Alabama Journal of Mathematics

Alabama Journal of Mathematics
Volume 49 (1) (2026), 1 – 14

ORIGINAL PAPER

Local higher integrability for a class of multi-phase functionals with logarithmic growths

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Abstract

In this paper, we establish the local higher integrability for minimizers of a multi-phase functional involving logarithmic growth. The problem is investigated under standard assumptions regarding the Hölder continuity of the coefficients and the exponents within the multi-phase operator. Our proof follows the framework developed by De Filippis and Oh in [14], combined with a careful treatment of the logarithmic growth term. More specifically, we derive a reverse Hölder inequality for the gradient of the minimizers, which represents a crucial step in the regularity theory for non-uniformly elliptic problems. By employing a delicate iteration technique and handling the additional complexity introduced by the logarithmic factor, we demonstrate that the gradient belongs to a slightly better Lebesgue space than the one initially prescribed by the natural energy. The obtained results may provide insights into generalizing these findings to multi-phase problems involving more general N-functions.

Keywords: Higher integrability; Minimizer; Multi-phase functionals; Logarithmic growths.

MSC Classification (2020): Primary 49N60, 35J60; Secondary 35B65, 35B45, 35D30, 46E30.

1. Introduction

The primary objective of this study is to explore the regularity of local minimizers for the multi-phase variational energy functional $\mathcal{G}$, defined on the Orlicz-Sobolev space $W^{1,G(\cdot)}(\Omega)$, given by

$$\mathcal{G}(w,\Omega) := \int_{\Omega} G(x, Dw)\, dx, \text{ for every } w \in W^{1,G(\cdot)}(\Omega). \tag{1.1}$$

The function $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ was mentioned above, which exhibits a multi-phase structure combined with logarithmic growth, is formulated as

$$G(x,\xi) := |\xi|^p \log^{\gamma}(e + |\xi|) + a(x)|\xi|^q + b(x)|\xi|^s, \quad x \in \Omega, \xi \in \mathbb{R}^n. \tag{1.2}$$

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Received: January 7, 2026; Accepted: February 18, 2026

In the formulation above, $\Omega \subset \mathbb{R}^n$ denotes a bounded open set and γ is a positive parameter. We assume the exponents satisfy $1 < p < q \leq s$, while the weight functions $a(\cdot)$ and $b(\cdot)$ are required to be Hölder continuous, specifically:

$$a \in C^{0,\alpha}(\Omega), b \in C^{0,\beta}(\Omega), \quad \text{with} \quad \alpha, \beta \in (0,1]. \tag{1.3}$$

For the subsequent analysis, it is crucial to clarify the notion of a local minimizer associated with the map $\mathcal{G}$ in (1.1).

Definition 1.1. We say that a function $u \in W^{1,1}(\Omega)$ is a local minimizer of the functional $\mathcal{G}$ defined in (1.1) if $G(\cdot, |Du|) \in L^1(\Omega)$ and the following minimality property holds:

$$\mathcal{G}(u, \operatorname{supp}(u-v)) \leq \mathcal{G}(v, \operatorname{supp}(u-v))$$

for all $v \in W^{1,1}_{loc}(\Omega)$ such that the support of the function $u - v$ is a subset of Ω.

A central characteristic of the functional $\mathcal{G}$ involves the transition in its growth and ellipticity types, a phenomenon triggered on the sets where $a(x) = 0$ or $b(x) = 0$. Variational problems with non-standard growth conditions have attracted significant attention owing to their applications in mathematical physics and materials science, particularly in describing strongly anisotropic materials. A simple model of this problem occurs in the case $b \equiv 0$ and $\gamma = 0$, where (1.1) reduces to the classical double-phase problem. A well-known instance of this is the double-phase integral $\mathcal{P}$, defined as below, which provides a clear illustration of such behavior

$$\mathcal{P}(w, \Omega) := \int_\Omega (|Dw|^p + a(x)|Dw|^q)\, dx, \quad 1 < p < q. \tag{1.4}$$

The functional $\mathcal{P}$ defined in (1.4) serves as a fundamental model for nonautonomous problems characterized by nonstandard polynomial growth and a mild form of nonuniform ellipticity. This mathematical structure was pioneered by Zhikov [28–30], whose research offered an insightful framework for analyzing materials with pronounced directional properties. Following this, the regularity for local minimizers of the functional $\mathcal{P}$ and the related Euler-Lagrange form was explored by P. Marcellini in the seminal papers [19, 20]. Later, the regularity for double-phase functionals has been extensively developed in recent years, most notably through the contributions of Baroni, Colombo, and Mingione [1–4, 10, 11]. Consequently, there is now a vast literature dedicated to the existence and regularity of solutions for non-uniformly elliptic functionals, particularly those involving (p, q)-growth conditions. Specifically, it accounts for hardening effects that vary according to the exponents p and q. For a comprehensive overview of recent developments in the case of exponents, we refer the interested reader to [4, 6, 7, 9, 21, 24, 27] and the references therein.

In close connection with the double-phase operator $\mathcal{P}$ above, we recall the classical functional $\mathcal{L}$ associated with nearly linear growth, expressed as

$$w \mapsto \mathcal{L}(w, \Omega) := \int_\Omega |Dw| \log(1 + |Dw|)\, dx.$$

This kind of operator was presented by Mingione and Siepe in [22], which demonstrates, for instance, in the theory of plasticity with logarithmic hardening. More recently, the investigation of double-phase problems featuring logarithmic operators has gained interest, notably in [8, 12, 13, 18, 23, 25, 26]. Additionally, to generalize the double-phase operator, C. De Filippis and J. Oh [14] considered a multi-phase functional defined by the following energy:

$$w \in W^{1,H(\cdot)}(\Omega) \mapsto \mathcal{H}(w, \Omega) := \int_\Omega H(x, Dw)\, dx, \quad 1 < p < q \leq s, \tag{1.5}$$

with $H(x, \xi) = |\xi|^p + a(x)|\xi|^q + b(x)|\xi|^s$, for $x \in \Omega$ and $\xi \in \mathbb{R}^n$. They proved the regularity for local minimizers of this multi-phase energy under sharp assumptions relating a, b and p, q, s.

While the regularity theory for double-phase and multi-phase functions are now well-established, the transition to multi-phase scenarios with logarithmic perturbations presents new challenges. Motivated by the aforementioned studies, and particularly by [14], we investigate the minimizers of the functional involving the operator $\mathcal{G}$ as defined in (1.1). The presence of the term $\log^{\gamma}(e+|\xi|)$ of this functional requires a more delicate handling of using and combining inequalities. The central contribution of this work is the proof of a Gehring-type higher integrability result. This result is also known as a version of the reverse Hölder inequality, which is considered the first and most crucial step in investigating the regularity of solutions for variational inequalities and their corresponding Euler-Lagrange equations. We show that under suitable Hölder continuity assumptions on the coefficients $a(\cdot)$, $b(\cdot)$ and exponents p, q, s, the gradient of a local minimizer u possesses a higher degree of integrability than what is naturally required by the energy functional. This result serves as a fundamental step toward further regularity estimates, such as local boundedness of the local minimizers, gradient continuity or investigating the regularity of solutions for several classes of corresponding problems.

Let us introduce appropriate conditions between the Hölder constants of coefficients $a(\cdot), b(\cdot)$ and exponents p, q, s. A key aspect of the functional $\mathcal{P}$ in (1.4) for the case $p < q$ is how the ratio $\frac{q}{p}$ relates to the Hölder continuity of the coefficient $a(x)$. In this context, Baroni, Colombo, and Mingione [2,3] proved that the necessary and sufficient requirements for the regularity of minimizers associated with the funtionals in (1.4) are expressed by

$$a(\cdot) \in C^{0,\alpha}(\Omega) \quad \text{and} \quad \frac{q}{p} \leq 1 + \frac{\alpha}{n}.$$

Moreover, C. De Filippis and J. Oh [14] studied about the multi-phase variational energy function (1.5) under following sharp conditions

$$\frac{q}{p} \leq 1 + \frac{\alpha}{n} \quad \text{and} \quad \frac{s}{p} \leq 1 + \frac{\beta}{n}. \tag{1.6}$$

In this paper, we show that assumptions (1.6) are also sufficient conditions to obtain a Gehring-type inequality for problem with logarithmic growth (1.1). While our techniques and theorems largely align with previous results, particularly in the investigation of terms in the q- and s-phases, the primary challenge arises from analysing and estimating the term coupled with the logarithmic function. However, by treating this term through the lens of N-functions, it can be fully addressed using an extended version of the classical Poincaré-Sobolev inequality, combined with several subtle transformations. An interesting observation is that the logarithmic term does not seem to significantly affect the regularity of solutions for this multi-phase problem. This suggests the possibility of generalizing the logarithmic operator to more general N-functions.

The remainder of the paper is organized as follows. In Section 2, we introduce some general notation and recall basic preliminary results on N-functions and several function spaces, including Hölder and Orlicz-Sobolev spaces. The main result of the paper is presented in Section 3. The proof scheme consists of three steps: (i) establishing a Caccioppoli-type inequality for operators with logarithmic growth; (ii) proving the corresponding Sobolev-Poincaré inequality; and (iii) deriving the local higher integrability result using Gehring's Lemma.

2. Preliminaries

This section is primarily devoted to introducing the fundamental notation, lemmas, and theorems that will be employed throughout the paper. To begin, let Ω be an open bounded domain in $\mathbb{R}^n$ for $n \geq 2$. We adopt the convention $C := C(n, p, q, s, \gamma)$ to indicate that the constant C depends exclusively on the parameters n, p, q, s, and γ. The symbol

$$B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$$

represents an open ball of radius $r > 0$ centered at x_0 in n-dimensional space. Where no confusion arises, we will remove the center and write B_r instead of $B_r(x_0)$. Specifically, if the particular information about the center is not mentioned, balls with different radii are assumed to be concentric. Furthermore, for any measurable set $A \subset \mathbb{R}^n$ with $0 < |A| < \infty$ and a measurable mapping $f : A \to \mathbb{R}$, the symbol $(f)_A$ represents the average integral of the function f over A as

$$(f)_A := \fint_A f(x)\,dx = \frac{1}{|A|}\int_A f(x)\,dx.$$

For $A = B_r$, we simplify the notation by setting $(f)_r := (f)_{B_r}$. In addition, the symbols below will be utilized throughout this work:

$$\texttt{in-data}_1 = (n, p, q, s, \gamma) \text{ and } \texttt{in-data}_2 = (n, p, q, s, \gamma, [a]_{0,\alpha}, [b]_{0,\beta}).$$

2.1. N-functions and properties

A thorough understanding of N-functions and their properties is essential, as the function G defined in (1.2) - the central model of this study - exhibits the structure of an N-function with respect to the variable ξ. Furthermore, our analysis throughout this paper focuses primarily on Orlicz-Sobolev spaces, which are fundamentally constructed based on the framework of N-functions.

Definition 2.1. A function $M : \mathbb{R} \to \mathbb{R}$ is called an N-function if it can be represented as

$$M(u) = \int_0^{|u|} \varphi(x)\,dx,$$

where $\varphi : [0, \infty) \to [0, \infty)$ is right continuous in $[0, +\infty)$ and positive in $(0, +\infty)$. Furthermore φ is a non-decreasing function satisfying the conditions

$$\varphi(0) = 0, \quad \text{and} \quad \lim_{t\to\infty} \varphi(t) = \infty. \tag{2.1}$$

Consider a mapping $q : [0, \infty) \to [0, \infty)$ which is given by

$$q(s) = \sup\{t \in [0, \infty) : \varphi(t) \le s\}.$$

It is well known that q is a non-decreasing function satisfying the conditions (2.1). Furthermore, q is right continuous in $[0, +\infty)$ and positive in $(0, +\infty)$.

Definition 2.2. The function M^*, determined by

$$M^*(u) = \int_0^{|u|} q(x)\,dx,$$

where q is given in Definition (2.1), is defined as the complementary function to M.

We say an N-function M fulfills the Δ_2-condition provided that a constant $c > 0$ can be found such that

$$M(2t) \le cM(t), \quad \text{for all } t \ge 0.$$

Let $\Delta_2(M)$ denote the smallest constant c satisfying the last inequality. Furthermore, in the case of a collection $\{M_\lambda\}$ of N-functions, we set

$$\Delta_2(\{M_\lambda\}) := \sup_\lambda \Delta_2(M_\lambda).$$

Reference [17, Theorem 4.2] provides a detailed proof for the following result.

Theorem 2.3. *The Δ_2 condition is satisfied by the complementary function M^* of an N-function M if and only if one can find a fixed constant $l > 1$ yielding the inequality*

$$M(t) \le \frac{1}{2l} M(lt), \quad \textit{for every } t \ge 0.$$

2.2. Some important function spaces

In this subsection, we collect the fundamental definitions and notation regarding the function spaces that will be employed throughout this paper, including Hölder, Sobolev, Orlicz and Musielak-Orlicz spaces.

Given an open set $\Omega \subset \mathbb{R}^n$ and a real number $\gamma \in (0,1]$, a function $f : \Omega \to \mathbb{R}$ is said to be Hölder continuous with exponent γ if there exists a constant $C > 0$ such that the following inequality holds

$$|f(x) - f(y)| \leq C|x-y|^\gamma, \text{ for all } x, y \in \Omega.$$

If f is Hölder continuous with exponent γ on Ω, we write

$$\|f\|_{C(\Omega)} = \sup_{x\in\Omega} \|f(x)\|, \quad \text{and} \quad [f]_{0,\gamma,\Omega} = \sup_{x,y\in\Omega, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\gamma}.$$

When the domain Ω is clear from the context, we may simplify the notation $[f]_{0,\gamma,\Omega}$ to $[f]_{0,\gamma}$.

Definition 2.4. The Hölder space $C^{k,\gamma}(\Omega)$ comprises all functions f for which the norm is finte

$$\|f\|_{C^{k,\gamma}(\Omega)} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{C(\Omega)} + \sum_{|\alpha|\leq k} [D^\alpha f]_{0,\gamma,\Omega} < \infty.$$

Definition 2.5. For any non-negative integer k and $1 \leq p < \infty$, the Sobolev space $W^{k,p}(\Omega)$ is comprised of all locally summable functions f whose weak derivatives $D^\alpha f$ belong to the $L^p(\Omega)$ for all $|\alpha| \leq k$. We equip this space with the norm defined by

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha f|^p \, dx \right)^{\frac{1}{p}}.$$

We also define two related spaces. The space $W_0^{k,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$. The local Sobolev space $W_{loc}^{k,p}(\Omega)$ is comprised of all locally integrable functions f, with the property that $D^\alpha f \in L_{loc}^p(\Omega)$ for every multi-index $|\alpha| \leq k$.

Next, we provide definitions of Orlicz, Orlicz-Sobolev and Musielak-Orlicz-Sobolev spaces, along with related function spaces. The main results of this paper focus on the regularity of functions belonging to these spaces.

Consider an N-function M and a bounded open set $\Omega \subset \mathbb{R}^n$. We denote the Orlicz class by $\mathcal{L}^M(\Omega)$, which gathers all real-valued measurable functions u on Ω providing a finite modular $\rho_M(u) < \infty$. Here, the modular functional is given by

$$\rho_M(u) := \int_\Omega M(|u(x)|) \, dx.$$

Definition 2.6. The Orlicz space $L^M(\Omega)$ is the collection of all Lebesgue measurable functions f defined in Ω such that there exists $k > 0$ for which $\rho_M(\frac{f}{k}) < \infty$.

Definition 2.7. The Orlicz-Sobolev space $W^{k,M}(\Omega)$ is identified as the subspace of $L^M(\Omega)$ containing functions f such that all weak derivatives $D^\alpha f$ also belong to $L^M(\Omega)$ for all $|\alpha| \leq k$.

The two related spaces $W_0^{k,M}(\Omega)$ and $W_{loc}^{k,M}(\Omega)$ are defined in a similar way. We now consider an extended definition of N-functions, which are Musielak-Orlicz functions.

Definition 2.8. Let Ω be an open subset of $\mathbb{R}^n$. A function $\varphi : \Omega \times [0, +\infty) \to \mathbb{R}$ is called a Musielak-Orlicz function if it satisfies both conditions below:

i) $\varphi(x, \cdot)$ is an N-function for all $x \in \Omega$.

ii) $\varphi(\cdot, t)$ is a Lebesgue measurable function for all $t \in [0, +\infty)$.

In a similar way to Orlicz and Orlicz-Sobolev spaces, we have the following definitions.

Definition 2.9. The Musielak-Orlicz space $L^{\varphi}(\Omega)$ is the collection of all Lebesgue measurable functions f defined in Ω such that there exists $k > 0$ for which

$$\int_{\Omega} \varphi\left(x, \frac{|f(x)|}{k}\right) dx < \infty.$$

Definition 2.10. The Musielak-Orlicz-Sobolev space $W^{k,\varphi}(\Omega)$ is defined as the set of all functions f such that all weak derivatives $D^{\alpha} f$ also belong to $L^{\varphi}(\Omega)$ for all $|\alpha| \le k$.

Finally, the two related spaces $W_0^{k,\varphi}(\Omega)$ and $W_{loc}^{k,\varphi}(\Omega)$ are defined in a similar way.

3. Higher integrability for logarithmic multi-phase functionals

Our proof scheme consists of three steps. The initial step is establishing Caccioppoli inequality, which means estimating the average integral of $G(\cdot, Du)$ on some ball in terms of the average integral of $G\left(\cdot, \frac{u-(u)_r}{r}\right)$ on a larger ball. In the next step, we prove a Sobolev-Poincaré type inequality in order to derive an inverse estimate for a lower integrability of $G(\cdot, Du)$. Finally, the main result is obtained by applying the Gehring's lemma to the function $G(\cdot, Du)$, which is shown to satisfy a weak reverse Hölder inequality.

3.1. Caccioppoli inequality via logarithmic growth

In this subsection, we execute the first step of our proof scheme. To prove the Caccioppoli inequality, we employ some basic estimates for functions expressing logarithmic growth. The two lemmas below are quite simple and can be proved by some elementary calculations. However, we present the proofs in detail so that readers can follow them more easily.

Lemma 3.1. *Let* $\gamma > 0$ *and* $\rho > 0$. *Then, there holds*

$$\log^{\gamma}(e+t) \le \left(\frac{\gamma}{e\rho}\right)^{\gamma} (e+t)^{\rho}, \quad \textit{for all } t \ge 0. \tag{3.1}$$

Proof. Let us set $\varphi : [1,\infty) \to \mathbb{R}$ to be the function formulated as

$$\varphi(t) = \frac{\gamma}{e\rho} t^{\frac{\rho}{\gamma}} - \log t, \quad t \in [1,\infty).$$

One can easily see that $\varphi'(t) = \frac{1}{e} t^{\frac{\rho}{\gamma}-1} - \frac{1}{t}$, so $\varphi'(t) = 0$ if and only if $t = e^{\frac{\gamma}{\rho}}$. In addition, we have $\varphi(1) = \frac{\gamma}{e\rho}, \varphi(e^{\frac{\gamma}{\rho}}) = 0$ and $\lim\limits_{t\to+\infty} \varphi(t) = +\infty$. For this reason, we may observe that $\varphi(t) \ge 0$ for all $t \ge 1$. This implies the desired inequality

$$\log^{\gamma}(t) \le \left(\frac{\gamma}{e\rho}\right)^{\gamma} t^{\rho}, \quad \text{for all } t \ge 1,$$

which allows us to conclude (3.1). ☐

Lemma 3.2. *Let* G *be the function considered in* (1.2). *Then there exists a constant* $c := c(\texttt{in-data}_1) > 0$ *such that*

$$G(x, \xi_1 + \xi_2) \le c\big[G(x,\xi_1) + G(x,\xi_2)\big], \quad \textit{for all } \xi_1, \xi_2 \in \mathbb{R}^n.$$

Proof. For a fixed $x \in \Omega$, let us consider a function $g : [0,+\infty) \to \mathbb{R}$ defined by

$$g(t) = t^p \log^{\gamma}(e+t) + a(x)t^q + b(x)t^s, \quad t \ge 0.$$

For every $t_1, t_2 \ge 0$, the direct computation gives us

$$g\left(\frac{t_1+t_2}{2}\right) \ge \frac{g(t_1+t_2)}{2^{\sigma}},$$

where $\sigma = \max\{p+\gamma, q, s\}$. Moreover, it is easy to check that g is convex. It follows that

$$g\left(\frac{t_1+t_2}{2}\right) \le \frac{g(t_1)+g(t_2)}{2}.$$

Combining two above inequalities, it follows that

$$g(t_1+t_2) \le 2^{\sigma-1}\big[g(t_1)+g(t_2)\big], \quad \text{for all } t_1, t_2 \ge 0.$$

Thus, the proof is complete by noting that $G(x,\xi) = g(|\xi|)$ for all $\xi \in \mathbb{R}^n$. $\square$

The main idea to prove the Caccioppoli inequality is the lemma below, which has been proved by Giusti in [16].

Lemma 3.3. *Let $Z : [\varrho, r] \to \mathbb{R}^+$ be a continuous and bounded function with given $0 < \varrho < r$. Assume that for all $\varrho \le t < k \le r$, the following inequality holds:*

$$Z(t) \le \theta Z(k) + \frac{A_1}{(k-t)^{\alpha}} + \frac{A_2}{(k-t)^{\beta}} + \frac{A_3}{(k-t)^{\gamma}}, \tag{3.2}$$

where $A_1, A_2, A_3, \alpha, \beta, \gamma \ge 0$ and $0 \le \theta < 1$ are given constants. Then the following estimate holds:

$$Z(\varrho) \le c(\alpha,\beta,\gamma,\theta)\left[\frac{A_1}{(r-\varrho)^{\alpha}} + \frac{A_2}{(r-\varrho)^{\beta}} + \frac{A_3}{(r-\varrho)^{\gamma}}\right].$$

Proof. Let us consider an increasing sequence $(t_j)_{j\ge 0} \subset [\varrho, r]$ defined by

$$t_0 = \varrho \text{ and } t_{j+1} = t_j + (1-\lambda)\lambda^j (r-\varrho),\ j \ge 0.$$

Here, $\lambda \in (0,1)$ represents a parameter that will be fixed in the sequel. For every $i \ge 0$, assumption (3.2) gives us

$$Z(t_i) \le \theta Z(t_{i+1}) + \frac{\lambda^{-i\alpha}}{(1-\lambda)^{\alpha}}\frac{A_1}{(r-\varrho)^{\alpha}} + \frac{\lambda^{-i\beta}}{(1-\lambda)^{\beta}}\frac{A_2}{(r-\varrho)^{\beta}} + \frac{\lambda^{-i\gamma}}{(1-\lambda)^{\gamma}}\frac{A_3}{(r-\varrho)^{\gamma}}.$$

Let us define $\kappa = \max\{\alpha,\beta,\gamma\}$, it follows that

$$Z(t_i) \le \theta Z(t_{i+1}) + \frac{\lambda^{-i\kappa}}{(1-\lambda)^{\kappa}}\left[\frac{A_1}{(r-\varrho)^{\alpha}} + \frac{A_2}{(r-\varrho)^{\beta}} + \frac{A_3}{(r-\varrho)^{\gamma}}\right].$$

By iterating the inequality on the intervals $[t_{j-1}, t_j]$ for each $j \ge 1$ and using induction, we obtain the estimate

$$Z(\varrho) \le \theta^j Z(t_j) + \frac{1}{(1-\lambda)^{\kappa}}\sum_{i=0}^{j-1}\left(\theta\lambda^{-\kappa}\right)^i\left[\frac{A_1}{(r-\varrho)^{\alpha}} + \frac{A_2}{(r-\varrho)^{\beta}} + \frac{A_3}{(r-\varrho)^{\gamma}}\right].$$

We now choose λ such that $\theta\lambda^{-\kappa} < 1$. Taking the limit as $j \to \infty$ in the above inequality, we get the conclusion with $c = (1-\theta\lambda^{-\kappa})^{-1}(1-\lambda)^{-\kappa}$. $\square$

Remark 3.4. Note that the number of constants A_i does not change the form of the final result. The same estimate holds for l such constants $A_1, A_2, ..., A_l$. We will use this lemma for $l = 5$ in the following theorem.

Theorem 3.5. *Let $u \in W^{1,G(\cdot)}_{loc}(\Omega)$ be a local minimizer of the map $\mathcal{G}$ given by (1.1), where conditions (1.3) and (1.6) hold. Then there exists a constant $C := C(\texttt{in-data}_1) > 0$ such that for any ball $B_r \subset \Omega$, the following inequality holds:*

$$\fint_{B_{\frac{r}{2}}} G(x, Du)\,dx \le C\fint_{B_r} G\left(x, \frac{u-(u)_r}{r}\right)dx. \tag{3.3}$$

Proof. As mentioned above, we prove (3.3) by applying Lemma 3.3 and Remark 3.4 to the map $t \mapsto \int_{B_t} G(x, Du)\,dx$. In other words, we will show that this map satisfies an inequality of the form (3.2).

Firstly, we use a well-known technique which was presented by Giusti [16, Theorem 7.1]. Let us fix a ball $B_r \subset \Omega$. For $\frac{r}{2} \le t < k \le r$, we consider η as a function in $C_0^\infty(B_k)$ which fulfills

$$|D\eta| \le \frac{4}{k-t}, \ 0 \le \eta \le 1, \text{ and } \eta \equiv 1 \text{ on } B_t. \tag{3.4}$$

We set $w := u - \eta(u - (u)_r)$ as a comparison function. It is clear to see that

$$w \in W^{1,1}_{\mathrm{loc}}(\Omega),\ Dw \equiv 0 \text{ on } B_t \text{ and } \operatorname{supp}(u - w) \subset B_k.$$

Then the minimality of u gives $\mathcal{G}(u, B_k) \le \mathcal{G}(w, B_k)$, which is equivalent to

$$\int_{B_k} G(x, Du)\,dx \le \int_{B_k} G(x, Dw)\,dx = \int_{B_k\setminus B_t} G(x, (1-\eta)Du - (u - (u)_r)D\eta)\,dx.$$

Applying Lemma 3.2 and (3.4) on the right-hand side of this inequality, it yields

$$\begin{aligned} \int_{B_k} G(x, Du)\,dx \le c \int_{B_k\setminus B_t} & G(x, Du)\,dx \\ &+ \frac{c_*}{(k-t)^p}\int_{B_r} |u-(u)_r|^p \log^\gamma(e + |D\eta(u-(u)_r)|)\,dx \\ &\quad + \frac{c_*}{(k-t)^q}\int_{B_r} a(x)|u-(u)_r|^q\,dx \\ &\qquad + \frac{c_*}{(k-t)^s}\int_{B_r} b(x)|u-(u)_r|^s\,dx. \end{aligned} \tag{3.5}$$

On the other hand, by applying the fundamental inequality

$$\log^\gamma(e + ab) \le c(\gamma)\left[\log^\gamma(e+a) + \log^\gamma(e+b)\right],\ \forall a, b \ge 0,$$

one gets that

$$\log^\gamma(e + |D\eta(u-(u)_r)|) \le C\left[\log^\gamma(e + r|D\eta|) + \log^\gamma\left(e + \left|\frac{u-(u)_r}{r}\right|\right)\right].$$

Moreover, Lemma 3.1 and (3.4) allow us to estimate

$$\log^\gamma(e + r|D\eta|) \le C\left(1 + \frac{r}{k-t}\right),$$

which guarantees that

$$\begin{aligned} \int_{B_r} |u-(u)_r|^p \log^\gamma(e + |D\eta(u-(u)_r)|)\,dx &\le c\left(1 + \frac{r}{k-t}\right)\int_{B_r} |u-(u)_r|^p\,dx \\ &+ c\int_{B_r} |u-(u)_r|^p \log^\gamma\left(e + \left|\frac{u-(u)_r}{r}\right|\right)\,dx. \end{aligned} \tag{3.6}$$

Combining estimates (3.5) and (3.6), it gives us the following result

$$\begin{aligned}\int_{B_k} G(x,Du)\,dx \le c\int_{B_k\setminus B_t} G(x,Du)\,dx &+ \frac{c}{(k-t)^p}\int_{B_r}|u-(u)_r|^p\,dx\\ &+ \frac{cr}{(k-t)^{p+1}}\int_{B_r}|u-(u)_r|^p\,dx\\ &+ \frac{c}{(k-t)^p}\int_{B_r}|u-(u)_r|^p\log^\gamma\left(e+\left|\frac{u-(u)_r}{r}\right|\right)dx\\ &+ \frac{c}{(k-t)^q}\int_{B_r}a(x)|u-(u)_r|^q\,dx\\ &+ \frac{c}{(k-t)^s}\int_{B_r}b(x)|u-(u)_r|^s\,dx, \end{aligned}\tag{3.7}$$

where $c := c(\texttt{in-data}_1) > 0$. By adding to both sides of (3.7) the quantity $c\int_{B_t} G(x,Du)\,dx$, we come up with

$$\begin{aligned}\int_{B_t} G(x,Du)\,dx \le \theta\int_{B_k} G(x,Du)\,dx &+ \frac{c}{(k-t)^p}\int_{B_r}|u-(u)_r|^p\,dx\\ &+ \frac{cr}{(k-t)^{p+1}}\int_{B_r}|u-(u)_r|^p\,dx\\ &+ \frac{c}{(k-t)^p}\int_{B_r}|u-(u)_r|^p\log^\gamma\left(e+\left|\frac{u-(u)_r}{r}\right|\right)dx\\ &+ \frac{c}{(k-t)^q}\int_{B_r}a(x)|u-(u)_r|^q\,dx\\ &+ \frac{c}{(k-t)^s}\int_{B_r}b(x)|u-(u)_r|^s\,dx, \end{aligned}$$

where $\theta = \frac{c}{c+1} < 1$. We can now utilize Lemma 3.3 along with Remark 3.4 for $Z(t) = \int_{B_t} G(x,Du)\,dx$, so that the subsequent estimate can be established

$$\begin{aligned}\int_{B_{\frac{r}{2}}} G(x,Du)\,dx \le \frac{c}{r^p}\int_{B_r}&|u-(u)_r|^p\log^\gamma\left(e+\left|\frac{(u-(u)_r)}{r}\right|\right)dx\\ &+\frac{c}{r^q}\int_{B_r}a(x)|u-(u)_r|^q\,dx+\frac{c}{r^s}\int_{B_r}b(x)|u-(u)_r|^s\,dx.\end{aligned}$$

This inequality allows us to conclude (3.3). The proof is complete. □

3.2. Sobolev-Poincaré inequality

Next, we prove a Sobolev-Poincaré type inequality for the function G. Polynomial growth terms can be addressed similarly to the approach in [14]. Hence, our primary focus here is handling the logarithmic growth term.

Theorem 3.6. *Under all assumptions of Theorem 3.5, then there exists a constant* $C = C(\texttt{in-data}_2, \|Du\|_{L^p(B_r)}) > 0$ *and an exponent* $d = d(\texttt{in-data}_1) \in (0,1)$ *such that*

$$⨍_{B_r} G\left(x,\frac{u-(u)_r}{r}\right)dx \le C\left(⨍_{B_r} G(x,Du)^d\,dx\right)^{\frac{1}{d}},\tag{3.8}$$

for all ball $B_r \subset \Omega$ *with* $r \le 1$.

Proof. Taking $r \in (0,1]$ such that $B_r \subset \Omega$, let us denote

$$a_r = r^{-\alpha} \sup_{x\in B_r} a(x), \text{ and } b_r = r^{-\beta} \sup_{x\in B_r} b(x).$$

By (1.2), it is possible to decompose the average integral on the left-hand side of (3.8) as follows

$$\fint_{B_r} G\left(x, \frac{u-(u)_r}{r}\right) dx = \mathcal{I}_{p,\log} + \mathcal{I}_q + \mathcal{I}_s, \tag{3.9}$$

where $\mathcal{I}_{p,\log}$, $\mathcal{I}_q$ and $\mathcal{I}_s$ are given by

$$\mathcal{I}_{p,\log} := \fint_{B_r} \frac{|u-(u)_r|^p}{r^p} \log^\gamma\left(e + \frac{|u-(u)_r|}{r}\right) dx,$$

$$\mathcal{I}_q := \fint_{B_r} a(x)\frac{|u-(u)_r|^q}{r^q}\, dx, \quad \mathcal{I}_s := \fint_{B_r} b(x)\frac{|u-(u)_r|^s}{r^s}\, dx.$$

Similar to the proof in [14], we also prove (3.8) in four following cases

$$\text{Case 1: } a_r \le 4[a]_{0,\alpha} \text{ and } b_r \le 4[b]_{0,\beta}, \tag{3.10}$$
$$\text{Case 2: } a_r > 4[a]_{0,\alpha} \text{ and } b_r > 4[b]_{0,\beta}, \tag{3.11}$$
$$\text{Case 3: } a_r > 4[a]_{0,\alpha} \text{ and } b_r \le 4[b]_{0,\beta}, \tag{3.12}$$
$$\text{Case 4: } a_r \le 4[a]_{0,\alpha} \text{ and } b_r > 4[b]_{0,\beta}. \tag{3.13}$$

Let us now consider the first case (3.10), which directly leads to

$$\mathcal{I}_q \le 4[a]_{0,\alpha} r^\alpha \fint_{B_r} \frac{|u-(u)_r|^q}{r^q}\, dx.$$

Using the following notation

$$\tilde{t} := \max\left\{\frac{nt}{n+t}, 1\right\}, \qquad \text{for all } t > 1, \tag{3.14}$$

one can check that $1 \le \tilde{q} < p$ due to assumption (1.6). Thus, applying the classical Sobolev-Poincaré inequality, it yields

$$\mathcal{I}_q \le c[a]_{0,\alpha} r^\alpha \left(\fint_{B_r} |Du|^{\tilde{q}}\, dx\right)^{\frac{q}{\tilde{q}}} = c[a]_{0,\alpha} r^\alpha \left(\fint_{B_r} |Du|^{\tilde{q}}\, dx\right)^{\frac{q-p}{\tilde{q}}} \left(\fint_{B_r} |Du|^{\tilde{q}}\, dx\right)^{\frac{p}{\tilde{q}}}.$$

Next, thanks to Hölder inequality, we observe that

$$\begin{aligned}
\mathcal{I}_q &\le c[a]_{0,\alpha} r^\alpha \left(\fint_{B_r} |Du|^p\, dx\right)^{\frac{q-p}{p}} \left(\fint_{B_r} |Du|^{\tilde{q}}\, dx\right)^{\frac{p}{\tilde{q}}} \\
&\le c[a]_{0,\alpha} r^{\alpha - n\left(\frac{q-p}{p}\right)} \|Du\|^{q-p}_{L^p(B_r)} \left(\fint_{B_r} |Du|^{\tilde{q}}\, dx\right)^{\frac{p}{\tilde{q}}} \\
&\le c \left(\fint_{B_r} |Du|^{\tilde{q}}\, dx\right)^{\frac{p}{\tilde{q}}},
\end{aligned}$$

where $c = c(n, p, q, [a]_{0,\alpha}, \|Du\|_{L^p(B_r)})$. Since $\log(e + |Du|) \ge 1$, this inequality implies that

$$\mathcal{I}_q \le c \left(\fint_{B_r} \left[|Du|^p \log^\gamma(e+|Du|)\right]^{\frac{\tilde{q}}{p}} dx\right)^{\frac{p}{\tilde{q}}} \le c \left(\fint_{B_r} \left[G(x, Du)\right]^{\frac{\tilde{q}}{p}} dx\right)^{\frac{p}{\tilde{q}}}. \tag{3.15}$$

Similarly, we also obtain the following estimate

$$\mathcal{I}_s \le c\left(\fint_{B_r} [G(x,Du)]^{\frac{\tilde{s}}{p}}\,dx\right)^{\frac{p}{\tilde{s}}}, \tag{3.16}$$

where $\tilde{s} \in [1,p)$ is defined as in (3.14) and $c = c(n,p,s,[b]_{0,\beta},\|Du\|_{L^p(B_r)})$. Let us now estimate the last integral $\mathcal{I}_{p,\log}$. We now introduce the following function

$$\Phi(t) = t^p \log^\gamma(e+t), \quad t \ge 0.$$

According to Theorem 2.3, It is straightforward to verify that Φ is an N-function satisfying the condition $\Delta_2(\{\Phi,\Phi^*\}) < \infty$. Applying Poincaré inequality in [15, Theorem 7], there exists $\theta := \theta(p,\gamma) \in (0,1)$ and $c := c(n,p,\gamma)$ such that

$$\begin{aligned}\mathcal{I}_{p,\log} &= \fint_{B_r} \Phi\left(\frac{|u-(u)_r|}{r}\right)dx \\ &\le c\left(\fint_{B_r} [\Phi(|Du|)]^\theta dx\right)^{\frac{1}{\theta}} \\ &\le c\left(\fint_{B_r} [G(x,Du)]^\theta\,dx\right)^{\frac{1}{\theta}}.\end{aligned} \tag{3.17}$$

Collecting all of estimates in (3.15), (3.16), (3.17) and then substituting them into (3.9), one gets

$$\fint_{B_r} G\left(x,\frac{u-(u)_r}{r}\right)dx \le C\left(\fint_{B_r} [G(x,Du)]^{d_1}\,dx\right)^{\frac{1}{d_1}}, \tag{3.18}$$

where $C := C(\texttt{in-data}_2,\|Du\|_{L^p(B_r)}) > 0$ and $d_1 = \max\left\{\theta,\frac{\tilde{q}}{p},\frac{\tilde{s}}{p}\right\} \in (0,1)$.

We now direct our attention to the second case (3.11), which implies that

$$\begin{cases}\sup\limits_{x\in B_r} a(x) > 4\dfrac{|a(k)-a(t)|}{|k-t|^\alpha}r^\alpha > 2|a(k)-a(t)|, \\ \sup\limits_{x\in B_r} b(x) > 4\dfrac{|b(k)-b(t)|}{|k-t|^\beta}r^\beta > 2|b(k)-b(t)|,\end{cases}$$

for all $k,t \in B_r$. From this point, $\sup\limits_{x\in B_r} a(x) \ge 2(\sup\limits_{x\in B_r} a(x) - \inf\limits_{x\in B_r} a(x))$, so that $\inf\limits_{x\in B_r} a(x) \ge \dfrac{1}{2}\sup\limits_{x\in B_r} a(x)$. Similarly, we also have $\inf\limits_{x\in B_r} b(x) \ge \dfrac{1}{2}\sup\limits_{x\in B_r} b(x)$. Therefore, if we fix $x_0 \in B_r$ arbitrarily, from the last inequality, we have the following estimates

$$\begin{cases}a(x)t^q \le 2\left[\inf_{x\in B_r} a(x)\right]t^q \le 2a(x_0)t^q \le 4a(x)t^q, \\ b(x)t^s \le 2\left[\inf_{x\in B_r} b(x)\right]t^s \le 2b(x_0)t^s \le 4b(x)t^s,\end{cases} \tag{3.19}$$

for all $x \in B_r$ and $t \ge 0$. We now introduce a new function

$$\xi \in \mathbb{R}^n \mapsto G_0(\xi) := |\xi|^p \log^\gamma(e+|\xi|) + a(x_0)|\xi|^q + b(x_0)|\xi|^s.$$

For the sake of simplicity, we shall use the same notation G_0 to denote both the function $G_0 : \mathbb{R}^n \to [0,+\infty)$ defined above and its associated N-function $G_0 : [0,+\infty) \to [0,+\infty)$. Specifically, we identify $G_0(\xi)$ with $G_0(t)$ where $t = |\xi|$. It is easy to verify that G_0 is an N-function, so that applying the Poincaré inequality in [15, Theorem 7] for the function

G_0 and the inequality (3.19). It gives

$$\begin{aligned}\fint_{B_r} G\left(x, \frac{u-(u)_r}{r}\right) dx &\le 2 \fint_{B_r} G_0\left(\frac{u-(u)_r}{r}\right) dx \\ &\le c\left(\fint_{B_r} G_0(Du)^{d_2}\, dx\right)^{\frac{1}{d_2}} \\ &\le C\left(\fint_{B_r} G(x,Du)^{d_2}\, dx\right)^{\frac{1}{d_2}}, \end{aligned} \tag{3.20}$$

where $C := C(\texttt{in-data}_1)$ and $d_2 := d_2(\texttt{in-data}_1) < 1$.

Let us consider the third case (3.12). For $x_0 \in B_r$, we set

$$\xi \in \mathbb{R}^n \mapsto G_0^q(\xi) = |\xi|^p \log^\gamma(e+|\xi|) + a(x_0)|\xi|^q.$$

We recognize that in this case, (3.16) and $(3.19)_1$ are still true. Hence, we again apply the Poincaré inequality in [15, Theorem 7] for the N-function G_0^q and the estimate (3.16), which shows that

$$\begin{aligned}\fint_{B_r} G\left(x, \frac{u-(u)_r}{r}\right) dx &\le 2 \fint_{B_r} G_0^q\left(\frac{u-(u)_r}{r}\right) dx + c\left(\fint_{B_r} [G(x,Du)]^{\frac{\tilde{s}}{p}}\, dx\right)^{\frac{p}{\tilde{s}}} \\ &\le c'\left(\fint_{B_r} G_0^q(Du)^{d_q}\, dx\right)^{\frac{1}{d_q}} + c\left(\fint_{B_r} [G(x,Du)]^{\frac{\tilde{s}}{p}}\, dx\right)^{\frac{p}{\tilde{s}}} \\ &\le C\left(\fint_{B_r} G(x,Du)^{d_3}\, dx\right)^{\frac{1}{d_3}}, \end{aligned} \tag{3.21}$$

where $C := C(\texttt{in-data}_2, ||Du||_{L^p(B_r)}), d_q := d_q(\texttt{in-data}_1)$ and $d_3 = \max\{d_q, \frac{\tilde{s}}{p}\} < 1$.

The final case (3.13) can be handled in the same manner as the previous case. For $x_0 \in B_r$, we set

$$G_0^s(\xi) = |\xi|^p \log^\gamma(e+|\xi|) + b_0|\xi|^s.$$

Then we recognize that (3.15) and $(3.19)_2$ are still true. Therefore, using these estimates and the Poincaré inequality in [15, Theorem 7] for the N-function G_0^s, we get

$$\begin{aligned}\fint_{B_r} G\left(x, \frac{u-(u)_r}{r}\right) dx &\le 2 \fint_{B_r} G_0^s\left(\frac{u-(u)_r}{r}\right) dx + c\left(\fint_{B_r} [G(x,Du)]^{\frac{\tilde{q}}{p}}\, dx\right)^{\frac{p}{\tilde{q}}} \\ &\le c'\left(\fint_{B_r} G_0^s(Du)^{d_s}\, dx\right)^{\frac{1}{d_s}} + c\left(\fint_{B_r} [G(x,Du)]^{\frac{\tilde{q}}{p}}\, dx\right)^{\frac{p}{\tilde{q}}} \\ &\le C\left(\fint_{B_r} G(x,Du)^{d_4}\, dx\right)^{\frac{1}{d_4}}, \end{aligned} \tag{3.22}$$

where $C := C(\texttt{in-data}_2, ||Du||_{L^p(B_r)}), d_s := d_s(\texttt{in-data}_1)$ and $d_4 = \max\{d_s, \frac{\tilde{q}}{p}\} < 1$. Setting $d = \max\{d_1, d_2, d_3, d_4\} < 1$, from estimates (3.18), (3.20), (3.21) and (3.22), we obtain (3.8). The proof is complete. □

3.3. Local higher integrability

From this point, Theorems 3.6 and 3.5 give us the main result by applying the following Gehring's Lemma, which is well-known in [5, Theorem 4.2].

Lemma 3.7. *Let $w \in L^p_{loc}(\Omega)$ for $1 < p < \infty$. Suppose that there exists a constant $C_1 > 0$ such that*

$$\left(\fint_{B_{\frac{r}{2}}} |w|^p \, dx\right)^{\frac{1}{p}} \leq C_1 \fint_{B_r} |w| \, dx,$$

for all $B_r \subset \Omega$. Then, it follows that there exist a value $q = q(p, n) > p$ and a positive constant C_2 depending solely on n, p, q and C_1, such that for every ball B_r contained in Ω, we have:

$$\left(\fint_{B_{\frac{r}{2}}} |w|^q \, dx\right)^{\frac{1}{q}} \leq C_2 \left(\fint_{B_r} |w|^p \, dx\right)^{\frac{1}{p}}.$$

Theorem 3.8. *Let $u \in W^{1,p}_{loc}(\Omega)$ be a local minimizer of the integral functional $\mathcal{G}$ defined in* (1.1)*. Suppose that assumptions* (1.3) *and* (1.6) *are satisfied. Then, for any ball $B_r \subset \Omega$ with radius $r \leq 1$, there exist two constants $c > 0$ and $\delta > 1$, both depending on* `in-data`$_2$ *and $\|Du\|_{L^p(B_r)}$, such that $G(\cdot, Du) \in L^\delta_{loc}(\Omega)$ and the following estimate holds:*

$$\left(\fint_{B_{\frac{r}{2}}} [G(x, Du)]^\delta \, dx\right)^{\frac{1}{\delta}} \leq c \fint_{B_r} G(x, Du) \, dx.$$

Proof. Let $r \in (0, 1]$ such that $B_r \subset \Omega$. By combining the estimates provided in Theorems 3.5 and 3.6, we arrive at the weak reverse Hölder inequality below

$$\fint_{B_{r/2}} G(x, Du) \, dx \leq C \left(\fint_{B_r} [G(x, Du)]^d \, dx\right)^{\frac{1}{d}},$$

for some constants $C = C(\texttt{in-data}_2, \|Du\|_{L^p(B_r)}) > 0$ and $0 < d < 1$. Therefore, applying Lemma 3.7 to the weak reverse Hölder inequality, it completes the proof and yields the desired higher integrability result. ☐

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