies provide a systematic way to incorporate previous information and quantify uncertainty below such censoring schemes, even though they regularly require computationally intensive algorithms, including Markov Chain Monte Carlo (MCMC). Frequentist strategies, in comparison, are computationally easier; however, they can be sensitive to small sample sizes and heavy censoring.

Therefore, there remains a regular demand for lifetime fashions that are both flexible and computationally plausible, able to address parameter estimation demanding situations underneath censoring schemes. This has precipitated ongoing efforts to broaden new distributions that integrate the adaptability of generalized fashions with the interpretability and realistic relevance of classical ones.

In this context, the Akash distribution has emerged as a promising opportunity to model lifetime data. This paper focuses on the Bayesian and Frequentist reliability evaluation of the Akash distribution under modern first-failure censoring, with particular attention to parameter estimation and inferential performance.

The remainder of this paper is organized as follows. Section 2 introduces the Akash distribution along with its fundamental properties, including key reliability characteristics. Section 3 presents the framework of Progressive First-Failure Censoring and highlights the advantages of adopting this censoring scheme. Section 4 is devoted to classical estimation methods; in particular, the maximum likelihood estimation of the distributional parameter, reliability characteristics, and the construction of asymptotic confidence intervals are discussed. In addition, we examine Bootstrap-based methods, including percentile (Bootstrap-p) and Studentized (Bootstrap-t) confidence intervals. Section 5 develops Bayesian estimation procedures for reliability characteristics under the squared error loss function. Several Bayesian approaches are considered, such as the Tierney–Kadane approximation, importance sampling, and the Metropolis–Hastings (M–H) algorithm. The highest posterior density (HPD) credible interval is also constructed using samples generated via the M–H algorithm. Section 6 reports the results of extensive simulation studies, while Section 7 illustrates the proposed methods through practical applications. Finally, Section 8 provides concluding remarks and directions for future research.

2. Akash Distribution and Its Properties

(13) introduced the one-parameter Akash distribution as an effective and versatile tool for modeling and studying lifetime facts. Its mixture of an exponential and a gamma distribution allows it to model a wide range of decay rates. The Akash distribution is more flexible and may represent a wide variety of hazard rate shapes, including increasing, decreasing, and bathtub-shaped patterns. The distribution's mathematical properties, parameter estimation, and applications in a range of domains, including engineering and medical research, make it a valuable tool for researchers and practitioners.

The Akash distribution is a convex combination of the exponential(θ) and gamma($3,\theta$) distributions. Depending on the mixing proportions, its distributions are $\frac{\theta^2}{\theta^2+2}$ and $\frac{1}{\theta^2+2}$, respectively ((14)). These proportions determine the relative contributions of the exponential and gamma additives to the distribution. The study of (13) shows that Akash distributions display superiority over Lindley distributions and exponential distributions in terms of their chance charge characteristic and mean residual lifetime characteristic. (13) proving that the Akash distribution presents a higher match than the Lindley and exponential distributions in positive lifetime information scenarios. This demonstrates the effectiveness of the Akash distribution in capturing the complexities of actual-world facts and making statistical inferences.

2.1. The model

The probability density function (pdf) of a random variable (r.v.) X with an Akash distribution is defined as follows:

$$f(x;\theta) = \frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}; \quad x>0,\ \theta>0. \tag{2.1}$$

The corresponding cumulative distribution function (cdf) is given by

$$F(x;\theta) = 1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}; \quad x>0,\ \theta>0. \tag{2.2}$$

2.2. Construction of the Akash Distribution

Let Y be a mixture distribution of

(1) an *exponential distribution* with scale parameter θ, and
(2) a *gamma distribution* with shape parameter 3 and scale parameter θ.

and their mixing proportions be p and $1-p$, respectively. Then, the mixture pdf is given by

$$f(x;\theta) = p \cdot \theta e^{-\theta x} + (1-p) \cdot \frac{\theta^3}{2} x^2 e^{-\theta x}, \quad x > 0,\ \theta > 0.$$

If we choose

$$p = \frac{\theta^2}{\theta^2+2}, \quad 1-p = \frac{2}{\theta^2+2},$$

then, the expression simplifies to the one-parameter Akash distribution with pdf given by equation (2.1).

2.3. Moments and shape measures of the Akash distribution

The rth raw moment is,

$$\mu_r' = \mathbb{E}[X^r] = \frac{\theta^3}{\theta^2+2}\left(\frac{\Gamma(r+1)}{\theta^{r+1}} + \frac{\Gamma(r+3)}{\theta^{r+3}}\right); \quad r = 1,2,3,...$$

In particular,

$$\mu_1' = \mathbb{E}[X] = \frac{\theta^2+6}{\theta(\theta^2+2)}, \qquad \mu_2' = \mathbb{E}[X^2] = \frac{2(\theta^2+12)}{\theta^2(\theta^2+2)}.$$

Hence the variance is

$$\operatorname{Var}(X) = \mu_2' - (\mu_1')^2 = \frac{\theta^4+16\theta^2+12}{\theta^2(\theta^2+2)^2}.$$

The skewness γ_1 and excess kurtosis γ_2 are

$$\gamma_1(\theta) = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2(\theta^4+4\theta^2+4)^{3/2}(\theta^6+30\theta^4+36\theta^2+24)}{(\theta^4+16\theta^2+12)^{3/2}(\theta^6+6\theta^4+12\theta^2+8)},$$

and

$$\gamma_2 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{6(\theta^8+48\theta^6+64\theta^4+96\theta^2+48)}{\theta^8+32\theta^6+280\theta^4+384\theta^2+144}.$$

Numerically, for typical $\theta > 0$ these produce $\gamma_1 > 0$ and $\gamma_2 > 0$, i.e. the Akash distribution is positively skewed and leptokurtic for most parameter values. Median and other quantiles must be computed numerically.

2.4. Reliability characteristics

In this study, the Akash distribution is considered as a lifetime model and its various reliability characteristics are examined. The reliability and hazard rate functions (also known as the failure rate function) of the Akash distribution, respectively, are given by

$$R(t) = \left[1 + \frac{\theta t(\theta t+2)}{\theta^2+2}\right] e^{-\theta t}; \quad t > 0,\ \theta > 0, \tag{2.3}$$

$$H(t) = \frac{\theta^3(1+t^2)}{\theta t(\theta t+2) + (\theta^2+2)}; \quad t > 0,\ \theta > 0. \tag{2.4}$$

In life testing situations, the expected additional lifetime given that a component has survived until time t is a function of t, called the mean residual life. The mean residual life function, $m(t)$ of the Akash distribution is given by

$$m(t) = \frac{\theta^2 t^2 + 4\theta t + (\theta^2 + 6)}{\theta[\theta t(\theta t + 2) + (\theta^2 + 2)]}; \quad t > 0, \ \theta > 0. \tag{2.5}$$

This model is a strong contender for predicting how long things last because it has nice mathematical properties, especially concerning its reliability, failure rate, and expected remaining life. The catch is that when some of our data is incomplete (due to censoring), it becomes quite tricky to pinpoint its exact values. That's the reason we need to create robust and efficient methods to estimate them.

3. Progressive First-Failure Censoring

Censoring is a basic concept in reliability theory and life testing, in which the exact lifetime (or other outcome) of an item is only partially known. This happens when objects are taken out of an experiment before they fail by design or accident. Censoring is an important concept in various fields such as medicine, survival analysis, and engineering.

There are several types of censoring schemes, but the most common censoring schemes are **Type-I** and **Type-II**.

- **Type-I censoring** occurs when a study is designed to end at a predetermined fixed time. If a subject does not experience a failure by the end of the study, then that subject is censored.
- **Type-II censoring** is when an experiment is terminated after a fixed number of subjects have failed. The other subjects are right-censored thereafter.

These simple strategies are inefficient for long-lived items and experiments. They may become quite long, thus requiring more efficient solutions.

More advanced schemes have been developed to address these limitations:

- **First Failure Censoring**: Introduced by Balasooriya (1), this plan offers a time-saving solution. It involves testing n groups, each with k items, simultaneously until the first failure is observed in each group.While the plan is cost-effective, it does not permit intermittent removal of units.
- **Progressive Censoring**: Cohen (3) proposes a progressive censoring scheme, which allows the removal of units or objects during the test. This adds flexibility to the experiment.

  Cohen (3) proposed the scheme, which allows for the removal of units from the test at various failure times, adding flexibility to the experiment.

Since first failure censoring is cost-effective and time-saving, and progressive censoring has intermittent removal properties, Wu and Kus (17) combined these two strategies to develop the progressive first failure censoring scheme, a more efficient life testing plan. Compared to other censoring plans, it has a more flexible characteristic. Progressive first-failure censoring has become more popular because of the practical benefits of reducing test time and cost. At each failure time, we deplete a fixed number of groups out of the censoring mechanism, and we only observe the first failure in each group.

The following is a description of the progressive first-failure censoring scheme:

Suppose that n independent groups with k items within each group are put on a life test.

- When the first failure occurs (say at time $X^R_{1:m:n:k}$), the group with the failed item is removed, along with a specified number of other groups, R_1.
- At the second failure ($X^R_{2:m:n:k}$), again, the failed item is removed and another R_2 group.
- This continues until the m-th failure ($X^R_{m:m:n:k}$), at which point all remaining groups (R_m) are removed and the test is terminated.

These progressive first-failure-censored order statistics are denoted as $X^R_{1:m:n:k} < X^R_{2:m:n:k} < \cdots < X^R_{m:m:n:k}$, with the censoring scheme defined as $R = (R_1, R_2, \cdots, R_m)$. The total number of groups is given by the equation:

$$n = m + \sum_{i=1}^{m} R_i. \tag{3.1}$$

If the failure times of the $n \times k$ items originally in the test are from a continuous population with cdf $F(x)$ and pdf $f(x)$, the joint PDF for $X^R_{1:m:n:k}, X^R_{2:m:n:k}, \cdots, X^R_{m:m:n:k}$ is given by

$$f_{1,2,\cdots,m}(x_1, x_2, \cdots, x_m) = Ck^m \prod_{j=1}^{m} f(x_j)[1 - F(x_j)]^{k(R_j+1)-1};$$
$$0 < x_1 < \cdots < x_m < \infty, \tag{3.2}$$

where $C = n(n - R_1 - 1)\cdots(n - R_1 - R_2 - \cdots - R_{m-1} - m + 1)$.
This scheme is highly flexible and generalizes many other censoring plans:

- If $R = (0, \cdots, 0)$, the equation reduces to the joint pdf of a first-failure censored sample.
- If $k = 1$ and $R = (0, \cdots, 0)$, then $n = m$, corresponding to a complete sample.
- If $k = 1$, then the type II censored order statistics are obtained.

Although more items are used (only m of $n \times k$ items are failures) in the progressive first-failure-censoring plan, its primary advantage is the significant reduction in test time and cost. This censoring poses some unique challenges in estimating model parameters and reliability functions. The likelihood function is sufficiently complex, making it difficult to apply standard estimation procedures directly. Therefore, specialized statistical analysis techniques need to be developed or adopted. In this paper, both classical estimation methods and Bayesian estimation methods are used to estimate parameters and reliability characteristics.

4. Classical Estimation Methods

The first step in conducting any statistical inference is to estimate the unknown parameters of the model. In our study, Akash distribution is taken into account, and the only unknown parameter that needs to be estimated is θ. Initially, the maximum likelihood estimation method is used to estimate the parameter θ, followed by a determination of reliability characteristics, such as $R(t)$, $H(t)$, and $m(t)$. Furthermore, the asymptotic confidence interval (ACI) and bootstrap confidence intervals of θ are examined.

4.1. Maximum likelihood estimation

4.1.1. Point estimation. Let $\tilde{\mathbf{x}} = (x_1, x_2, ..., x_m)$ be a progressive first failure censored sample drawn from a continuous population with pdf and cdf given by (2.1) and (2.2) respectively. Then, the likelihood function using Equations (3.2), (2.1), and (2.2) is given by

$$L(\tilde{\mathbf{x}}, \theta) = Ck^m \left(\frac{\theta^3}{\theta^2 + 2}\right)^m \prod_{i=1}^{m} (1 + x_i^2) \left[1 + \frac{\theta x_i(\theta x_i + 2)}{\theta^2 + 2}\right]^{k(R_i+1)-1} e^{-\theta k(R_i+1)x_i} \tag{4.1}$$

The corresponding log-likelihood function is,

$$l(\tilde{\mathbf{x}},\theta) = A + 3m\ ln\ \theta - m\ ln\ (\theta^2+2) - \theta k\sum_{i=1}^{m}(R_i+1)x_i + \sum_{i=1}^{m}(k(R_i+1)-1)ln\ \left(1+\frac{\theta x_i(\theta x_i+2)}{\theta^2+2}\right), \tag{4.2}$$

where $A = ln\ C + m\ ln\ k + \sum_{i=1}^{m} ln\ (1+x_i^2)$ and

$$C = \prod_{k=1}^{m}\left(n-k+1-\sum_{l=0}^{k-1}R_l\right)$$

The MLE of θ is the solution of the following equation:

$$\frac{\partial(l(\tilde{\mathbf{x}},\theta))}{\partial(\theta)} = \frac{3m}{\theta} - \frac{2m\theta}{\theta^2+2} - k\sum_{i=1}^{m}(R_i+1)x_i + \sum_{i=1}^{m}\left(k(R_i+1)-1\right)\left[\frac{4x_i(\theta x_i+1)-2\theta^2 x_i}{(\theta^2+2)\left[(\theta^2+2)+\theta x_i(\theta x_i+2)\right]}\right] = 0, \tag{4.3}$$

The solution of equation (4.3) can be evaluated numerically using an iterative procedure, such as the Newton-Raphson method, for the given values of $(n, m, \tilde{\mathbf{R}}, \tilde{\mathbf{x}})$. By obtaining the MLE of θ as $\hat{\theta}$ from (4.3), the MLEs of $R(t)$, $H(t)$, and $m(t)$ can be evaluated by using the invariance property of MLEs as

$$\hat{R}(t) = \left[1+\frac{\hat{\theta}t(\theta t+2)}{\hat{\theta}^2+2}\right]e^{-\hat{\theta}t}; \quad t>0, \tag{4.4}$$

$$\hat{H}(t) = \frac{\hat{\theta}^3(1+t^2)}{\hat{\theta}t(\hat{\theta}t+2)+(\hat{\theta}^2+2)}; \quad t>0, \tag{4.5}$$

$$\hat{m}(t) = \frac{\hat{\theta}^2t^2+4\hat{\theta}t+(\hat{\theta}^2+6)}{\hat{\theta}[\hat{\theta}t(\hat{\theta}t+2)+(\hat{\theta}^2+2)]}; \quad t>0. \tag{4.6}$$

4.1.2. Confidence Intervals. To calculate the confidence interval, the Fisher information must first be calculated. The asymptotic variance-covariance matrix of the MLE of the parameter θ is the inverse of the following Fisher information matrix:

$$I(\hat{\theta}) = -E\left[\frac{\partial^2 l(\mathbf{x},\theta)}{\partial\theta^2}\right]_{\theta=\hat{\theta}} \tag{4.7}$$

where

$$\frac{\partial^2 l(x,\theta)}{\partial\theta^2} = -\frac{3m}{\theta^2} - \frac{2m(2-\theta^2)}{(\theta^2+2)^2} + \sum_{i=1}^{m}\left(k(R_i+1)-1\right)\left[\frac{4x_i(\theta^2+2)(x_i-\theta)\left[(\theta^2+2)+\theta x_i(\theta x_i+2)\right]}{(\theta^2+2)^2\left[(\theta^2+2)+\theta x_i(\theta x_i+2)\right]^2}\right. - \left.\frac{\left[4x_i(\theta x_i+1)-2\theta^2 x_i\right]\left[(\theta^2+2)[4\theta+2\theta x_i^2+2x_i]+2\theta^2 x_i(\theta x_i+2)\right]}{(\theta^2+2)^2\left[(\theta^2+2)+\theta x_i(\theta x_i+2)\right]^2}\right], \tag{4.8}$$

The variance of $\hat{\theta}$, $Var(\hat{\theta})$ is given by the inverse of the Fisher information matrix, $I^{-1}(\hat{\theta})$.

Then the asymptotic confidence interval of θ can be obtained as

$$\hat{\theta} \pm z_{\alpha/2}\sqrt{Var(\hat{\theta})}$$

where, $z_{\alpha/2}$ is the upper $(\alpha/2)^{th}$ percentile of the standard normal distribution $N(0,1)$.

Also, the Monte Carlo simulation can be used to find the coverage probabilities (CP),

$$CP = P\left[\left|\frac{(\hat{\theta}-\theta)}{\sqrt{Var(\hat{\theta})}}\right| \leq z_{\alpha/2}\right]$$

4.2. Bootstrap Methods

Bootstrap confidence intervals are statistical techniques used to determine the range of values for a population parameter based on a sample. In the case of insufficient sample sizes, bootstrap sampling techniques may be appropriate. The following steps (5) are used to construct the bootstrap confidence intervals of θ using two methods, percentile bootstrap (boot-p) (4) and bootstrap-t (boot-t) (7).

Algorithm Bootstrap-p confidence interval (PCI)

1: Generate a progressive first-failure censoring sample $\tilde{x} = (x_1, x_2, \ldots, x_m)$ with n groups of size k, censoring scheme $\tilde{R} = (R_1, R_2, \ldots, R_m)$, and effective sample size m from the Akash distribution. Compute the MLE of θ, denoted by $\hat{\theta}$.

2: Generate an independent bootstrap sample $\tilde{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)$ from the same censoring scheme using $\hat{\theta}$, and compute the corresponding MLE $\hat{\theta}^*$.

3: Repeat Step 2, M times, to obtain bootstrap MLEs $\hat{\theta}_1^*, \hat{\theta}_2^*, \ldots, \hat{\theta}_M^*$.

4: Let $\hat{\theta}_{(1)}^* \leq \hat{\theta}_{(2)}^* \leq \cdots \leq \hat{\theta}_{(M)}^*$ be the ordered values of $\hat{\theta}_i^*$. Then the approximate $100(1-\alpha)\%$ PCI of θ is

$$\left(\hat{\theta}^*_{\lfloor(\alpha/2)M\rfloor},\ \hat{\theta}^*_{\lfloor(1-\alpha/2)M\rfloor}\right),$$

where $\lfloor b \rfloor$ denotes the integer part of b.

Algorithm Bootstrap-t confidence interval (TCI)

1: Generate a progressive first-failure censoring sample $\tilde{x} = (x_1, x_2, \ldots, x_m)$ with n groups of size k, censoring scheme $\tilde{R} = (R_1, R_2, \ldots, R_m)$, and effective sample size m from the Akash distribution. Compute the MLE of θ, denoted by $\hat{\theta}$.

2: Generate an independent bootstrap sample $\tilde{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)$ from the same censoring scheme using $\hat{\theta}$, and compute the corresponding MLE $\hat{\theta}^*$.

3: Obtain the bootstrap-t statistic

$$\tau^* = \frac{\hat{\theta}^* - \hat{\theta}}{\sqrt{I^{-1}(\hat{\theta}^*)}},$$

where $I^{-1}(\hat{\theta}^*)$ is the estimated variance.

4: Repeat Steps 2–3, M times, to obtain bootstrap statistics $\tau_1^*, \tau_2^*, \ldots, \tau_M^*$.

5: Let $\tau_{(1)}^* \leq \tau_{(2)}^* \leq \cdots \leq \tau_{(M)}^*$ be the ordered values of τ_i^*. The approximate $100(1-\alpha)\%$ TCI of θ is

$$\left(\hat{\theta} - \tau^*_{\lfloor(1-\alpha/2)M\rfloor}\sqrt{I^{-1}(\hat{\theta}^*)},\ \hat{\theta} - \tau^*_{\lfloor(\alpha/2)M\rfloor}\sqrt{I^{-1}(\hat{\theta}^*)}\right).$$

The next section presents Bayes estimates for unknown parameters, θ, and reliability characteristics.

5. Bayesian Estimation Methods

The purpose of this section is to derive Bayesian estimates for the parameter, θ, and the reliability characteristics $R(t), H(t)$ and $m(t)$ of an Akash distribution. Bayes estimation is a unique method that differs from other estimation techniques. Unlike traditional estimation approaches, it does not solely rely on the observed sample data. Bayes estimation considers both the prior information and the sample data to derive reliable estimates.

5.1. Bayes Estimator under Squared Error Loss

The Bayesian inference under the squared error loss function (SELF), also known as a quadratic loss function, is considered. It is a symmetric loss function in which underestimation and overestimation are equally penalized. The squared loss function can simply be expressed as follows:

$$L_s(\nu, \eta) = (\eta - \nu)^2. \tag{5.1}$$

It is not possible to obtain a conjugate prior to the Akash distribution since it does not belong to an exponential family. However, a two-parameter gamma prior would be a reasonable choice for θ. Assume that the prior belief regarding the unknown parameter θ follows a gamma distribution with hyperparameters a and b. Therefore, the corresponding prior distribution for θ is as follows:

$$\pi(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}; \quad \theta > 0,\ a, b > 0. \tag{5.2}$$

Combining the prior distribution and the likelihood function in (4.1), the posterior distribution of θ can be written as

$$\pi(\theta|\tilde{x}) \propto \pi(\theta)L(\tilde{x}, \theta) \tag{5.3}$$

After a simple calculation, the posterior distribution of θ is obtained as

$$\begin{aligned}\pi(\theta|\tilde{x}) \propto \theta^{3m+a-1} \exp\left\{-\theta\left[b + k\sum_{i=1}^{m}(R_i+1)x_i\right] - k\sum_{i=1}^{m}(R_i+1)\ln(\theta^2+2)\right\} \\ \times \exp\left\{\sum_{i=1}^{m}[k(R_i+1)-1]\ln[\theta^2(1+x_i^2)+2(1+\theta x_i)]\right\}.\end{aligned} \tag{5.4}$$

Now, the corresponding Bayes estimate of θ against the loss function L_s is obtained as

$$\begin{aligned}E(\theta|\tilde{x}) = \int_0^\infty \theta^{3m+a} \exp\left\{-\theta\left[b + k\sum_{i=1}^{m}(R_i+1)x_i\right] - k\sum_{i=1}^{m}(R_i+1)\ln(\theta^2+2)\right\} \\ \times \exp\left\{\sum_{i=1}^{m}[k(R_i+1)-1]\ln[\theta^2(1+x_i^2)+2(1+\theta x_i)]\right\} d\theta.\end{aligned} \tag{5.5}$$

Similarly, the corresponding Bayes estimators of the reliability characteristics $R(t)$, $H(t)$, and $m(t)$ of an Akash distribution can be obtained as follows:

$$\begin{aligned}E(R(t)|\tilde{x}) = \int_0^\infty R(t)\theta^{3m+a-1} \exp\left\{-\theta\left[b + k\sum_{i=1}^{m}(R_i+1)x_i\right] - k\sum_{i=1}^{m}(R_i+1)\ln(\theta^2+2)\right\} \\ \times \exp\left\{\sum_{i=1}^{m}[k(R_i+1)-1]\ln[\theta^2(1+x_i^2)+2(1+\theta x_i)]\right\} d\theta.\end{aligned} \tag{5.6}$$

$$E(H(t)|\tilde{x}) = \int_0^\infty H(t)\theta^{3m+a-1} \exp\left\{-\theta\left[b + k\sum_{i=1}^{m}(R_i+1)x_i\right] - k\sum_{i=1}^{m}(R_i+1)\ln(\theta^2+2)\right\}$$
$$\times \exp\left\{\sum_{i=1}^{m}[k(R_i+1)-1]\ln[\theta^2(1+x_i^2)+2(1+\theta x_i)]\right\} d\theta. \tag{5.7}$$

$$E(m(t)|\tilde{x}) = \int_0^\infty m(t)\theta^{3m+a-1} \exp\left\{-\theta\left[b + k\sum_{i=1}^{m}(R_i+1)x_i\right] - k\sum_{i=1}^{m}(R_i+1)\ln(\theta^2+2)\right\}$$
$$\times \exp\left\{\sum_{i=1}^{m}[k(R_i+1)-1]\ln[\theta^2(1+x_i^2)+2(1+\theta x_i)]\right\} d\theta. \tag{5.8}$$

Where, $R(t), H(t)$ and $m(t)$ are given by equations (2.3), (2.4) and (2.5) respectively.

As seen from the above, the analytical solution to the Bayesian estimators cannot be achieved. Thus, the Tierney and Kadane (T-K) approximation (Tierney & Kadane, 1986), the importance sampling procedure, and the Metropolis-Hastings algorithm will be discussed in the following sections.

5.2. T-K Approximation

A method of approximation known as the Tierney-Kadane (T-K) Approximation method was proposed by (16). This subsection uses the T-K approximation procedure to compute the Bayes point estimates of the parameter and the reliability characteristics. The posterior mean of the parametric function $\omega(\theta)$ is given by

$$E(\omega(\theta)|\tilde{x}) = \frac{\int_0^\infty \omega(\theta)exp\{l(\tilde{\mathbf{x}},\theta)+\Pi(\theta)\}d\theta}{\int_0^\infty exp\{l(\tilde{\mathbf{x}},\theta)+\Pi(\theta)\}d\theta}, \tag{5.9}$$

where, $\Pi(\theta) = ln\ \pi(\theta)$.

Let's define the following two functions:

$$\Delta(\theta) = \frac{l(\tilde{\mathbf{x}},\theta)+\Pi(\theta)}{m}, \tag{5.10}$$

$$\Delta^*(\theta) = \Delta(\theta) + \frac{ln(\omega(\theta))}{m}, \tag{5.11}$$

If $\hat{\theta}$ and $\hat{\theta}^*$ are the values which maximizes the Equations (5.10) and (5.11) respectively, the posterior mean of the parametric function $\omega(\theta)$ is given by (5.9) is approximated by

$$E(\omega(\theta)|\tilde{x}) = \sqrt{\frac{|\Sigma^*|}{|\Sigma|}}\ exp\Big[m\Big(\Delta^*(\hat{\theta}^*) - \Delta(\hat{\theta})\Big)\Big] \tag{5.12}$$

Where, $|\Sigma^*|$ and $|\Sigma|$ are the determinants of negative inverse Hessian of $\Delta^*(\theta)$ and $\Delta(\theta)$. Using the T-K approximation, Bayes estimate of θ is obtained by incorporating prior distributions into the log-likelihood function, where $\omega^*(\theta)$ is defined as,

$$\Delta(\theta) = \frac{1}{m}\Bigg[(3m+a-1)\ ln\ \theta - \theta\Big\{b + k\sum_{i=1}^{m}(R_i+1)x_i\Big\} - m\ ln\ (\theta^2+2)$$
$$+ \sum_{i=1}^{m}(k(R_i+1)-)ln\ \left(1+\frac{\theta x_i(\theta x_i+2)}{\theta^2+2}\right)\Bigg] \tag{5.13}$$

$$\frac{\partial\Delta(\theta)}{\partial\theta} = \frac{3m+a-1}{\theta} - b - k\sum_{i=1}^{m}(R_i+1)x_i - \frac{2m\theta}{\theta^2+2} + \sum_{i=1}^{m}\Big(k(R_i+1)-1\Big)\Bigg[\frac{4x_i(\theta x_i+1)-2\theta^2 x_i}{(\theta^2+2)\big[(\theta^2+2)+\theta x_i(\theta x_i+2)\big]}\Bigg]. \tag{5.14}$$

Also,

$$\frac{\partial^2\Delta(\theta)}{\partial\theta^2} = -\frac{3m+a-1}{\theta^2} - \frac{2m(2-\theta^2)}{(\theta^2+2)^2} + \sum_{i=1}^{m}\Big(k(R_i+1)-1\Big) \Bigg[\frac{4x_i(\theta^2+2)(x_i-\theta)\big[(\theta^2+2)+\theta x_i(\theta x_i+2)\big] - \big[4x_i(\theta x_i+1)-2\theta^2 x_i\big]\big[(\theta^2+2)[4\theta+2\theta x_i^2+2x_i]+2\theta}{(\theta^2+2)^2\big[(\theta^2+2)+\theta x_i(\theta x_i+2)\big]^2} \tag{5.15}$$

Hence,

$$|\Sigma| = \left|\frac{\partial^2\Delta(\theta)}{\partial\theta^2}\right|^{-1}_{\theta=\hat{\theta}}. \tag{5.16}$$

To compute the approximate Bayes estimates of θ using the T-K Approximation, the following can be considered;

$$\Delta^*(\theta) = \Delta(\theta) + \frac{ln(\theta)}{m}. \tag{5.17}$$

The T-K approximation estimator θ, $\hat{\theta}_{TK}$ is obtained by solving:

$$\frac{\partial(\Delta^*(\theta))}{\partial(\theta)} = \frac{1}{m}\Bigg[\frac{3m+a}{\theta} - \frac{2m\theta}{\theta^2+2} - b - k\sum_{i=1}^{m}(R_i+1)x_i + \sum_{i=1}^{m}\Big(k(R_i+1)-1\Big)\Bigg[\frac{4x_i(\theta x_i+1)-2\theta^2 x_i}{(\theta^2+2)\big[(\theta^2+2)+\theta x_i(\theta x_i+2)\big]}\Bigg]\Bigg] = 0, \tag{5.18}$$

Similarly, the the approximate Bayes estimates of $R(t)$, $H(t)$, and $m(t)$ which are represented by $\hat{R}_{TK}(t)$, $\hat{H}_{TK}(t)$, and $\hat{m}_{TK}(t)$ respectively can be derived.

5.3. Importance sampling procedure

A Bayesian estimate of the parameter and the reliability characteristics can also be computed using importance sampling (IS). The posterior distribution shown in (44) can be rewritten as follows:

$$\pi(\theta|\underset{\sim}{x}) \propto f_{GD}(\theta; 3m+a, S)U(\theta), \tag{5.19}$$

where, $S = \left[b + k\sum_{i=1}^{m}(R_i+1)x_i\right]$,

$$U(\theta) = exp\Bigg\{\sum_{i=1}^{m}\big[k(R_i+1)-1\big]ln[\theta^2(1+x_i^2)+2(1+\theta x_i)] - k\sum_{i=1}^{m}(R_i+1)ln(\theta^2+2)\Bigg\}$$

and

$f_{GD}(.; p, q)$ is a gamma density with shape parameters p and scale parameters q. Now, the Bayes estimate of $\phi(\theta)$, a function of θ, under the SELF is given by,

$$\hat{\omega}_{IS}(\theta) = E[\omega(\theta)|\underset{\sim}{x}] = \frac{\int_0^\infty \omega(\theta)\pi(\theta|\underset{\sim}{x})d\theta}{\int_0^\infty \pi(\theta|\underset{\sim}{x})d\theta}, \tag{5.20}$$

Using the following algorithm, Bayesian estimates of the parameters can be approximated.

Algorithm Importance Sampling (I-S) Algorithm

1: Generate $\theta^{(1)}$ from $f_{GD}(\theta; 3m+a, S)$.
2: Repeat Step 1, M times to obtain $\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(M)}$.
3: The approximate Bayesian estimate of a function of the parameter $\omega(\theta)$ is

$$\hat{\omega}_{IS}(\theta) = \frac{\sum_{j=1}^{M} \omega(\theta^{(j)})\, U(\theta^{(j)})}{\sum_{j=1}^{M} U(\theta^{(j)})}.$$

4: The Bayesian estimates of θ and the associated reliability characteristics under the SELF are

$$\hat{\theta}_{IS} = \frac{\sum_{j=1}^{M} \theta^{(j)} U(\theta^{(j)})}{\sum_{j=1}^{M} U(\theta^{(j)})}, \quad \hat{R}_{IS}(t) = \frac{\sum_{j=1}^{M} \left[(1 + \frac{\theta^{(j)}t(\theta^{(j)}t+2)}{(\theta^{(j)})^2+2})e^{-\theta^{(j)}t}\right] U(\theta^{(j)})}{\sum_{j=1}^{M} U(\theta^{(j)})},$$

$$\hat{H}_{IS}(t) = \frac{\sum_{j=1}^{M} \frac{(\theta^{(j)})^3(1+t^2)}{\theta^{(j)}t(\theta^{(j)}t+2)+((\theta^{(j)})^2+2)} U(\theta^{(j)})}{\sum_{j=1}^{M} U(\theta^{(j)})}, \quad \hat{m}_{IS}(t) = \frac{\sum_{j=1}^{M} \frac{(\theta^{(j)})^2t^2+4\theta^{(j)}t+((\theta^{(j)})^2+6)}{\theta^{(j)}[(\theta^{(j)}t(\theta^{(j)}t+2)+(\theta^{(j)})^2+2)]} U(\theta^{(j)})}{\sum_{j=1}^{M} U(\theta^{(j)})}.$$

5.4. Metropolis-Hastings (M-H) Algorithm

The M-H algorithm is one of the widely used MCMC techniques. For more information, see (6) and the references cited therein. This method is used to generate a sequence of samples based on a posterior distribution. There are several applications of this algorithm described in (10) and Hastings (8). The steps of a M-H algorithm are summarized below.

Algorithm Metropolis-Hastings (M-H) Algorithm

1: Consider an initial guess of θ, say $\theta^{(0)}$.
2: Generate a candidate point $\theta_c^{(j)}$ from the proposal density $\eta(\theta^{(j)}|\theta^{(j-1)})$.
3: Generate u from a uniform distribution $U(0,1)$.
4: Compute the acceptance probability

$$A(\theta^{(j)}|\theta^{(j-1)}) = \min\left\{\frac{\pi(\theta_c^{(j)}|\underset{\sim}{x})\,\eta(\theta^{(j-1)}|\theta_c^{(j)})}{\pi(\theta^{(j-1)}|\underset{\sim}{x})\,\eta(\theta_c^{(j)}|\theta^{(j-1)})}, 1\right\}.$$

5: If $u \le A$, set $\theta^{(j)} = \theta_c^{(j)}$; otherwise, set $\theta^{(j)} = \theta^{(j-1)}$.
6: Repeat steps 2–5 for $j = 1, 2, \ldots, M$ to obtain the sequence $(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(M)})$.

Therefore, to obtain the estimate, we use the $(M - M_0)$ observations, where M_0 is the burn-in period. Using the M-H algorithm under SELF, the approximate Bayes estimate of $\phi(\theta)$ can be calculated as follows:

$$\hat{\omega}_{MH}(\theta) = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \omega(\theta^{(j)}).$$

Therefore, the Bayesian estimates of the parameter θ and the reliability characteristics $R(t), H(t)$ and $m(t)$ of an Akash distribution under SELF are computed as follows:

$$\hat{\theta}_{MH} = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\theta^{(j)}$$

$$\hat{R}_{MH}(t) = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\left[1+\frac{\theta^{(j)}t(\theta^{(j)}t+2)}{{\theta^{(j)}}^2+2}\right]e^{-\theta^{(j)}t},$$

$$\hat{H}_{MH}(t) = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\frac{{\theta^{(j)}}^3(1+t^2)}{\theta^{(j)}t(\theta^{(j)}t+2)+({\theta^{(j)}}^2+2)},$$

and

$$\hat{m}_{MH}(t) = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\frac{{\theta^{(j)}}^2t^2+4\theta^{(j)}t+({\theta^{(j)}}^2+6)}{\theta^{(j)}[\theta^{(j)}t(\theta^{(j)}t+2)+({\theta^{(j)}}^2+2)]}.$$

As a next step, we will calculate the HPD credible interval.

5.5. HPD credible interval

The Highest Posterior Density (HPD) credible interval for the parameter θ can indeed be obtained using samples generated by the M-H algorithm. When dealing with complex Bayesian models, where direct analytical solutions are not possible, this method is particularly useful.

Let $\theta_{(1)} < \theta_{(2)} < ... < \theta_{(M)}$ be the orders value of $\theta^{(1)}, \theta^{(2)}, ..., \theta^{(M)}$. Now, using the algorithm proposed by (2), the $100(1-\xi)\%$, where, $0 < \xi < 1$, HPD credible interval of θ will be $(\theta_{(j)}, \theta_{(j+[(1-\xi)M])})$, where j is chosen such that

$$\theta_{(j+[(1-\xi)M])} - \theta_{(j)} = min_{1\le i\le \xi M}(\theta_{(j+[(1-\xi)M])} - \theta_{(j)}),\ j = 1, 2, ..., M,$$

where, $[x]$ is the integer part of x.

6. Simulation Study

In this section, we present and evaluate the effectiveness of the various estimation methods investigated for the Akash distribution. Thus, we conduct a Monte Carlo simulation to determine the effectiveness of the evaluation method. In this comparison study, we used bias and mean squared errors (MSEs) to measure the effectiveness of each of the proposed estimation methods. All programs for this computational study were coded in the statistical software R version 4.4.1 (2024-06-14 ucrt).

In this study, progressive first failure-censored samples of the Akash distribution are generated for several combinations of (n, m, k) with a predetermined censoring plan R and for different values of the model parameter θ. With some modifications, the algorithm proposed by (1) is employed to generate these samples. These modifications allow the progressive first failure-censored sample $(x_1, x_2, \ldots, x_m)$ to be viewed as a progressively censored sample from a population with cumulative distribution function (CDF) $[1-(1-F(x))^k]$ (see (17)).

For simulation purposes, a large number $N = 2000$ of progressive first failure-censored samples with various combinations of (n, m, k) and R (as shown in Table 1) are generated from the Akash distribution. Note that the notation used in censoring schemes, such as (0*4), denotes $(0, 0, 0, 0)$. Two different representative values of θ are considered: $\theta = 0.67$ and $\theta = 1.2$. However, due to space limitations, only the set with $\theta = 0.67$ is used for simulation purposes. For each n, four different failure plans are adopted, with three of these plans being common across all values of n. The three common failure plans are as follows:

Plan 1: $\{(k,n,m),(R_1 = n-m,\ W_i = 0,\ \ \forall\ \ i = 2,3,...,m)\}$. In this case, $(n-m)$ groups are discarded from the experiment at the first failure only.

Plan 2: $\{(k,n,m),(W_i = 0,\ \ \forall\ \ i = 1,2,...,m-1,\ W_m = n-m)\}$. In this case, $(n-m)$ groups are removed at the m^{th} failure, and

Plan 3: $\{(k,n = m),(W_i = 0,\ \ \forall\ \ i = 1,2,...,m)\}$. This is the case of the first failure censored sample.

Two different priors were used for the Bayes estimates as follows.

(i) A prior distribution that is almost non-informative was specified by setting $a = b = 0.001$, where these small values were chosen to ensure that the posterior distribution remains integrable.
(ii) An informative prior with $a = 2, b = 3$.

The average confidence level (AL) and coverage probability (CP) of five different confidence intervals for θ—namely, the asymptotic confidence interval (ACI), PCI, TCI, and HPD—are calculated for comparison purposes. In addition, four different estimates of $\theta, R(t), H(t)$ and $m(t)$ were calculated from each sample. The bias, that is, $\left(E(\hat{\theta}) - \theta\right)$, and MSE were calculated for the four different estimators for $\theta, R(t), H(t)$ and $m(t)$. For the Bayesian methods, both informative and non-informative cases are considered. For the Bayesian methods, both informative and noninformative prior cases were considered as discussed above.

Classification of Censoring Levels: Classifying censoring levels is useful for interpreting simulation results and ensuring balanced coverage of different censoring intensities. First, the percent censored is calculated by

$$\text{Percent Censored (p)} = \frac{\sum_{i=1}^{m} R_i}{n} \times 100\%.$$

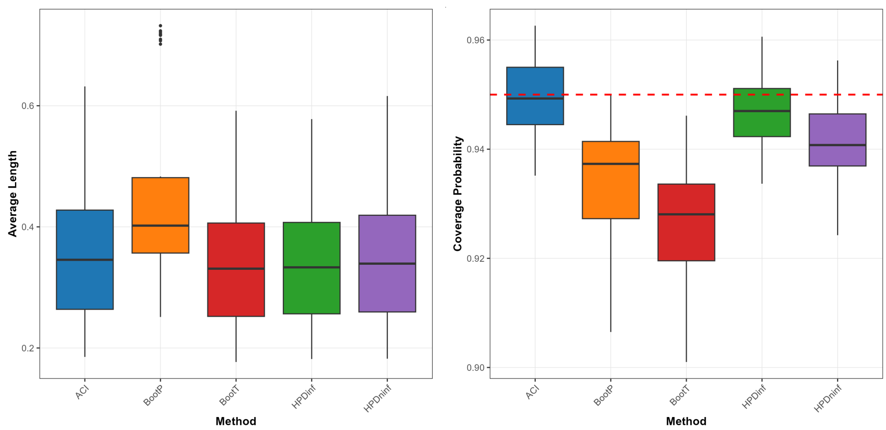
Based on the proportion censored p, the censoring schemes are classified as follows:

$$\begin{cases} \text{Light censoring,} & p \leq 0.30, \\ \text{Moderate censoring,} & 0.30 < p \leq 0.60, \\ \text{Heavy censoring,} & p > 0.60. \end{cases}$$

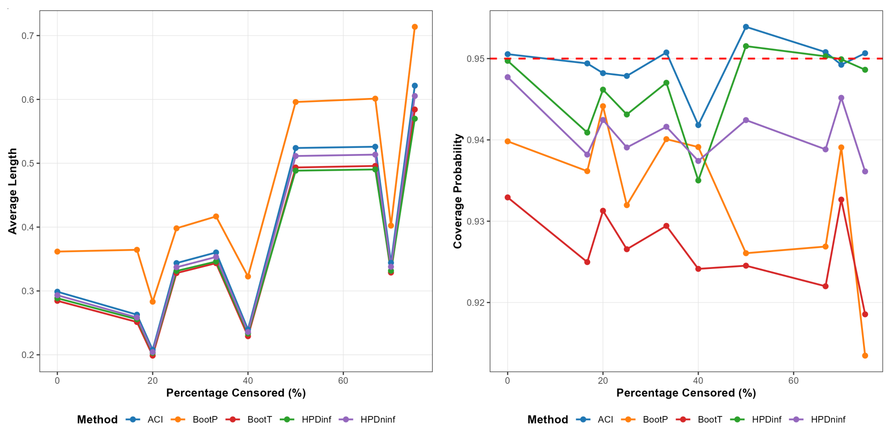
The low proportion of censoring ($p \leq 0.30$) schemes are named as light censoring. This shows retention of the majority of the corresponding data. The medium proportion of censoring ($0.30 < p \leq 0.60$) that reflects a partial loss of information is classified as moderate censoring. The schemes, ($p > 0.60$) are categorized as heavy censoring, representing severe information loss where estimation becomes more challenging. See Table 1 for the classification of R.

Table 1. Progressive first-failure censoring schemes with %Censored

CS	n	m	R	%CS
[1]	10	5	(3,0*3,2)	50%
[2]	10	5	(1*5)	50%
[3]	10	5	(2,1*3,0)	50%
[4]	10	10	(0*10)	0%
[5]	15	5	(7,3,0*3)	66.7%
[6]	15	5	(4,0*2,2,4)	66.7%
[7]	15	5	(0,1,2,3,4)	66.7%
[8]	15	10	(3,0*8,2)	33.3%
[9]	15	10	((1,0)*5)	33.3%
[10]	15	10	(5,0*9)	33.3%
[11]	15	15	(0*15)	0%
[12]	20	5	(3,0*3,12)	75%
[13]	20	5	(0*2,15,0*2)	75%
[14]	20	5	(3*5)	75%
[15]	20	10	(0*9,10)	50%
[16]	20	10	(1*10)	50%
[17]	20	10	(1*4,0*4,3*2)	50%
[18]	20	15	(0*10,1*5)	25%
[19]	20	15	(1*3,0*10,1*2)	25%
[20]	20	15	(3,0*13,2)	25%
[21]	20	20	(0*20)	0%
[22]	30	10	(20,0*9)	66.7%
[23]	30	10	(5*3,0*6,5)	66.7%
[24]	30	10	(0*4,10*2,0*4)	66.7%
[25]	30	20	(5*2,0*18)	33.3%
[26]	30	20	(10,0*19)	33.3%
[27]	30	20	((0,1)*10)	33.3%
[28]	30	25	(2*2,0*22,1)	16.7%
[29]	30	30	(1,0*22,2*2)	16.7%
[30]	30	30	(0*10,1*5,0*10)	16.7%
[31]	30	30	(0*30)	0%
[32]	50	15	(35,0*14)	70%
[33]	50	15	(0*14,35)	70%
[34]	50	15	(0*7,35,0*7)	70%
[35]	50	30	((2,0,0)*10)	40%
[36]	50	30	(1*20,0*10)	40%
[37]	50	30	(20,0*29)	40%
[38]	50	40	(10,0*39)	20%
[39]	50	40	(0*39,10)	20%
[40]	50	40	(0*15,1*10,0*15)	20%
[41]	50	50	(0*50)	0%



(a) Distribution of Length and Coverage



(b) Impact of Censoring on Performance

Figure 1. Performance of Confidence Interval Methods

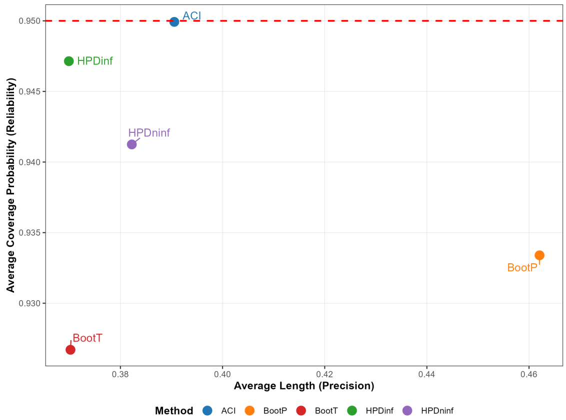


Figure 2. Precision-Reliability Trade-off of Confidence Intervals

The relative comparison of confidence interval methods illustrates explicit trade-offs involving precision and reliability. According to the Fig.(1), asymptotic Confidence Interval (ACI) always provides coverage probabilities with the closest convergence to the desired level (0.95) while having comparatively moderate interval length - thus providing the best balance of reliability and precision. Bayesian HPD intervals (HPD_{inf} and HPD_{ninf}) also act similarly and have the potential to offer quite short intervals with a slight loss in coverage, which also reflects their ability to balance interval length and precision. On the other hand, bootstraps seem limited; BootP gives the highest intervals with lower coverage, and BootT gives the lowest coverage probabilities in all the conditions, which means that bootstrap methods are not as accurate in the current context concerning censoring. Furthermore, the censoring aspect of the analyses corroborates these claims. All methods increase in the size of the interval with a greater percentage of censoring. As illustrated in Fig. 2, the HPD method's average length is shorter — indicating higher precision — than that of the ACI method; however, it does not achieve the target reliability of 0.95. By contrast, the ACI method offers the best overall performance, emphasizing the target reliability as a primary method for establishing a confidence interval along with an acceptable average length.

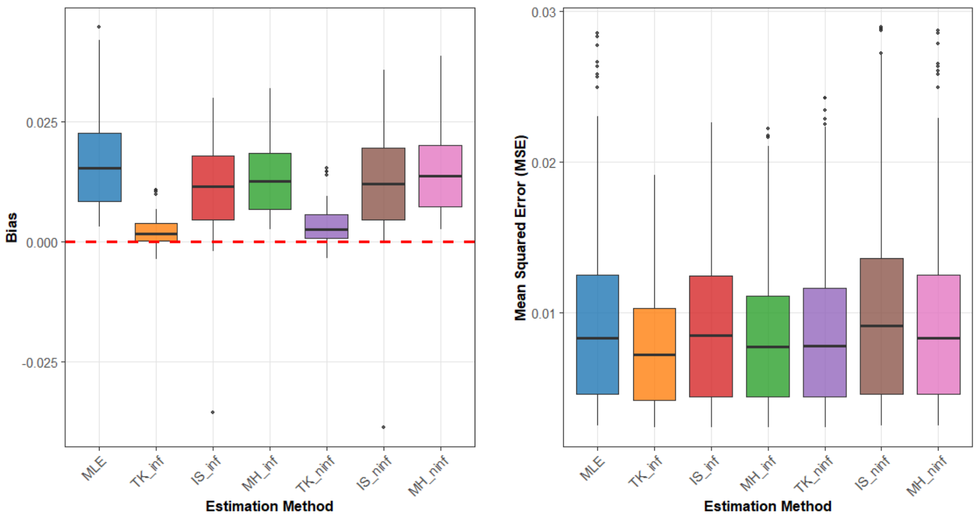
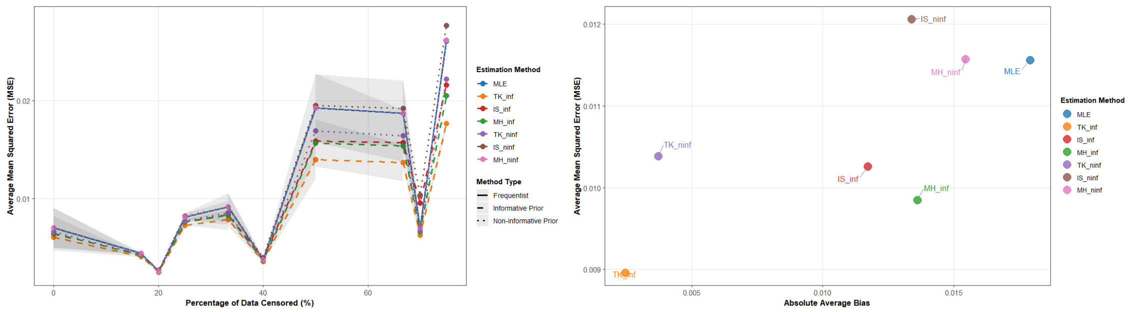


Figure 3. Distribution of Bias/MSE across all estimation methods.



(a) Impact of Censoring on Estimation. **(b)** All Estimation Methods.

Figure 4. Performance of Estimation Methods of θ.

Figure 3 and Figure 4 illustrate the θ estimation methods, highlighting trade-offs in bias and efficiency between frequentist and Bayesian approaches. The maximum likelihood estimator (MLE) has low mean square error (MSE); however, it has considerable bias, which limits its usefulness. Among all the Bayesian estimators with informative priors, the best is $\mathrm{TK}_{\mathrm{inf}}$, which exhibits nearly zero bias and a low MSE, consistent with other methods. The $\mathrm{IS}_{\mathrm{inf}}$ and $\mathrm{MH}_{\mathrm{inf}}$ had slight additional bias and moderate efficiency. The non-informative prior estimators provided larger variance and MSE than estimators with informative priors. Regarding transfer estimates, as more censoring occurred, all the estimates lost accuracy; however, Bayesian informative statistics were low but still more stable overall. Overall, while MLE is a suitable option for frequentist estimation of such parameters, the $\mathrm{TK}_{\mathrm{inf}}$ offers improved bias and efficiency compared to the MLE and other Bayesian options in the presence of more maximum censoring.

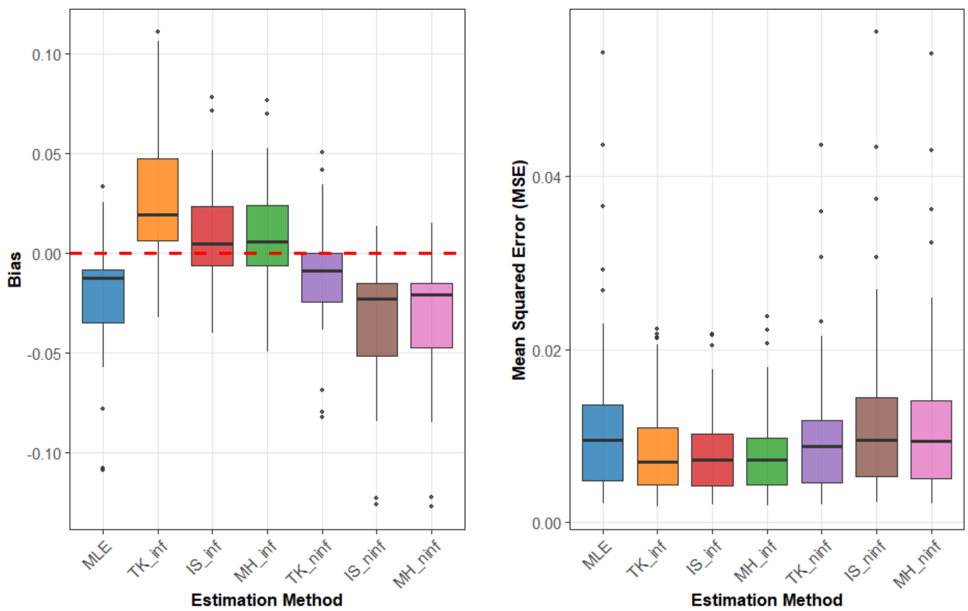
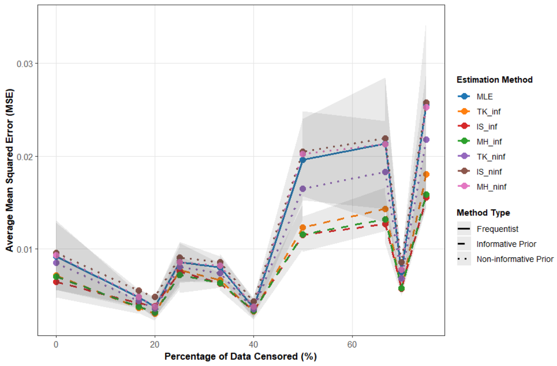
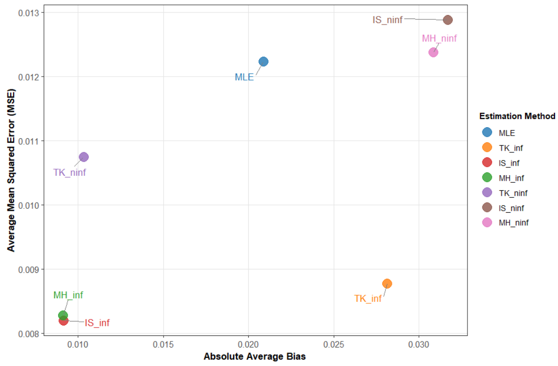


Figure 5. Distribution of Bias/MSE across all estimation methods.



(a) Impact of Censoring on Estimation.



(b) All Estimation Methods.

Figure 6. Performance of Estimation Methods of $R(t)$.

Figure 5 and Figure 6 illustrate the $R(t)$ estimation processes, with bias-efficiency trade-offs for Bayesian and frequentist approaches. The trade-off for bias and MSE is noted from Figure 5. We can see that the Bayesian informative prior clustering near the origin has a better trade-off towards reducing bias over MSE. The non-informative Bayesian prior is a larger bias and MSE. The MLE estimator has smaller bias but with larger MSE.

Figure 6a shows an apparent connection between the efficiencies of estimations and proportions of censoring. In general, mean squared error (MSE) increased with the increase of censoring proportions. There were comparable trends of performances between all the estimation processes, but estimation processes with Bayesian and informative prior resulted in the lowest total MSE. The result shows that there is greater robustness against data loss with the usage of a Bayesian informative prior. Figure 6b gives support for the conclusions above based on the facts that MH_{inf} and IS_{inf} are offering lower MSE and lower bias.

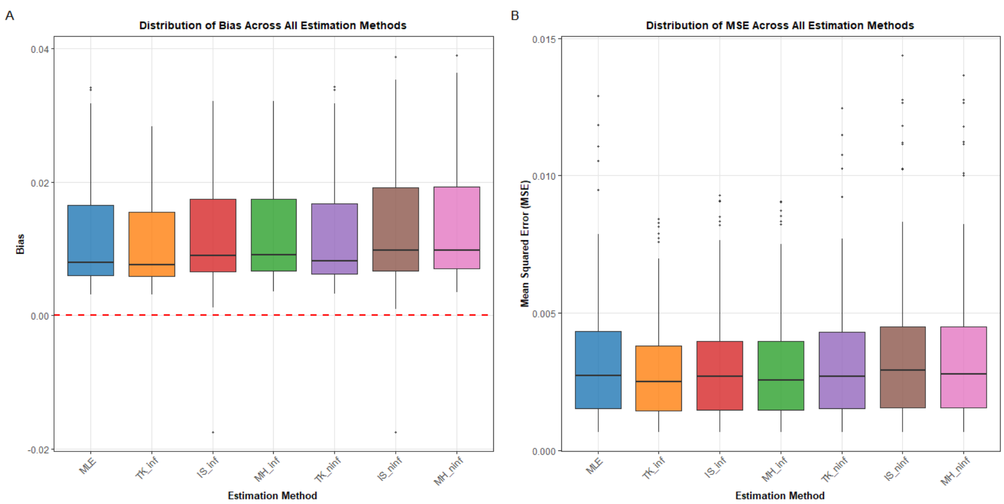
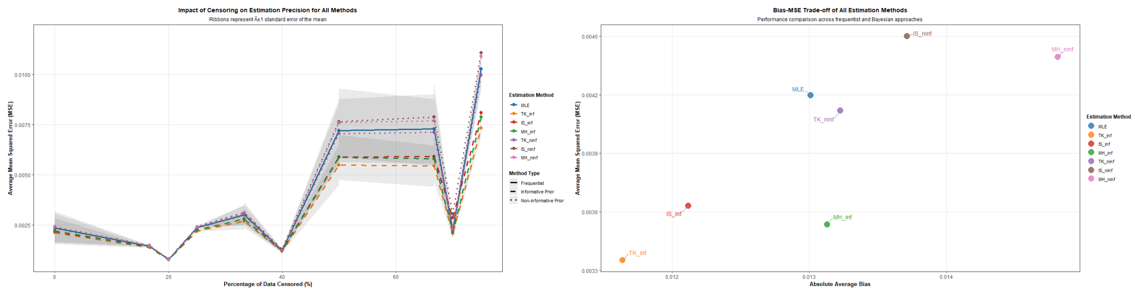


Figure 7. Distribution of Bias/MSE across all estimation methods.



(a) Impact of Censoring on Estimation. **(b)** All Estimation Methods.

Figure 8. Performance of Estimation Methods of $H(t)$.

Figure 7 and Figure 8 illustrate the $H(t)$ estimation methods, comparing the efficiencies of frequentist and Bayesian approaches. Figures 7 summarize the bias and MSE of different approaches. Both the bias and MSE factors are narrower for informative prior Bayesian procedures. All three methods rather are wider in range/or variability due to non-informative priors; while MLE doesn't suffer from bias and is expected to be narrower in distribution, it provides more variation than informative prior approaches. The comparative performance of estimating procedures is depicted through 8a, with MSE being observed to increase with growing rates of censoring for all procedures. The MSE is observed to always remain lower with respect to non-informative prior Bayesian procedures and maximum likelihood estimator techniques for all informative prior Bayesian procedures considered herein. These observations confirm that using prior information provides greater precision and accuracy to estimating procedures, especially in situations with widespread data loss. 8b gives support for the conclusions above based on the facts that TK_{inf} is offering lower MSE and lower bias.

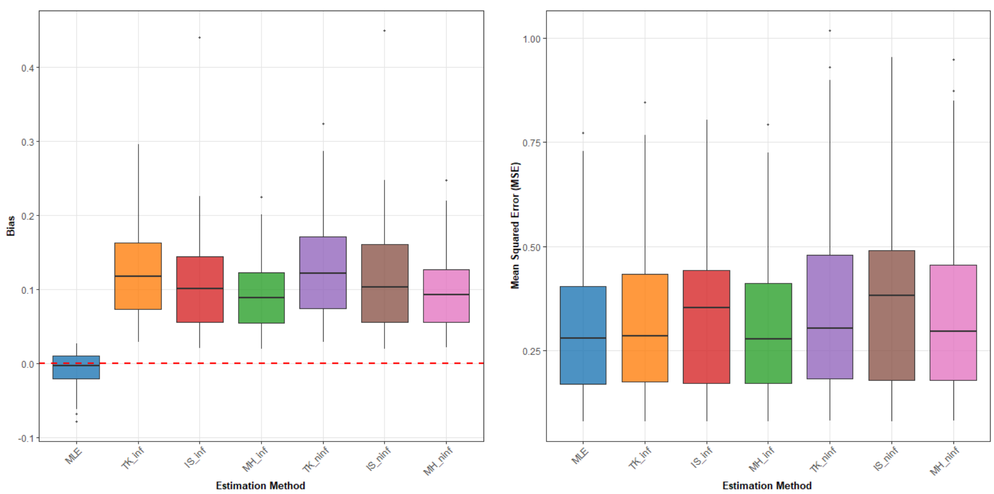
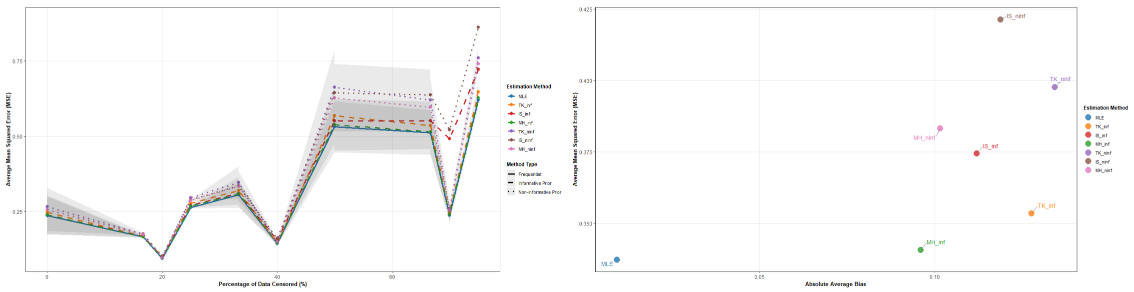


Figure 9. Distribution of Bias/MSE across all estimation methods.



(a) Impact of Censoring on Estimation. **(b)** All Estimation Methods.

Figure 10. Performance of Estimation Methods of $m(t)$.

Figure 9 and Figure 10 illustrate the $m(t)$ estimation methods, comparing the efficiencies of frequentist and Bayesian approaches. Based on the Figure 9, MLE exhibits the lowest bias, while Bayesian procedures under informative priors, in particular $\mathrm{MH}_{\mathrm{inf}}$ and $\mathrm{TK}_{\mathrm{inf}}$, exhibit enhanced efficiency by lowering MSE, though accompanied by slightly higher bias levels. On the other hand, Bayesian procedures under non-informative priors exhibit always higher MSE and bias. This Figure 10a again demonstrates these tendencies under varying levels of censoring. MSE increases with increasing percentages of censoring, while informative Bayesian procedures remain relatively more efficient and stabler than those of both MLE and non-informative Bayesian estimations. Most importantly, although MLE is relatively unbiased, it is no longer efficient under severe censoring, while informative Bayesian procedures show increased resistance against data loss. Figure 10b gives support for the conclusions above based on the facts that MLE is offering lower MSE and lower bias.

7. Real Data modelling

We now illustrate the methods proposed in the previous sections using real data sets.

First, a progressive first-failure-censored sample is generated from an Akash distribution with parameters $\theta = 1.8,\ k = 5, n = 30$ and $m = 8$ using the algorithm of (1). The generated data and the censoring scheme are presented in Table 2. To make the comparison more meaningful, we assume both non-informative and informative priors on the parameters.

Table 2. Simulated progressive first-failure-censored sample.

i	1	2	3	4	5	6	7	8
R_i	0	0	5	0	0	3	0	14
X_{R_i}	0.00134	0.00552	0.00790	0.02752	0.03909	0.03992	0.04454	0.06555

From the tables (3,4) and figure (11) above, Bayesian methods with informative priors excel others in point estimation, especially the Metropolis–Hastings (MH) approach, which showed the smallest error (DTV = 0.0241). In summary, methods using non-informative priors performed poorly, highlighting the value of prior information. For functions, reliability, hazard rate, and MRL, Bayesian estimates were closer to true values than those of MLE, which showed higher bias.

In interval estimation, the Bayesian HPD interval with an informative prior was the most efficient (shortest length = 0.6515), outperforming frequentist and noninformative Bayesian intervals. The commonly used bootstrap-p method produced overly wide intervals, limiting its usefulness.

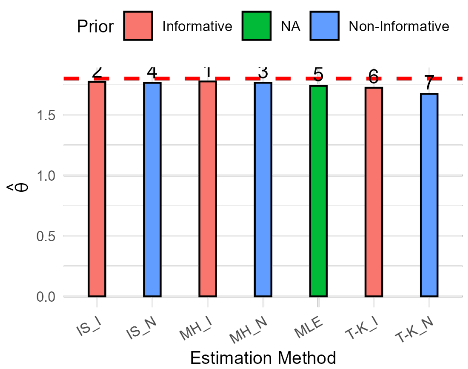
Table 3. Comparison of parameter estimation methods for θ, reliability, hazard rate, and MRL.

Method	Prior	θ	Reliability	Hazard	MRL	DTV	Rank
True Values		**1.8000**	**0.1325**	**1.1480**	**0.7141**		
MLE		1.7394	0.1478	1.0925	0.7186	0.0606	5
T-K	Non-Inf.	1.6732	0.1939	1.0322	0.7238	0.1268	7
IS	Non-Inf.	1.7645	0.1656	1.1198	0.7191	0.0355	4
MH	Non-Inf.	1.7651	0.1658	1.1214	0.7190	0.0349	3
T-K	Inf.	1.7236	0.1687	1.0781	0.7198	0.0764	6
IS	Inf.	1.7728	0.1541	1.1267	0.7175	0.0272	2
MH	Inf.	1.7759	0.1531	1.1290	0.7173	0.0241	1

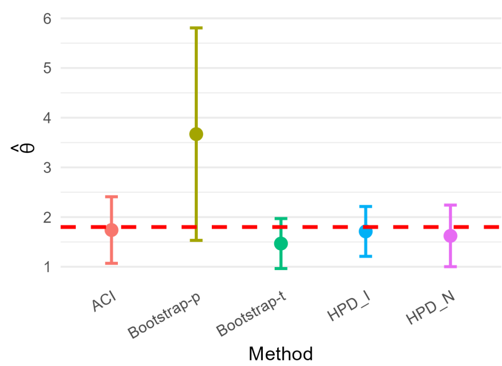
Note: Non-Inf. = Non-Informative Prior ($a = 0.001$, $b = 0.001$); Inf. = Informative Prior ($a = 16$, $b = 10.4$); DTV = Distance to True Value of θ.

Table 4. Comparison of Confidence Interval Methods

Method	Lower	Upper	Length	Rank
ACI	1.1417	2.7007	1.5590	4
Bootstrap-p	1.7496	6.2795	4.5299	5
Bootstrap-t	1.0497	2.1088	1.0591	2
HPD_Non-Inf.	1.0685	2.5421	1.4737	3
HPD_Inf.	0.8049	1.4564	0.6515	1



(a) Comparison of Estimation Methods.



(b) Comparison of Interval Estimation Methods for θ.

Figure 11

Estimation methods were ranked according to their DTV values in ascending order, whereas confidence interval methods were ranked based on the length of the interval, also in ascending order.

7.1. Application 2

Now, we consider a real-life data set and illustrate the methods proposed in the previous sections. The real data set is from (12) and has been analyzed using the Weibull and exponentiated exponential distributions in (11). The data concerning the tensile strength of 100 observations of carbon fibers are:

3.7, 3.11, 4.42,3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

Table 5. Model comparison using log-likelihood, information criteria, and K–S goodness-of-fit test.

Model	k	LogLik	AIC	BIC	AICc	K–S p-value
Weibull	2	-140.9957	285.9915	291.2018	286.1152	0.8194
Akash	1	-173.1686	348.3371	350.9423	348.3779	0.0001
Lindley	1	-181.3755	364.7510	367.3561	364.7918	0.0000016
Exponential	1	-195.9886	393.9773	396.5825	394.0181	0.0000000020

Note: k = number of parameters.

From Table 5, although the Weibull distribution provides the best overall fit with the lowest AIC, BIC, and AICc values and the highest K-S p-value, the Akash distribution remains significant. As a one-parameter model, it achieves a much better fit than the Lindley and Exponential distributions while maintaining simplicity. Thus, while the Weibull is statistically superior, we emphasize the Akash distribution for its competitive performance among one-parameter models and its value as a simpler, more interpretable alternative to the two-parameter Weibull.

The data set is first arranged in ascending order and subsequently partitioned into 25 randomized groups with $k = 4$. The following shows the groups of the data set. $\{0.39, 0.81, 0.85, 0.98\}, \{1.08, 1.12, 1.17, 1.18\}, \{1.22, 1.25, 1.36, 1.41\}, \{1.47, 1.57, 1.57, 1.59\}, \{1.59, 1.61, 1.61, 1.69\}, \{1.69, 1.71, 1.73, 1.80\}, \{1.84, 1.84, 1.87, 1.89\}, \{1.92, 2.00, 2.03, 2.03\}, \{2.05, 2.12, 2.17, 2.17\}, \{2.17, 2.35, 2.38, 2.41\}, \{2.43, 2.48, 2.48, 2.50\}, \{2.53, 2.55, 2.55, 2.56\}, \{2.59, 2.67, 2.68, 2.73\}, \{2.74, 2.76, 2.77, 2.79\}, \{2.81, 2.81, 2.82, 2.83\}, \{2.85, 2.87, 2.88, 2.93\}, \{2.95, 2.96, 2.97, 2.97\}, \{3.09, 3.11, 3.11, 3.15\}, \{3.15, 3.19, 3.19, 3.22\}, \{3.22, 3.27, 3.28, 3.31\}, \{3.31, 3.33, 3.39, 3.39\}, \{3.51, 3.56, 3.60, 3.65\}, \{3.68, 3.68, 3.70, 3.75\}, \{4.20, 4.38, 4.42, 4.70\}, \{4.90, 4.91, 5.08, 5.56\}$.

The table 6 shows that the progressive first failure censored size data (m = 25) of the 25 carbon fiber groups were observed.

Table 6. Simulated progressive first-failure censored sample.

i	1	2	3	4	5	6	7	8	9	10
R_i	0	0	1	0	0	0	2	2	0	0
X_{R_i}	0.39	1.08	1.22	1.59	1.69	1.84	1.92	2.05	2.17	2.53
i	11	12	13	14	15	16	17	18	19	20
R_i	0	0	0	0	0	0	0	0	0	0
X_{R_i}	2.74	2.81	2.85	3.09	3.15	3.22	3.31	3.51	3.65	4.90

This application involves 5 censored groups and 20 first observed failures.

Based on tables (7) and (8), all three methods (frequentist (MLE) and Bayesian with both non-informative and informative priors) yield similar estimates of θ, along with consistent reliability, hazard, and mean residual life (MRL) results at $t = 1.3$. Bayesian estimates via informative priors are somewhat more stable, while the MLE results are more stable, confirming the consistency of the estimates across methodologies. Confidence interval assessments reveal that the bootstrap percentile method yields the narrowest and most accurate intervals, while their HPD counterparts are wider and more conservative.

Table 7. Estimates of θ, reliability, hazard, and mean residual life (MRL) under different methods and priors.

Method	Prior	$\hat{\theta}$	Reliability	Hazard	MRL
MLE		0.4642	0.9348	0.0711	5.1164
T-K	Non-Inf	0.4595	0.9365	0.0694	5.1809
IS	Non-Inf	0.4627	0.9342	0.0715	5.2018
MH	Non-Inf	0.4620	0.9345	0.0712	5.2146
T-K	Inf	0.4589	0.9367	0.0691	5.1893
IS	Inf	0.4621	0.9345	0.0712	5.2135
MH	Inf	0.4620	0.9345	0.0713	5.2146

Note: Non-Inf. = Non-Informative Prior ($a = 0.001$, $b = 0.001$); Inf. = Informative Prior ($a = 1.25$, $b = 3$).

Table 8. Comparison of confidence intervals and their lengths.

Method	**Lower**	**Upper**	**Length**	**Rank**
ACI	0.3726	0.5558	0.1833	4
Bootstrap-p	0.3930	0.5188	0.1258	1
Bootstrap-t	0.4156	0.5484	0.1327	2
HPD_Non-Inf.	0.3720	0.5553	0.1833	3
HPD_Inf.	0.3725	0.5423	0.1698	5

Note: Non-Inf. = Non-Informative Prior ($a = 0.001$, $b = 0.001$); Inf. = Informative Prior ($a = 3$, $b = 2$). ACI = Asymptotic Confidence Interval; HPD = Highest Posterior Density. Rank is based on the shortest interval length (smallest length = 1).

7.2. Application 3

The following data set reports the failure times of 23 ball bearings tested for life. Each observation represents the number of million revolutions completed before failure.

$17.88, 28.92, 33, 41.5, 42.12, 45.6, 48.8, 51.8, 52, 54.12, 55.56, 67.8, 68.44, 68.64, 68.9, 84.1, 93.12, 98.6, 105, 106, 128, 128, 173.4$

The first step is to check whether the Akash distribution is a reasonable model for the data. The Q–Q and P–P plots in 12 visually compared the sample quantiles and empirical probabilities against the theoretical quantiles and probabilities, respectively. The points in both plots are lying near the line, which supports the reasonability of the fits.

Then, we fitted the candidate distributions to the data, which were Akash, Weibull, Lindley, and Exponential. We used the log-likelihood, AIC, BIC, and AICc values for model comparison (Table 9). The Akash distribution has the largest log-likelihood value, and the smallest AIC, BIC, and AICc values among one-parameter distributions.

Table 9. Comparison of different models using Log-Likelihood, AIC, BIC, AICc, and K–S p-values.

Model	LogLik	AIC	BIC	AICc	K–S p-value
Akash	-113.5302	229.0605	230.1960	229.2510	0.9010
Weibull	-113.6893	231.3786	233.6496	231.9786	0.6710
Lindley	-115.7363	233.4727	234.6081	233.6631	0.3597
Exponential	-121.4368	244.8736	246.0091	245.0641	0.0265

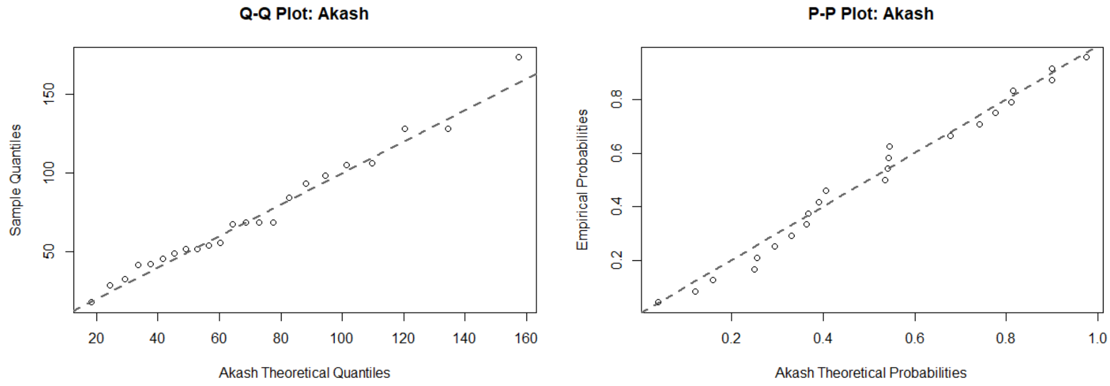


Figure 12. Q–Q and P–P plots for Akash distribution.

In addition, we have conducted the Kolmogorov–Smirnov tests, and the p-value for Akash is 0.901, while they are 0.671, 0.3597, and 0.0265 for Weibull, Lindley, and Exponential, respectively.

All the results suggest that the Akash distribution fits the data very well and can be used for further reliability analysis.

Table 10 shows the progressive first-failure–censored sample and the corresponding censoring scheme used for the analysis with $k = 1$ and $m = 12$.

Table 10. Simulated progressive first-failure-censored sample.

i	1	2	3	4	5	6	7	8	9	10	11	12
R_i	3	0	0	0	0	0	0	3	0	0	0	5
X_{R_i}	28.92	41.52	51.84	54.12	55.56	68.88	84.12	93.12	98.64	106	128	173.4

Table 11. Estimates of θ, reliability, hazard, and mean residual life (MRL) under different methods and priors.

Method	Prior	θ	Reliability	Hazard	MRL
MLE		0.0221	0.8995	0.0050	97.1320
T-K	Non-Inf.	0.0216	0.9045	0.0047	99.9369
IS	Non-Inf.	0.0220	0.9002	0.0049	97.4934
MH	Non-Inf.	0.0220	0.8999	0.0049	97.3332
T-K	Inf.	0.0230	0.8895	0.0054	91.9736
IS	Inf.	0.0235	0.8844	0.0057	89.5724
MH	Inf.	0.0233	0.8870	0.0056	90.7734

Note: Non-Inf. = Non-Informative Prior ($a = 0.001$, $b = 0.001$); Inf. = Informative Prior ($a = 3$, $b = 2$). Estimates of $R(t)$, $H(t)$, & $m(t)$ are for t=50.

In Tables (11) and (12), it is clear that almost all the MLE and Bayesian estimates with non-informative prior give consistent estimates for θ. The estimates obtained by the Tierney-Kadane and the Importance Sampling methods with a non-informative prior are quite close to the MLE estimates, but all these methods estimate the mean residual life to be slightly higher. For estimates with an informative prior, the estimates are upward shifted, which leads to a larger hazard rate. This deviation from the non-informative prior case illustrates the dependence of the posterior estimates on the structure of prior information put into the model. For the estimation intervals, the Bootstrap-t and the Bayesian HPD (non-informative prior) intervals are the most efficient as they have the

Table 12. Comparison of confidence intervals and their lengths.

Method	**Lower**	**Upper**	**Length**	**Rank**
ACI	0.0155	0.0286	0.0131	4
Bootstrap-p	0.0190	0.0367	0.0178	5
Bootstrap-t	0.0134	0.0257	0.0123	1
HPD_Non-Inf.	0.0149	0.0274	0.0125	2
HPD_Inf.	0.0149	0.0284	0.0135	3

Note: Non-Inf. = Non-Informative Prior ($a = 0.001$, $b = 0.001$); Inf. = Informative Prior ($a = 3$, $b = 2$). ACI = Asymptotic Confidence Interval; HPD = Highest Posterior Density. Rank is based on the shortest interval length (smallest length = 1).

smallest lengths; hence, they yield the most accurate inferences. As a conclusion, the Bayesian method with a non-informative prior is accurate and efficient for inferences about the Akash distribution under progressive first-failure censoring.

8. Conclusion

This study focused on the estimation of the parameters and the reliability characteristics of the Akash distribution under the progressive first-failure censoring plan. The Akash distribution proves to be a flexible model for lifetime data analysis, and the adopted censoring plan offers a practical framework for efficient experimentation.

The simulation results lead to several key conclusions.

Regarding Interval Estimation.

- The asymptotic confidence intervals (ACI) achieved the best balance, providing coverage probabilities closest to the nominal 95% level with reasonable interval lengths.
- The Bayesian Highest Posterior Density (HPD) intervals offered the shortest lengths, but at a slight cost to coverage reliability.
- The bootstrap methods were generally found to be less accurate in this context.

Regarding Parameter Estimation.

- Although the MLE works, it can be extremely biased.
- Bayesian methods, especially those which make use of informative priors, are better than the frequentist methods on all accounts, as they produce lower bias and MSE than the frequentist methods.
- The simplest, and of the highest accuracy estimator, within the Bayesian paradigm, is the Tierney-Kadane approximation.
- The Tierney-Kadane approximation emerged as the most efficient and computationally straightforward method among the Bayesian techniques.
- All estimators understandably lose precision, as the proportion of censoring increases; however, Bayesian estimators with informative priors demonstrate greater robustness and stability under these conditions of heavy data loss.

In summary, the Bayesian approach, particularly with an informative prior (when justified) and the Tierney–Kadane approximation, is recommended for point estimation due to its accuracy and efficiency in analyzing censored lifetime data from the Akash distribution. For interval estimation, both ACIs and HPD intervals are feasible options, depending on whether the priority is strict coverage probability or interval precision. Future work could extend this analysis to other loss functions, multi-parameter distributions, or different progressive censoring schemes.

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