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Original Paper

Advancing 8th Graders' Number Sense: Using Learning Trajectories and SOLO Taxonomy to Support Understanding of Irrational Numbers

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Abstract

This action research case study examined how instructional strategies grounded in Learning Trajectories (Clements & Sarama, 2023) and Vergnaud's Conceptual Fields Theory (1996, 2009), assessed through the Structure of Observed Learning Outcomes (SOLO) Taxonomy (Biggs & Collis, 1982), supported 8th graders' understanding of irrational numbers over a four-day instructional sequence. Six middle school students in Huntsville, Alabama, completed a series of tasks aligned with Alabama Standard 8.NSO.2. Data sources included post-assessments, Progress Learning mastery reports, and SOLO-coded reasoning samples collected during instructional tasks. Instruction emphasized classification, comparison, and ordering of rational and irrational numbers through number line activities, contextual reasoning, and the integration of digital tools. Quantitative results showed gains in number line accuracy, with students advancing from pre/unistructural to multistructural or relational SOLO levels. Qualitative analysis revealed shifts from fragmented strategies (e.g., rote decimal recall) to integrated reasoning (e.g., using benchmark squares to locate irrational values). These findings highlight how developmental frameworks can inform targeted interventions that scaffold students toward deeper numerical reasoning, with implications for classroom practice, particularly in addressing persistent number sense gaps among Title I middle school learners.

Keywords: action research; irrational numbers; Learning Trajectories; number sense; numerical reasoning; SOLO Taxonomy; Vergnaud's Conceptual Fields Theory; middle school mathematics; instructional intervention; number line reasoning.

MSC Classification (2020): Primary 97D40, 97C30; Secondary 97B50, 97D50.

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1. Introduction

1.1. Background

Mathematical proficiency in middle school serves as a cornerstone for students' long-term academic success, particularly as they transition into algebra and higher-level mathematics (National Council of Teachers of Mathematics [NCTM], 2014). A critical component of this proficiency is numerical thinking and reasoning—the ability to make sense of numbers, their relationships, and their structures. Within this domain, number sense and operations play a vital role, forming the conceptual foundation for algebraic reasoning, problem solving, and higher mathematics. The Alabama Course of Study emphasizes this progression in its Grade 8 mathematics standards. Specifically, Standard 8.NSO.2 states: "Locate rational approximations of irrational numbers on a number line, compare their sizes, and estimate the values of the irrational numbers" (Alabama Department of Education, 2021). This standard requires students to move beyond familiar rational numbers to engage with irrational numbers, which often pose significant cognitive challenges. Learners must approximate, estimate, and visualize numbers such as , and π, thereby building conceptual bridges to algebraic, quadratic, and exponential reasoning—key skills for success in secondary mathematics. Despite its importance, research indicates that students often struggle with the transition from rational to irrational numbers (Lin, 2022). Many rely on memorized decimal expansions or superficial classification rules without integrating these ideas into broader conceptual frameworks. These gaps are particularly pronounced among students who score below grade level on diagnostic assessments such as i-Ready, where disparities in number sense development become evident across demographic and school contexts. This has direct implications for equity, as students in Title I schools disproportionately experience these conceptual gaps, limiting their readiness for algebra and higher mathematics. Together, Learning Trajectories (LT) and Vergnaud's Conceptual Fields Theory (CFT) inform developmental progressions, while the Structure of Observed Learning Outcomes (SOLO) Taxonomy provides a measurement lens for analyzing the depth and structure of students' reasoning. LT describes the predictable pathways students follow as their mathematical thinking develops; CFT explains how students construct understanding through the coordination of problem situations, strategies, and conceptual structures; and the SOLO Taxonomy classifies the qualitative complexity of their responses, showing how reasoning evolves from surface to relational and abstract understanding.

1.2. Research Problem

The problem addressed in this case study is that fragmented or limited understandings of irrational numbers contribute to persistent underperformance on mandated assessments and reduce readiness for advanced mathematics. Standardized assessments such as i-Ready and Alabama Comprehensive Assessment Program (ACAP) provide broad diagnostic information but often fail to illuminate how students reason about numbers. Without insight into their developmental trajectories or reasoning structures, teachers may not implement targeted instructional strategies that address underlying conceptual barriers. Therefore, this case study draws on three theoretical and conceptual frameworks:

1. Learning Trajectories (LT) (Clements & Sarama, 2023): Students progress along predictable developmental pathways in mathematical thinking, moving from informal strategies to increasing formal reasoning. LT emphasizes diagnosing students' current positions on these trajectories and designing tasks to move them forward.
2. Vergnaud's Conceptual Fields Theory (CFT) (Vergnaud, 1996, 2009): Mathematical understanding develops through the interaction of problem situations, operative

schemes (strategies), and conceptual structures. In the context of irrational numbers, CFT highlights how learners must coordinate decimal, radical, and number-line representations to develop robust concepts.

3. SOLO Taxonomy (Biggs & Collis, 1982): This framework classifies student responses into five levels of increasing complexity—prestructural, unistructural, multistructural, relational, and extended abstract—providing a systematic way to evaluate the qualitative sophistication of students' reasoning.

Together, LT and CFT inform developmental progressions, while SOLO provides a measurement lens for analyzing the depth and structure of students' reasoning. This combination supports both instructional design and assessment.

1.3. Purpose of the Study

The purpose of this action research study is to investigate how instructional strategies grounded in developmental frameworks can support students' numerical thinking and reasoning about irrational numbers. Specifically, the study aims to:

1. Assess students' baseline reasoning using SOLO taxonomy and LT/CFT frameworks on pre-tasks.
2. Implement targeted instructional strategies (e.g., number classification, comparison, ordering tasks) designed to move students along trajectories.
3. Analyze changes in reasoning and performance, using pre- and post-task data, mastery reports, and SOLO-coded observations.
4. Draw instructional implications for middle school mathematics teachers seeking to address conceptual gaps in number sense.

By systematically applying these frameworks to instructional design, data collection, and analysis, this case study seeks to contribute to both practical classroom strategies and the research base on numerical thinking development.

1.4. Research Question

How can instructional strategies, guided by the Learning Trajectories Framework and assessed through the SOLO Taxonomy, support 8th-grade students' progression in understanding and applying Standard 8.NSO.2 (locating, comparing, and estimating irrational numbers)?

2. Literature Review

The development of numerical thinking is foundational to mathematics learning, serving as the conceptual bridge between arithmetic fluency and higher mathematical reasoning (Siegler & Lortie-Forgues, 2017). Number sense encompasses students' abilities to flexibly interpret, estimate, compare, and manipulate numbers in a variety of contexts. Rather than being a discrete skill, number sense develops gradually over time through interaction with mathematical ideas and experiences, beginning with early counting and quantity discrimination and extending into more abstract reasoning about number structures in later grades. Research has consistently shown that students who demonstrate strong number sense in the upper elementary and middle grades are better equipped to handle algebraic thinking, problem solving, and advanced mathematics courses (Barrera-Mora & Reyes-Rodriguez, 2019; Boaler, 2016). Despite this, many students experience persistent gaps in numerical reasoning as they progress through school. These gaps often manifest most clearly when students encounter irrational numbers, values that cannot be expressed as terminating or repeating decimals. Lin (2022) found that students frequently treat irrational numbers as inaccessible, relying on rote decimal approximations or superficial classification rules rather than understanding their structure. For example, students may memorize $\sqrt{2} \approx 1.414$ but fail to reason about why $\sqrt{2}$ lies between 1 and 2 or how it can be accurately approximated on a number line. Such patterns reflect a broader issue: students' reasoning about numbers often remains fragmented rather than integrated, limiting their ability to engage in algebraic reasoning later. Understanding this development requires frameworks that describe how students' thinking progresses. The Structure of Observed Learning Outcomes (SOLO) taxonomy, developed by Biggs and Collis (1982), provides one such lens. SOLO describes learning as a developmental progression across five hierarchical levels: prestructural, where responses are irrelevant or show no understanding; unistructural, where one relevant aspect is recognized; multistructural, where several relevant aspects are identified but not connected; relational, where ideas are integrated into a coherent structure; and extended abstract, where the learner generalizes to new situations. This taxonomy has been used widely in mathematics education research to classify the qualitative complexity of student responses (Adeniji et al., 2022). Applying SOLO taxonomy to number sense tasks allows teachers and researchers to analyze not just whether students are correct, but how they are reasoning. This approach enables educators to pinpoint students' developmental stages, providing targeted support to help them progress to higher levels of reasoning. Moreover, SOLO taxonomy aligns closely with constructivist perspectives on learning, emphasizing the growth of understanding from fragmented knowledge to integrated conceptual structures. While SOLO taxonomy provides a lens for assessing cognitive development, it does not by itself describe the trajectories students follow in developing mathematical understanding. This is where Learning Trajectories (LT) and Vergnaud's Conceptual Fields Theory (CFT) offer complementary perspectives. Learning Trajectories, articulated by Clements and Sarama (2023), describe students' learning as a progression through empirically validated developmental pathways. Each trajectory consists of three components: a mathematical goal, a developmental progression describing increasingly sophisticated levels of thinking, and a set of instructional tasks designed to move students along the progression. In the domain of number systems, learning trajectories trace students' movement from informal strategies, such as estimating quantities visually, to formal reasoning about rational and irrational numbers. Importantly, LT emphasizes the diagnosis of students' current developmental level and the intentional design of instructional experiences that build upon and extend their existing understandings. Vergnaud's Conceptual Fields Theory complements this developmental approach by focusing on the interaction between problem situations, operative schemes, and conceptual structures (Vergnaud, 1996, 2009). According to CFT, mathematical understanding emerges within conceptual fields—coherent domains of knowledge that involve multiple representations, operations, and problem types. For example, understanding irrational numbers involves coordinating radical notation (e.g., $\sqrt{50}$), decimal representations (e.g., 7.07), and spatial representations on the number line. Students' schemes, their strategies for solving problems, interact with these representations, leading to the construction of increasingly sophisticated conceptual structures. CFT thus provides a way to interpret students' reasoning behaviors. Together, Learning Trajectories and CFT provide a developmental map and interpretive framework. LT outlines the pathways students typically follow, while CFT explains how students construct understanding within a conceptual field. When integrated with SOLO taxonomy, these frameworks allow educators to both diagnose students' developmental levels and analyze the qualitative structure of their reasoning. This integrated approach has been shown to be particularly effective in identifying instructional leverage points, moments where targeted interventions can help students make conceptual breakthroughs (Clements & Sarama, 2023; Guss et al., 2022). A growing body of research highlights the critical role of instructional strategies in supporting students' progression along developmental trajectories of numerical reasoning. Effective strategies move students from reliance on rote procedures toward conceptual understanding and flexible reasoning (Barrera-Mora & Reyes-Rodriguez, 2019; Boaler, 2016). In the domain of irrational numbers, this often involves designing learning experiences that bridge concrete and abstract representations. For example, Barrera-Mora and Reyes-Rodriguez (2019) demonstrated that contextualized tasks fostered number sense by prompting students to invent and justify strategies rather than apply memorized rules. Humphreys and Parker (2015) emphasize the role of reasoning-focused instructional routines in encouraging students to verbalize their thought processes, make conjectures, and generalize patterns. Similarly, Guss, Clements, and Sarama (2022) argue that technology-based instructional tools, particularly those grounded in Learning Trajectories, provide structured yet adaptive pathways for students to develop number sense. Instructional strategies that integrate collaborative learning, contextualization, and representation coordination are especially powerful for students performing below grade level. Lin (2022) found that students with limited conceptual understanding of irrational numbers benefited from structured sorting and ordering activities that required them to classify numbers as rational or irrational and then justify placements. When such activities were scaffolded using visual supports like color-coded number lines or foldables, students were able to move from noticing isolated features to integrating multiple properties. These findings align with the Learning Trajectories model, which emphasizes providing tasks that are just beyond students' current developmental level. Parallel to instructional strategies, assessment practices play a vital role in supporting students' numerical development. Standardized diagnostic tools like i-Ready provide useful data about students' placement levels, but they often fail to capture how students are reasoning. Clements and Sarama (2023) argue that task-based assessments aligned with learning trajectories reveal deeper insights into students' conceptual understanding. By coding responses using the SOLO taxonomy, teachers can identify qualitative shifts in reasoning even when overall accuracy remains constant. This approach reflects a balanced assessment model, combining quantitative measures like pre- and post-test scores with qualitative analyses of student reasoning (Adeniji et al., 2022; Siegler & Lortie-Forgues, 2017). A final critical strand in the literature concerns equity and access in mathematics learning. Persistent opportunity gaps, often correlated with socioeconomic status, school funding, and demographic factors, contribute to disparities in number sense development (Bansilal, 2017; Song et al., 2023). Students in Title I schools frequently enter middle school with significant gaps in foundational numerical reasoning. Research grounded in Vygotskian theory highlights the importance of scaffolding within the Zone of Proximal Development (ZPD) to provide access to rigorous mathematical content (Vygotsky, 1978). The integration of Learning Trajectories, CFT, and SOLO taxonomy provides precisely such a framework, offering both a map of where students are and a vision of where they can go, while ensuring that instruction is both rigorous and responsive. Taken together, the reviewed research collectively supports the use of integrated developmental frameworks, specifically Learning Trajectories, Vergnaud's Conceptual Fields Theory, and SOLO Taxonomy, to design effective instructional interventions for middle school learners who struggle with abstract number concepts such as irrational numbers. This synthesis aligns directly with the purpose of the present case study: to investigate how applying these frameworks in tandem can strengthen students' numerical reasoning and conceptual understanding in a targeted instructional context.

3. Methodology

3.1. Research Design

This study employed a quantitative case study action research design (Mertler, 2017), situated within the researcher's own classroom context. Action research provides a structured process for teachers to systematically investigate and improve their instructional practices, while case study methodology allows for an in-depth examination of a small, bounded group of students over a defined period. The quantitative orientation reflects the use of pre- and post-assessment data, item analyses, and mastery reports to measure changes in student performance on numbers and operations tasks. The purpose of this design was twofold: (1) to implement and evaluate targeted instructional strategies informed by Learning Trajectories (Clements & Sarama, 2023), Vergnaud's Conceptual Fields Theory (Vergnaud, 1996, 2009), and the SOLO Taxonomy (Biggs & Collis, 1982) to support 8th-grade students' progression in numerical reasoning; and (2) to collect and analyze evidence of student growth in both performance and reasoning quality. This design is particularly suited for intervention contexts where the goal is to understand how a specific group of students responds to evidence-based instructional approaches, while maintaining ecological validity in real classroom settings.

3.2. Setting and Participants

The study was conducted in a public middle school in Huntsville, Alabama, within a fifth-period mathematics elective course. This course served as both an intervention and enrichment block aligned to grade-level standards, allowing targeted support for students identified as performing below grade level in mathematics. The participants were six 8th-grade students, identified through i-Ready fall diagnostic data as performing below grade level in mathematics. Their diagnostic placements were as follows:

- 1 student performing at a 7th-grade level
- 4 students performing at a 4th-grade level
- 1 student performing at a 3rd-grade level

All students were expected to master Alabama Grade 8 Standard 8.NSO.2, which focuses on locating, comparing, and estimating irrational numbers. Their diverse achievement levels made them a strategically important case for examining developmental progressions in numerical reasoning. Working with this small group allowed for close observation of individual learning trajectories, SOLO-coded reasoning patterns, and the effects of targeted instructional strategies.

3.3. Numbers and Operations Tasks

Three instructional tasks were developed to assess and support students' understanding of rational and irrational numbers, drawing from Learning Trajectories, CFT, and SOLO Taxonomy. These tasks, implemented sequentially over the intervention period, were designed to elicit different levels of reasoning and provide structured opportunities for growth.

Task 1: Classifying Real Numbers. Students engaged in a card-sorting activity to classify numbers as rational or irrational and identify all sets to which each number belonged. Students randomly selected cards, recorded numbers, and named the relevant number sets (e.g., integers, rational numbers, irrational numbers). This task targeted prestructural through multistructural levels on the SOLO taxonomy by prompting students to notice single features (terminating vs. non-terminating decimals) and then identify multiple properties. Within the Learning Trajectories framework, this task represented early stages in the trajectory: distinguishing familiar rational numbers from unfamiliar irrational values. From a CFT perspective, the task encouraged students to begin coordinating decimal and set-based representations.

Task 2: Comparing Real Numbers. In the second activity, students drew eight new cards and compared pairs of numbers by inserting the appropriate relational symbols ($<$, $>$, $=$). This task required students to compare rational and irrational numbers in various forms (e.g., fractions, decimals, radicals), integrating classification knowledge with comparison strategies. Students had to justify their choices during guided discussion. This activity was designed to elicit multistructural and early relational SOLO responses, as students had to consider multiple attributes simultaneously and make connections between decimal

approximations and symbolic representations. On the learning trajectory, this represented a transition phase, where students moved from identifying individual properties toward integrating them in meaningful ways.
Task 3: Ordering Real Numbers on a Number Line. The third activity required students to order mixed sets of rational and irrational numbers on a number line. Students selected cards, recorded numbers, and then graphed and labeled each number. Numbers included integers, fractions, terminating and repeating decimals, and common irrational numbers such as $\sqrt{2}$ and π. Students worked collaboratively to place these numbers accurately, drawing on both prior classification and comparison experiences. This task targeted relational and extended abstract levels on the SOLO taxonomy, as students were expected to integrate multiple representations (radical, decimal, spatial) and justify placements logically. Within the Learning Trajectories model, this task aligned with later developmental stages that emphasize estimation, number line reasoning, and conceptual coordination. From a CFT lens, this activity required students to activate and integrate schemes involving radicals, decimals, and spatial ordering, reflecting deeper conceptual field engagement.

3.4. Instructional Context

Across all tasks, instructional strategies were intentionally scaffolded to match students' developmental levels. For students at prestructural or unistructural levels, supports included foldables, guided classification charts, and color-coded number lines. For students at multistructural levels, instruction emphasized collaborative reasoning, contextual tasks, and number line placement. For students approaching relational levels, tasks incorporated real-world contexts (e.g., tiling with) and independent digital exploration using adaptive platforms like i-Ready. These strategies were explicitly grounded in Learning Trajectories and CFT to ensure that instruction targeted the "next step" in each student's conceptual development (see Appendix B: Numbers and Operations Instructional Tasks).

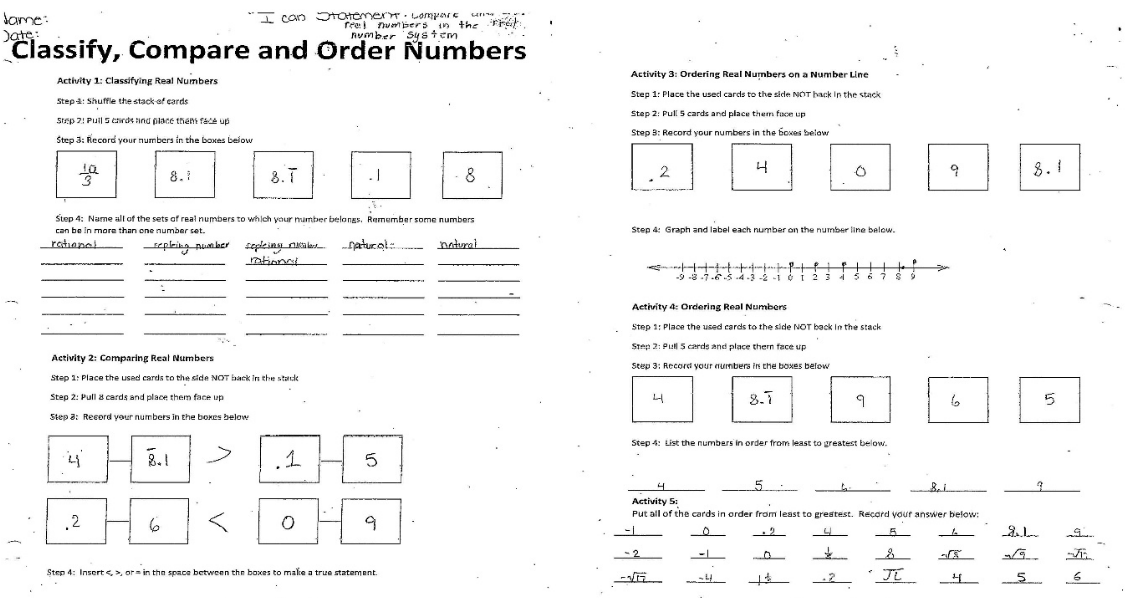


Figure 1. An anonymized student work sample showing classification, comparison, and number line placement activities.

3.5. Data Collection

To ensure inter-rater reliability in SOLO taxonomy coding, a second reviewer independently coded a sample of student responses from each task using a shared rubric. The reviewer participated in a brief calibration session to align interpretations of the SOLO levels. Coding discrepancies were discussed and resolved through consensus, ensuring consistent application of criteria across student work samples. Multiple data sources were used to assess changes in student performance and reasoning quality over the course of the intervention:

Table 1. Timeline of data collection procedures during the four-day instructional sequence.

Day 1	Day 2	Day 3	Day 4
• Consent & Tasks 1–2 • Collect consent forms • Provide task instructions to assess learning growth • Complete classification and comparison task	• Students continue instructional sequence with ordering and collaborative tasks	• Interactive Number Line • Students engage in Task 3 (interactive number line activity)	• Administer post-test

1. Post-Task Assessments: Students completed a Progress Learning Numbers and Operations assessment after the instructional sequence. These assessments included multiple-choice and drag-and-drop items on rational approximations of irrational numbers, ordering numbers, and number line placement.
2. Progress Learning Mastery Report: Class-level mastery data were generated through the Progress Learning platform, indicating overall performance levels (green/yellow/pink dot ranks) and mastery percentages for each skill.
3. Task-Based Observations: During each instructional task, the researcher collected field notes on students' strategies, language, and representations.
4. SOLO Taxonomy Coding: Student responses during tasks were coded using the SOLO taxonomy to determine their qualitative reasoning levels. Codes were validated through inter-rater discussions to ensure reliability.

Data were organized into anonymized student profiles in a spreadsheet to allow descriptive analysis of both accuracy growth and reasoning progression over time.

3.6. Method of Data Analysis

Data analysis in this case study followed a mixed-methods approach, integrating both quantitative and qualitative procedures to capture the full range of student learning. Quantitative analysis involved descriptive statistics derived from the Progress Learning post-assessment and mastery reports. Specifically, the researcher calculated each student's percentage correct on the post-assessment (aligned to Standard 8.NSO.2), overall class average mastery, and item-level accuracy rates across all four skill categories (interval identification, rational approximation, classification, and ordering). These statistics were used to identify patterns of growth and areas of persistent difficulty across the six participants. Because of the small sample size ($n = 6$), no inferential statistical tests were applied; instead, results are reported descriptively and interpreted in relation to individual student trajectories. Qualitative analysis centered on SOLO Taxonomy coding of student responses collected during the three instructional tasks. Each written or verbal response was coded independently by the primary researcher and a second reviewer using a shared SOLO rubric with defined criteria for each level (prestructural, unistructural, multistructural, relational, and extended abstract). After independent coding, discrepancies were discussed and resolved through consensus. Representative student responses were selected from each task to illustrate the qualitative progression observed across the intervention. These were further interpreted through the lenses of Learning Trajectories (to identify developmental stage shifts) and Vergnaud's Conceptual Fields Theory (to analyze coordination of representational schemes). This combination of quantitative performance data and qualitative reasoning data supported a balanced assessment approach consistent with the frameworks cited in the literature (Clements & Sarama, 2023; Adeniji et al., 2022).

3.7. Ethical Considerations

All ethical guidelines for classroom-based action research were followed. Students and their guardians were informed about the study's purpose, and parent/guardian consent and student assent forms were distributed prior to data collection. Of the seven students in the class, six returned signed consent and assent forms and were included in the study. One student, who was performing at a 2nd-grade mathematics level, did not return the signed forms and was therefore excluded from all data collection and analysis. Participation was voluntary, and no student grades were affected by participation. Data was anonymized using pseudonyms, and any identifying information was removed from work samples included in the analysis. The study was conducted within the bounds of the school's instructional improvement initiatives and followed university and district research ethics policies. A blank consent form is included in Appendix A: Parental/Guardian Consent Form.

3.8. Validity and Reliability

Given the small sample size of this study (n = 6), addressing the validity and reliability of the data collection tools, data collection process, and findings is essential.

Validity of the Data Collection Tools. The instructional tasks (classifying, comparing, and ordering real numbers) were designed to align directly with Alabama Grade 8 Standard 8.NSO.2, lending the instruments strong content validity. Each task was grounded in established theoretical frameworks: Learning Trajectories (Clements & Sarama, 2023), Vergnaud's Conceptual Fields Theory (Vergnaud, 1996, 2009), and the SOLO Taxonomy (Biggs & Collis, 1982). This theoretical alignment ensures that the tasks measured the intended construct—namely, students' developmental progression in numerical reasoning about irrational numbers—rather than peripheral skills. The SOLO taxonomy coding rubric, derived directly from Biggs and Collis (1982) and widely validated in mathematics education research (Adeniji et al., 2022), further strengthened the construct validity of the qualitative analysis. The use of Progress Learning as a standardized assessment platform also contributes to instrument validity, as the platform's items are developed and normed against grade-level standards. Together, these elements support the claim that the instruments measured what they were designed to measure, in a manner consistent with current theory and practice in mathematics education.

Reliability of the Data Collection Process. Maintaining reliability with a sample of only six students presents inherent challenges, particularly for any quantitative generalizations. To address this, the researcher employed multiple strategies. First, inter-rater reliability was established for SOLO coding: a second independent reviewer coded a representative sample of student responses using the shared rubric, and discrepancies were resolved through structured consensus discussion. This process ensured consistency in the application of coding criteria across tasks and reviewers. Second, multiple data sources (post-assessments, mastery reports, task-based observations, and SOLO coding) were triangulated to corroborate findings and reduce reliance on any single instrument. Triangulation is a recognized strategy for strengthening credibility and dependability in small-scale qualitative and mixed-methods research (Mertler, 2017). Third, the researcher maintained detailed field notes throughout the intervention, providing an audit trail of instructional decisions and observational data. While the small sample size limits the statistical reliability and external generalizability of the quantitative findings, the use of rich qualitative analysis, triangulation, and inter-rater verification supports confidence in the internal validity and interpretive credibility of the study's conclusions. Future replications with larger samples are needed to confirm these patterns and extend their generalizability.

4. Results

4.1. Overview of Student Performance

Post-assessment data were collected using the Progress Learning Numbers and Operations Task platform to measure students' performance on tasks aligned with Alabama Grade 8 Standard 8.NSO.2. The analysis focused on the six participating students who returned consent and assent forms. The Progress Learning Mastery Report indicated an overall class average of 59% across the Numbers and Operations tasks at the end of the instructional sequence, with 43 total questions completed, 14 of which were mastered (green dot) and 29 not yet mastered (yellow or pink dots). Mastery within the platform is defined as achieving $\geq$ 80% on an activity (green dot), while scores between 65–79% are indicated in yellow and below 65% in pink. The group's performance distribution reflected meaningful growth from initial diagnostic levels, particularly given that four of the six participating students began the intervention performing at a 4th-grade level, one at a 3rd-grade level, and one at a 7th-grade level. Table 2 summarizes the group's aggregate performance (see Appendix C: Supplemental Data Tables; While the class average remained below the 80% mastery threshold, qualitative analysis and item-level patterns indicated significant progress in reasoning sophistication and accuracy on key item types, particularly those involving number line placement and approximation of irrational numbers.

Table 2. Overall Mastery Report Summary.

Measure	Value
Number of Participants	6
Total Questions Completed	43
Number Mastered ($\geq$80%)	14
Number Not Mastered (<80%)	29
Class Average	59%

4.2. Item Analysis and Patterns of Growth

The Progress Learning Item Analysis revealed that students demonstrated the greatest growth on questions requiring identification of the interval between two integers containing a given square root and on ordering tasks involving both rational and irrational numbers. For example:

1. On an item asking students to determine between which two integers $\sqrt{73}$ lies, five out of six students selected the correct option (8 and 9) on the post-assessment.

Table 3. Post-assessment scores and SOLO taxonomy progression for each student pseudonym ($N = 6$).

Student	Post-Assessment Score (%)	SOLO Progression
Student 1	52	Unistructural → Relational
Student 2	62	Prestructural → Multistructural
Student 3	71	Unistructural → Relational
Student 4	83	Prestructural → Multistructural
Student 5	54	Unistructural → Relational
Student 6	67	Multistructural → Relational

2. On multiple-choice items requiring ordering of mixed sets of numbers, including radicals, decimals, and negative values, four students demonstrated improved scores on the post-assessment, moving from random guessing patterns to more consistent reasoning based on approximation.
3. Tasks requiring classification of numbers as rational or irrational showed more modest gains, suggesting that while students improved their ability to use properties of irrational numbers for placement and comparison, some conceptual distinctions remained fragile.

Items requiring direct approximation, like identifying the best rational approximation for $\sqrt{35}$, showed mixed results. Three students consistently selected the correct option (5.9), while two students occasionally reverted to choosing distractors close to integer boundaries (e.g., 5.4 or 6.7), indicating lingering reliance on superficial cues rather than conceptual estimation (see Appendix C: Supplemental Data Tables; Table C2 Representative Item Analysis Patterns (Post-Assessment)).

4.3. Representative Student Reasoning Across Tasks

To analyze qualitative changes in student thinking, representative responses were examined from the three instructional tasks (classification, comparison, ordering), coded using SOLO taxonomy levels and interpreted through Learning Trajectories and CFT lenses (see Appendix C: Supplemental Data Tables; Table C2 Representative Item Analysis Patterns (Post-Assessment).

Task 1: Classifying Real Numbers. At the outset, four students displayed prestructural or unistructural reasoning. For example, when classifying , one student initially stated, "It doesn't have a decimal, so it's rational," indicating reliance on superficial symbolic features rather than conceptual understanding. Another student identified irrational numbers by "numbers with weird symbols," a classic prestructural pattern. By the end of the instructional sequence, most students progressed to multistructural reasoning. When revisiting classification, a student explained, " is not a perfect square, so it is irrational. It cannot be written as a fraction, and the decimal does not stop or repeat." This reflects recognition of multiple relevant features (non-perfect square, non-repeating decimal) without yet integrating these features into broader conceptual structures.

Task 2: Comparing Real Numbers. In the comparison activity, students were asked to draw pairs of numbers (including fractions, decimals, and radicals) and insert relational symbols ($<$, $>$, $=$). Early in the sequence, several students compared numbers by superficial decimal inspection or symbol familiarity. For example, one student incorrectly compared radicals without justification. By the end of the intervention, students increasingly used estimation and reasoning with square numbers. A representative response came from a student at a 4th-grade diagnostic level: "$\sqrt{57}$ is between 7^2 and 8^2, but closer to 7. So, it is more than 7, less than 8." This response reflects multistructural reasoning with elements

of relational understanding. Another student, initially at unistructural level, began to use benchmark numbers: "$\sqrt{15}$ is a little under 4 because 4^2 is 16," showing clear progression along the trajectory toward relational thinking.
Task 3: Ordering Real Numbers on a Number Line. The ordering task provided the clearest evidence of conceptual breakthroughs. Initially, many students clustered irrational numbers at arbitrary positions or placed them at integer endpoints, indicating prestructural or unistructural reasoning. For example, two students placed $\sqrt{35}$ at 5, explaining, "I just guessed," or "because it's a square root, it goes somewhere around there." After targeted instruction and repeated exposure to number line reasoning, five of six students successfully ordered mixed sets of numbers, including fractions, decimals, and irrational numbers, using reasoning strategies aligned with relational SOLO levels. For example, one student explained their placement of $\sqrt{35}$: "5^2 is 25 and 6^2 is 36, so it's between 5 and 6. It's closer to 6 because 35 is close to 36." SOLO coding confirmed that three students reached relational levels, two achieved stable multistructural reasoning, and one remained at unistructural.

4.4. Key Trends and Misconceptions

Several key trends emerged across the data:

- Progression Along Developmental Pathways: Students demonstrated clear movement along Learning Trajectories, from initial reliance on superficial features to integrating multiple representations for reasoning. SOLO taxonomy provided fine-grained evidence of these shifts, particularly from unistructural to multistructural and relational levels.
- Strength in Number Line Reasoning: Ordering tasks yielded the most significant conceptual gains. Students found spatial representation tasks more accessible for integrating decimal and radical schemes, consistent with prior research on number line reasoning as a bridge to abstraction.
- Lingering Classification Confusions: Some students continued to conflate irrationality with "unusual symbols" or misunderstood the relationship between radicals and decimals, indicating that classification requires sustained emphasis. Common errors included assuming that all square roots less than 10 fall between 0 and 1, misclassifying repeating decimals, and misplacing negative radicals on the number line.

4.5. Summary of Findings

Overall, the results indicate that targeted instructional strategies, designed around Learning Trajectories and CFT and assessed using SOLO taxonomy, supported measurable progress in both performance and reasoning among participating students. Quantitatively, students improved on key item types related to approximation and ordering of irrational numbers. Qualitatively, students' reasoning shifted from fragmented, superficial strategies toward integrated, conceptually grounded approaches. These findings highlight the effectiveness of developmental frameworks for addressing persistent gaps in numerical reasoning in middle school mathematics classrooms.

5. Discussion

The findings of this action research case study provide evidence that integrating Learning Trajectories, Vergnaud's Conceptual Fields Theory (CFT), and the SOLO Taxonomy into instructional design and assessment can meaningfully support middle school students' numerical reasoning about irrational numbers. A notable finding was the substantial growth in students' ability to use number line reasoning to approximate and compare irrational numbers. Initially, many students relied on superficial cues or guessed placements, reflecting prestructural or unistructural SOLO levels. By the end of the intervention, most demonstrated multistructural or relational reasoning, accurately locating values such as $\sqrt{35}$ between 5 and 6 based on benchmark squares. This mirrors Lin's (2022) observation that number line tasks are particularly effective for helping students transition from rote decimal recall to flexible spatial reasoning. Similarly, Guss et al. (2022) emphasize that dynamic number line experiences provide a bridge from concrete representations to abstract reasoning, supporting conceptual integration of radicals, decimals, and spatial structures. The patterns observed in this study align closely with these findings, indicating that intentional sequencing of classification, comparison, and number line tasks can support significant developmental shifts. The progression of students across SOLO levels also confirms the utility of using the taxonomy as both a diagnostic and evaluative tool. Students who began at prestructural levels, unable to identify any systematic feature of irrational numbers, advanced to multistructural reasoning by the end of the intervention, demonstrating that even a brief, targeted instructional sequence can produce meaningful qualitative growth. The CFT lens further illuminated why certain transitions were more challenging than others: students who relied on a single representational scheme (e.g., decimal approximation alone) without coordinating it with radical or spatial representations tended to plateau at unistructural or early multistructural levels. This underscores the importance of designing instruction that explicitly bridges multiple representations and requires students to connect them in purposeful ways. Conversely, the persistence of classification confusions suggests that conceptual distinctions between rational and irrational numbers remain fragile for many students, even after targeted instruction. This aligns with Lin (2022), who found that students' understanding of irrationality is often procedural rather than conceptual. These results suggest that a four-day intervention, while impactful, is insufficient to consolidate robust classification understanding, and that sustained, distributed practice is needed.

6. Implications

The findings of this study offer several practical strategies that teachers can replicate in both traditional classroom and digital learning environments to strengthen students' understanding of irrational numbers and overall number sense. Central to this replication is the intentional use of developmental frameworks—Learning Trajectories (LT), Vergnaud's Conceptual Fields Theory (CFT), and the Structure of the Observed Learning Outcomes (SOLO) Taxonomy—to guide instructional design, formative assessment, and differentiation.

6.1. Classroom-Based Implementation

Teachers can adopt a similar three-task instructional sequence—classification, comparison, and ordering—to move students from surface-level recognition of irrational numbers toward relational reasoning. Beginning with tangible sorting activities (e.g., classifying cards into rational and irrational sets) helps students externalize their thinking and build shared mathematical language through structured discussion. As students progress, comparison and number line tasks can be incorporated into small-group rotations or math workshop models, allowing teachers to provide real-time feedback tailored to students' developmental positions on the Learning Trajectories. Using SOLO-coded observation checklists, teachers can systematically document each student's reasoning level and plan subsequent instruction accordingly.

6.2. Integration in Digital Environments

Teachers leveraging platforms such as Progress Learning or i-Ready can design tasks that mirror the developmental stages highlighted in this case study. For example, interactive number lines and dynamic graphing tools allow students to visualize relationships between radicals and decimals in real time, reinforcing spatial reasoning as emphasized by Guss et al. (2022). Embedding short reflection prompts or audio explanations after each task enables teachers to capture students' reasoning processes, which can then be analyzed for SOLO level progression. Adaptive technology data (e.g., mastery reports or item analyses) can also identify whether students remain at unistructural levels or have advanced to relational reasoning, informing targeted digital interventions.

6.3. Professional Collaboration and Data Reflection

Grade-level or departmental teams can replicate this model collaboratively by implementing the same developmental task sequence and meeting regularly to compare SOLO-coded student samples. This process supports consistent calibration across teachers, strengthens data-driven instructional planning, and promotes a culture of reflective practice grounded in evidence of student reasoning. Collaborative analysis sessions can also help teams refine instructional pacing and identify common misconceptions across classrooms.

6.4. Sustaining Growth Through Ongoing Assessment

Finally, teachers can embed these frameworks into regular formative assessments by including one reasoning prompt per lesson that requires students to justify the placement or comparison of irrational numbers. Over time, these collected responses provide longitudinal evidence of students' progression along Learning Trajectories and SOLO levels. This approach ensures that conceptual development is monitored continuously, rather than being assessed only at isolated points in time, supporting sustained growth in number sense.

7. Limitations and Future Research

7.1. Limitations

Several limitations should be acknowledged. First, the sample size was small (n = 6), reflecting the scope of classroom-based action research. While this allowed for detailed analysis of individual learning trajectories, the findings are not generalizable to larger populations without further replication. Second, the duration of the intervention was limited to a four-day instructional sequence rather than sustained instruction over a semester or year. Third, the study relied on a single type of assessment platform (Progress Learning) and task sequence; a more diverse set of instruments could yield richer data on students' reasoning processes. Finally, while SOLO coding was validated through inter-rater discussion, the study would benefit from more formal reliability procedures in future implementations.

7.2. Future Research

Future research can build on these findings in several ways. First, subsequent studies should administer a pre-test that is the same as or structurally similar to the post-test assessment, providing a clearer measure of conceptual growth over time and allowing finer-grained tracking of item-level gains. Second, incorporating questions that mirror assessment items into instructional activities would strengthen students' ability to transfer their developing reasoning to formal testing contexts. Third, future studies could explore longer intervention periods, tracking how sustained use of developmental frameworks affects growth over a semester or academic year. Expanding the sample size across multiple classrooms or schools, particularly including Title I and non-Title I contexts, would allow for examination of equity impacts and variability across demographic groups. Fourth, student surveys or brief reflection prompts asking students to describe their confidence, strategies, and difficulties would provide qualitative data to complement performance measures. Finally, further research might integrate mixed methods, combining quantitative growth measures with interviews or clinical interviews to provide deeper insight into students' evolving conceptual structures.

8. Conclusion

This study contributes to a growing body of evidence that developmentally grounded instruction, informed by Learning Trajectories, Vergnaud's Conceptual Fields Theory, and assessed using SOLO Taxonomy, can significantly support students' numerical reasoning about irrational numbers. For students performing below grade level, these frameworks provide both a diagnostic tool and a pathway for conceptual growth, enabling movement from fragmented strategies toward relational understanding. By designing tasks that align with students' developmental levels and analyzing their reasoning with structured frameworks, teachers can create more equitable and effective mathematics learning experiences that address persistent conceptual gaps in middle school classrooms. Although the small sample size and brief intervention period limit broad generalizability, the consistency of the qualitative and quantitative findings suggests that this integrated framework approach holds promise as a model for targeted instructional intervention in middle school mathematics.

References

[1] Adeniji, C., Oginni, O. I., & Agusiobo, B. C. (2022). Using SOLO taxonomy to assess students' cognitive development in mathematics. Journal of Research and Practice in Mathematics Education, 6(2), 45–59. https://doi.org/10.29329/jrpmath.2022.6.2.4

[2] Bansilal, S. (2017). The difficulty level of a national assessment of Grade 9 mathematics: The case of five schools. South African Journal of Childhood Education, 7(1), a412. https://doi.org/10.4102/sajce.v7i1.412

[3] Barrera-Mora, F., & Reyes-Rodriguez, A. (2019). Fostering middle school students' number sense. Journal of Mathematics Education, 12(3), 56–73.

[4] Baroody, A. J. (2006). Why children have difficulties mastering the basic number combinations and how to help them. Teaching Children Mathematics, 13(1), 22–31.

[5] Biggs, J. B., & Collis, K. F. (1982). Evaluating the quality of learning: The SOLO taxonomy (Structure of the Observed Learning Outcome). Academic Press.

[6] Boaler, J. (2016). Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages, and innovative teaching. Jossey-Bass.

[7] Clements, D. H., & Sarama, J. (2023). Learning and teaching early math: The learning trajectories approach (3rd ed.). Routledge. https://doi.org/10.4324/9781003313835

[8] Guss, S. S., Clements, D. H., & Sarama, J. H. (2022). High-quality early math: Learning and teaching with trajectories and technologies. In Handbook of research on innovative approaches to early childhood development and school readiness (pp. 349–372). IGI Global. https://doi.org/10.4018/978-1-7998-8649-5.ch015

[9] Humphreys, C., & Parker, R. (2015). Making number talks matter: Developing mathematical practices and deepening understanding, grades 4–10. Stenhouse Publishers.

[10] Lin, C. (2022). Students' misconceptions of irrational numbers: An analysis of strategies and reasoning. International Journal of Mathematical Education in Science and Technology, 53(7), 1032–1050. https://doi.org/10.1080/0020739X.2021.1875072

[11] Mertler, C. A. (2017). Action research: Improving schools and empowering educators (5th ed.). SAGE Publications.

[12] Siegler, R. S., & Lortie-Forgues, H. (2017). An integrative theory of numerical development. Child Development Perspectives, 11(3), 146–151. https://doi.org/10.1111/cdep.12221

[13] Song, X., Li, Q., Quintero, A., & Wang, L. (2023). The link between math anxiety and math achievement: The role of afterschool learning. Frontiers in Psychology, 14, 1138472. https://doi.org/10.3389/fpsyg.2023.1138472

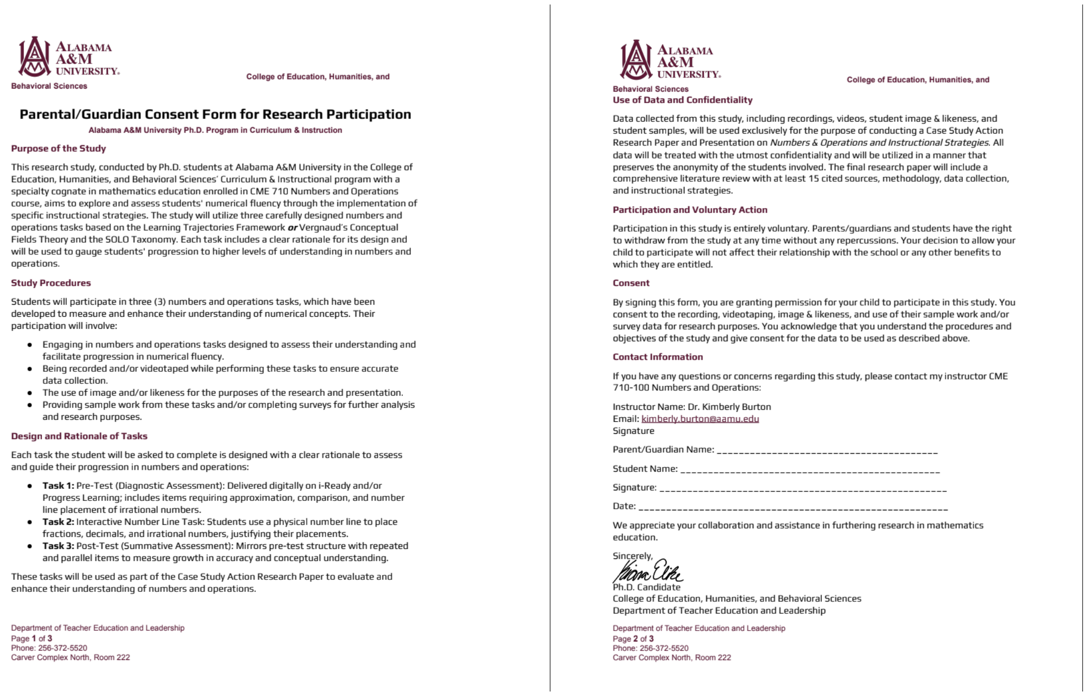
[14] Vergnaud, G. (1996). The theory of conceptual fields. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), Theories of mathematical learning (pp. 219–239). Routledge.

[15] Vergnaud, G. (2009). The theory of conceptual fields. Human Development, 52(2), 83–94. https://doi.org/10.1159/000202727

[16] Vygotsky, L. S. (1978). Mind in society: The development of higher psychological processes. Harvard University Press.

Appendix A. Parental/Guardian Consent Form

The following parental/guardian consent form was distributed to all students' families prior to data collection. Six of the seven students returned signed consent and assent forms; one student, performing at a 2nd-grade mathematics level, did not return the forms and was excluded from the study. The consent form outlines the purpose of the study, procedures, confidentiality measures, and voluntary nature of participation.



Alabama A&M University.

College of Education, Humanities, and Behavioral Sciences

Parental/Guardian Consent Form for Research Participation

Alabama A&M University Ph.D. Program in Curriculum & Instruction

Purpose of the Study

This research study, conducted by Ph.D. students at Alabama A&M University in the College of Education, Humanities, and Behavioral Sciences' Curriculum & Instructional program with a specialty cognate in mathematics education enrolled in CME 710 Numbers and Operations course, aims to explore and assess students' numerical fluency through the implementation of specific instructional strategies. The study will utilize three carefully designed numbers and operations tasks based on the Learning Trajectories Framework ***or*** Vergnaud's Conceptual Fields Theory and the SOLO Taxonomy. Each task includes a clear rationale for its design and will be used to gauge students' progression to higher levels of understanding in numbers and operations.

Study Procedures

Students will participate in three (3) numbers and operations tasks, which have been developed to measure and enhance their understanding of numerical concepts. Their participation will involve:

- Engaging in numbers and operations tasks designed to assess their understanding and facilitate progression in numerical fluency.
- Being recorded and/or videotaped while performing these tasks to ensure accurate data collection.
- The use of image and/or likeness for the purposes of the research and presentation.
- Providing sample work from these tasks and/or completing surveys for further analysis and research purposes.

Design and Rationale of Tasks

Each task the student will be asked to complete is designed with a clear rationale to assess and guide their progression in numbers and operations:

- **Task 1:** Pre-Test (Diagnostic Assessment): Delivered digitally on i-Ready and/or Progress Learning; includes items requiring approximation, comparison, and number line placement of irrational numbers.
- **Task 2:** Interactive Number Line Task: Students use a physical number line to place fractions, decimals, and irrational numbers, justifying their placements.
- **Task 3:** Post-Test (Summative Assessment): Mirrors pre-test structure with repeated and parallel items to measure growth in accuracy and conceptual understanding.

These tasks will be used as part of the Case Study Action Research Paper to evaluate and enhance their understanding of numbers and operations.

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College of Education, Humanities, and Behavioral Sciences

Use of Data and Confidentiality

Data collected from this study, including recordings, videos, student image & likeness, and student samples, will be used exclusively for the purpose of conducting a Case Study Action Research Paper and Presentation on *Numbers & Operations and Instructional Strategies*. All data will be treated with the utmost confidentiality and will be utilized in a manner that preserves the anonymity of the students involved. The final research paper will include a comprehensive literature review with at least 15 cited sources, methodology, data collection, and instructional strategies.

Participation and Voluntary Action

Participation in this study is entirely voluntary. Parents/guardians and students have the right to withdraw from the study at any time without any repercussions. Your decision to allow your child to participate will not affect their relationship with the school or any other benefits to which they are entitled.

Consent

By signing this form, you are granting permission for your child to participate in this study. You consent to the recording, videotaping, image & likeness, and use of their sample work and/or survey data for research purposes. You acknowledge that you understand the procedures and objectives of the study and give consent for the data to be used as described above.

Contact Information

If you have any questions or concerns regarding this study, please contact my instructor CME 710-100 Numbers and Operations:

Instructor Name: Dr. Kimberly Burton
Email: kimberly.burton@aamu.edu
Signature

Parent/Guardian Name: ______________________________________

Student Name: ____

Signature: ____

Date: ____

We appreciate your collaboration and assistance in furthering research in mathematics education.

Sincerely,
Ph.D. Candidate
College of Education, Humanities, and Behavioral Sciences
Department of Teacher Education and Leadership

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Figure 2. Parental/Guardian consent form.

Appendix B. Numbers and Operations Instructional Tasks

The instructional tasks were designed to align with Learning Trajectories, Vergnaud's Conceptual Fields Theory, and SOLO Taxonomy, and were implemented sequentially to scaffold students' reasoning development.

B.1. Classifying Real Numbers

- Purpose: Develop foundational understanding of number sets in the real number system.
- Activity: Students pulled five cards with different numbers, recorded them, and named all sets each number belonged to (e.g., natural, whole, integers, rational, irrational).
- Target SOLO levels: Prestructural to Multistructural.
- Developmental focus: Distinguishing rational vs. irrational using decimal and set representations.

B.2. Comparing Real Numbers

- Purpose: Build reasoning strategies for comparing rational and irrational numbers.
- Activity: Students selected eight cards, compared pairs of numbers using relational symbols ($<$, $>$, $=$), and justified their reasoning.
- Target SOLO levels: Multistructural to Early relational.
- Developmental focus: Coordinating properties of decimals, fractions, and radicals to support comparison.

B.3. Ordering Real Numbers on a Number Line

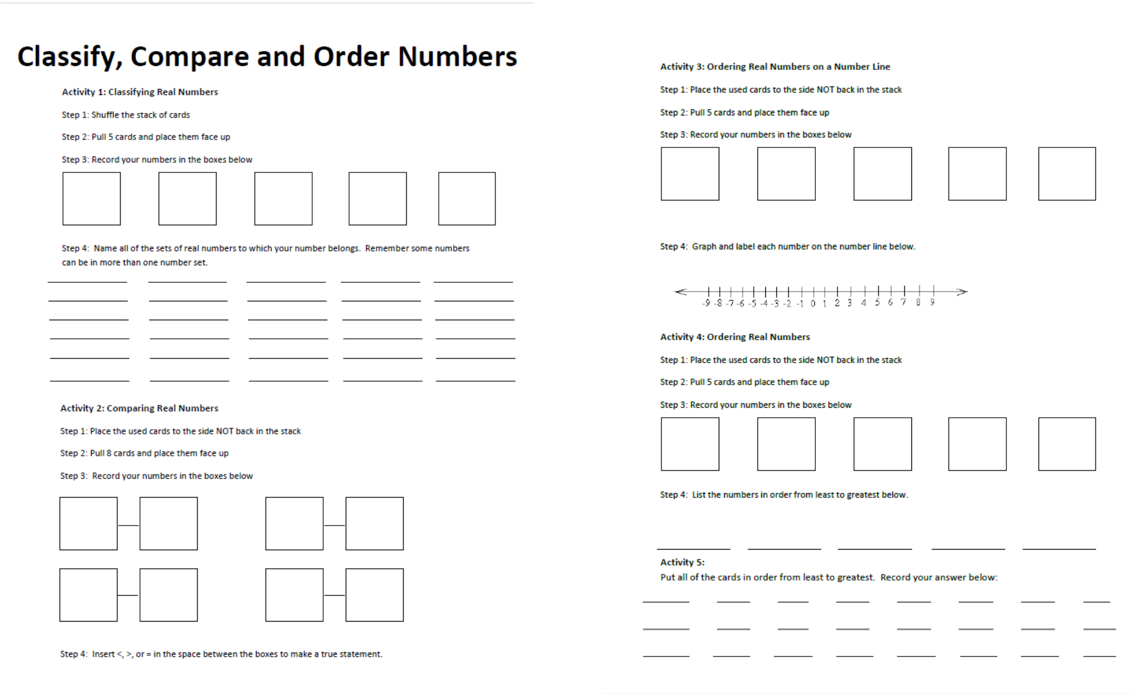
- Purpose: Integrate classification and comparison skills to accurately position numbers spatially.
- Activity: Students graphed and labeled a mix of rational and irrational numbers (e.g., $\sqrt{50}$, π, $-\sqrt{99}$, $\sqrt{35}$) on a number line.
- Target SOLO levels: Relational to Extended abstract.
- Developmental focus: Coordinating radical, decimal, and spatial representations to reason about number magnitude.

Appendix C. Supplemental Data Tables

Table 4. Progress Learning Mastery Report Summary.

Measure	Value
Number of Participants	6
Total Questions Completed	43
Number Mastered (≥80%)	14
Number Not Mastered (<80%)	29
Class Average	59%

Note. Mastery thresholds are based on the Progress Learning dot ranking system: Green ≥80%, Yellow 65–79%, Pink <65%. Note. Data drawn from Progress Learning Item Analysis Reports.



Classify, Compare and Order Numbers

Activity 1: Classifying Real Numbers

Step 1: Shuffle the stack of cards

Step 2: Pull 5 cards and place them face up

Step 3: Record your numbers in the boxes below

Step 4: Name all of the sets of real numbers to which your number belongs. Remember some numbers can be in more than one number set.

Activity 2: Comparing Real Numbers

Step 1: Place the used cards to the side NOT back in the stack

Step 2: Pull 8 cards and place them face up

Step 3: Record your numbers in the boxes below

Step 4: Insert <, >, or = in the space between the boxes to make a true statement.

Activity 3: Ordering Real Numbers on a Number Line

Step 1: Place the used cards to the side NOT back in the stack

Step 2: Pull 5 cards and place them face up

Step 3: Record your numbers in the boxes below

Step 4: Graph and label each number on the number line below.

Activity 4: Ordering Real Numbers

Step 1: Place the used cards to the side NOT back in the stack

Step 2: Pull 5 cards and place them face up

Step 3: Record your numbers in the boxes below

Step 4: List the numbers in order from least to greatest below.

Activity 5:

Put all of the cards in order from least to greatest. Record your answer below:

Figure 3. Representative instructional task sheets used during the intervention.

Table 5. Representative Item Analysis Patterns (Post-Assessment).

Item Type	Correct Responses ($n = 6$)	Common Strategies/Misconceptions
Interval Identification (e.g.,)	5	Using benchmark squares (e.g., $6^2 = 36$, $7^2 = 49$) to locate intervals
Rational approximation (e.g.,)	3	Some confusion between nearest whole number vs. decimal approximation
Classification of real numbers	4	Persistent misconceptions about symbols and decimal patterns
Ordering mixed numbers	4	Improved benchmark reasoning, but occasional misplacement of negatives

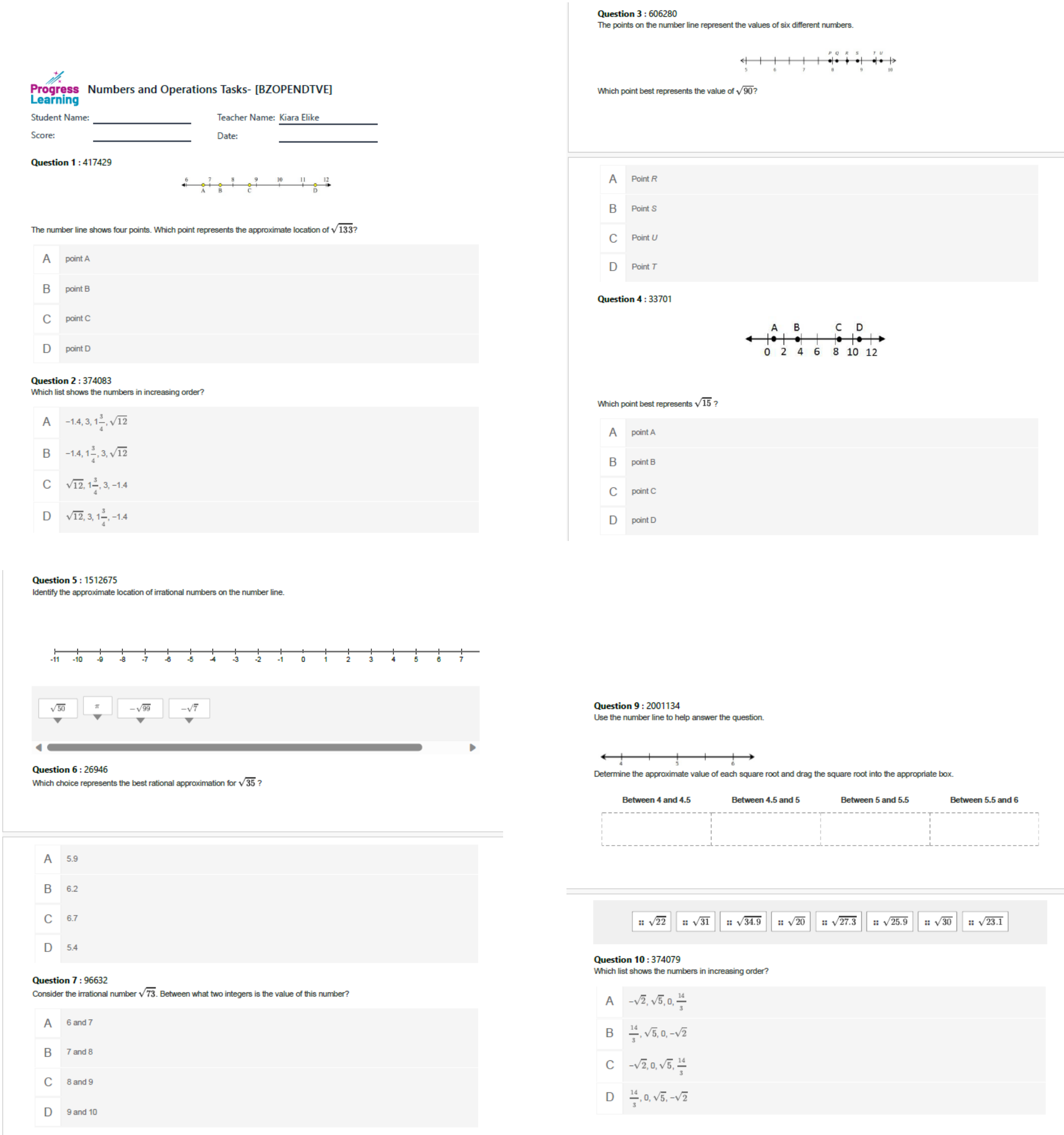


Figure 4. Representative Progress Learning item analysis screenshots.