Alabama Journal of Mathematics

Alabama Journal of Mathematics
Volume 48 (1) (2025), 9 – 16

ORIGINAL ARTICLE

Geodesic curves and Barycenters in $\mathbb{R}^n$

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Abstract

In this paper, we study geodesics in planes with respect to various distance functions. As an application, we show that for any interior point of a Euclidean triangle, there exists a metric on $\mathbb{R}^n$ with respect to which this point becomes the barycenter of the triangle. We also briefly discuss an extension to the case of positive definite matrices.

Keywords: geodesic curves; barycenters; distance functions.

MSC Classification: 51F30, 51F99

1. Introduction

Let $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ and $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$ be two sets of data points in $\mathbb{R}^n$. Let C_0 be a random data point, and let d be a distance function. In order to decide whether C_0 belongs to $\mathcal{A}$ or $\mathcal{B}$, we need to measure the distance from C_0 to both $\mathcal{A}$ and $\mathcal{B}$. To do this, we first find the averaging element of each set with respect to the distance d, and then calculate the distance from C_0 to these elements. However, solving two least squares problems can be computationally expensive.

Such a problem may arise in healthcare science, where $\mathcal{A}$ and $\mathcal{B}$ represent the sets of patients and healthy individuals, respectively. It is important to note that the distance d is given, but it can be modified.

This naturally raises the question: **How can we reduce the computational cost?**

One approach is to construct a new distance d_N such that a selected point in $\mathcal{A}$ becomes the barycenter of $\mathcal{A}$. Subsequently, we only need to solve the least squares problem for the set $\mathcal{B}$, using the distance d_N. This method effectively reduces the computational cost by half.

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Received: April 1, 2025; Accepted: April 15, 2025

This paper focuses on addressing the question posed above in $\mathbb{R}^2$. The organization of the paper is as follows: In the next section, we explore geodesic curves in the plane. Given a specific metric and two points in $\mathbb{R}^2$, we construct a geodesic connecting these points. Additionally, for a given increasing monotone function and two points on its graph, we demonstrate the existence of a metric under which the graph of the function becomes a geodesic. Using these results, we show that for any interior point within a Euclidean triangle—that is, a point inside the triangle but not on any of its edges—we can construct a metric such that this point becomes the barycenter of the triangle. As a consequence, it follows that any interior point of a data set can be designated as the barycenter of the set with respect to an appropriately constructed distance function.

2. Geodesic curves in $\mathbb{R}^2$

Let d be a metric in $\mathbb{R}^2$. A *geodesic curve* in the metric space $(\mathbb{R}^2, d)$ is the shortest path between two points in $\mathbb{R}^2$, with "shortest" defined according to the metric d. The shape of a geodesic depends on the metric of the space. Different metrics may define different geodesics between the same pair of points.

For example, in Euclidean space $(\mathbb{R}^2, d_E)$, geodesics are straight lines. Namely, given points $A(x_A, y_A)$ and $B(x_B, y_B)$, the geodesic line is given by

$$(1-\lambda)A + \lambda B, \quad \lambda \in [0,1].$$

Suppose that the coordinates of A and B are positive. Then, with respect to the hyperbolic distance

$$d_L(A,B) = d_E((\log x_A, \log y_A), (\log x_B, \log y_B)),$$

the geodesic curve joining A and B is the graph of the vector-valued function

$$r(\lambda) = (x(\lambda), y(\lambda)) = \left(x_A^{1-\lambda} x_B^{\lambda},\ y_A^{1-\lambda} y_B^{\lambda}\right), \quad \lambda \in [0,1].$$

In terms of scalar means, geodesics in Euclidean spaces correspond to weighted arithmetic means, whereas geodesics in log-spaces correspond to weighted geometric means. From this perspective, if we consider metrics in $\mathbb{R}^2$ defined by other scalar means, the geodesics are fundamentally determined by the specific mean used.

Now let $s_1, s_2 \in \mathbb{R} \setminus \{0\}$, and define $f(t) = t^{s_1}$, $h(t) = t^{s_2}$. Consider the function

$$\begin{aligned} G\colon (0,\infty) \times (0,\infty) &\to \mathbb{R}^2 \\ (x,y) &\mapsto G(x,y) = (f(x), h(y)). \end{aligned}$$

As can be seen, if $s_1, s_2 > 0$, then both $f(x)$ and $h(y)$ are strictly increasing. Hence, the function G is strictly monotonic.

Now, for two points A and B, we define the distance

$$d_G(A,B) = d_E(G(A), G(B)).$$

It is obvious that d_G is a metric in the plane. In the following theorem, we construct the geodesic with respect to the metric d_G.

Theorem 2.1. *The geodesic joining A and B with respect to d_G is given by the following parametric equations:*

$$\begin{cases} x(\lambda) = \left((1-\lambda)x_A^{s_1} + \lambda x_B^{s_1}\right)^{1/s_1}, \\ y(\lambda) = \left((1-\lambda)y_A^{s_2} + \lambda y_B^{s_2}\right)^{1/s_2}, \end{cases} \qquad \lambda \in [0,1].$$

Proof. We know that the geodesic connecting $G(A)$ and $G(B)$ with respect to the Euclidean distance d_E is

$$(1-\lambda)G(A) + \lambda G(B) = ((1-\lambda)f(x_A) + \lambda f(x_B),\ (1-\lambda)h(y_A) + \lambda h(y_B)). \tag{2.1}$$

Since $f(x)$ and $h(y)$ are strictly monotonic, the inverse functions $f^{-1}(x) = x^{1/s_1}$ and $h^{-1}(y) = y^{1/s_2}$ exist. Applying these inverse functions to (2.1), we obtain the geodesic with respect to d_G:

$$\begin{cases} x(\lambda) = ((1-\lambda)x_A^{s_1} + \lambda x_B^{s_1})^{1/s_1}, \\ y(\lambda) = ((1-\lambda)y_A^{s_2} + \lambda y_B^{s_2})^{1/s_2}, \end{cases} \qquad \lambda \in [0,1].$$

□

The functions $f(t) = t^{s_1}$ and $h(t) = t^{s_2}$ in Theorem 2.1 are bijections on their domains, in the sense that

$$(1-\lambda)f(x_A) + \lambda f(x_B) \in \mathrm{Im} f, \quad (1-\lambda)h(y_A) + \lambda h(y_B) \in \mathrm{Im} h, \quad \forall \lambda \in [0,1].$$

Thus, given two points A and B, if $f(x)$ and $h(x)$ are bijections (which means the above inclusion holds for all $\lambda \in [0,1]$), then by defining the metric

$$G\colon (0,\infty) \times (0,\infty) \to \mathbb{R}^2$$
$$(x,y) \mapsto G(x,y) = (f(x), h(y)),$$

the geodesic corresponding to d_G is given by:

$$\begin{cases} x(\lambda) = f^{-1}\left((1-\lambda)f(x_A) + \lambda f(x_B)\right), \\ y(\lambda) = h^{-1}\left((1-\lambda)h(y_A) + \lambda h(y_B)\right), \end{cases} \qquad \lambda \in [0,1].$$

Hence, we have the following theorem.

Theorem 2.2. *Given two points $A(x_A, y_A)$ and $B(x_B, y_B)$, let $f(x)$ and $h(x)$ be monotone functions such that*

$$(1-\lambda)f(x_A) + \lambda f(x_B) \in \mathrm{Im} f, \quad (1-\lambda)h(y_A) + \lambda h(y_B) \in \mathrm{Im} h$$

for all $\lambda \in [0,1]$. Then, the geodesic connecting A and B with respect to the distance d_G is given by the parametric equations:

$$\begin{cases} x(\lambda) = f^{-1}\left((1-\lambda)f(x_A) + \lambda f(x_B)\right), \\ y(\lambda) = h^{-1}\left((1-\lambda)h(y_A) + \lambda h(y_B)\right), \end{cases} \qquad \lambda \in [0,1].$$

The second part of this section focuses on how to make the graph of a monotonic function a geodesic.

Let $f(x)$ be a monotonic curve between two points A and B, meaning $A(x_A, f(x_A)) = (x_A, y_A)$ and $B(x_B, f(x_B)) = (x_B, y_B)$. Let $u(x)$ and $v(x)$ be monotonic functions. The parametric equations of the curve between two points $f(A) = (f(x_A), f(y_A))$ and $f(B) = (f(x_B), f(y_B))$ are given by

$$\begin{cases} x(\lambda) = (1-\lambda)u(x_A) + \lambda u(x_B), \\ y(\lambda) = (1-\lambda)v(y_A) + \lambda v(y_B), \end{cases} \qquad \lambda \in [0,1]. \tag{2.2}$$

If

$$(1-\lambda)\,u(x_A) + \lambda\,u(x_B) \in \mathrm{Im}\,u, \quad (1-\lambda)\,v(y_A) + \lambda\,v(y_B) \in \mathrm{Im}\,v$$

for all $\lambda \in [0,1]$, then $u(x)$ and $v(y)$ are bijections onto $\mathrm{Im}\,u$ and $\mathrm{Im}\,v$, respectively. Thus, by Theorem 2.2, the geodesic connecting A and B with respect to the distance d_G is given by the parametric equations

$$\begin{cases} x_\lambda = u^{-1}\left((1-\lambda)\,u(x_A) + \lambda\,u(x_B)\right), \\ y_\lambda = v^{-1}\left((1-\lambda)\,v(y_A) + \lambda\,v(y_B)\right), \end{cases} \qquad \lambda \in [0,1].$$

With respect to the Euclidean distance, the expression (2.2) represents a geodesic connecting the two points $f(A)$ and $f(B)$. Applying the inverse functions u^{-1} and v^{-1} to (2.2), we derive the geodesic between the points A and B with respect to d_G, which is precisely given by $f(x)$.

It is easy to see that there exist two monotonic functions $u(x)$ and $v(x)$ satisfying the required conditions for all $\lambda \in [0,1]$; for instance, $u(x) = x^{s_1}$ and $v(x) = x^{s_2}$ as in Theorem 2.1. Therefore, we can state the condition to make the graph of a monotonic function a geodesic.

Theorem 2.3. *Let $A(x_A, y_A)$ and $B(x_B, y_B)$ be two points lying on a monotonic curve represented by $f(x)$. Consider two monotonic functions $u(x)$ and $v(x)$. If*

$$(1-\lambda)u(x_A) + \lambda u(x_B) \in \operatorname{Im} u, \quad (1-\lambda)v(y_A) + \lambda v(y_B) \in \operatorname{Im} v$$

for all $\lambda \in [0,1]$, then $f(x)$ is a geodesic connecting A and B with respect to the distance d_G.

3. Barycenters

Let's begin this section by defining the concept of the barycenter of a set of points in the plane.

Definition 3.1. Let d be a distance in $\mathbb{R}^n$. The point X is called the **barycenter** (or **centroid**) of $A_1, A_2, \dots, A_n$ if the sum

$$\sum_{i=1}^{n} d^2(X, A_i)$$

is minimized.

We exclude the case of the discrete metric, as under this metric, any point in the plane could serve as the barycenter of the given points, making it an uninteresting scenario.

A distance d satisfies the **linearity property** if $d(A,B) + d(B,C) = d(A,C)$ for any collinear points A, B, and C. In $\mathbb{R}^2$, generalizing the idea in [1], one can see that the barycenter of two given points A and B with respect to the distance d is the point C such that $d(A,C) = d(B,C)$.

In the case of three non-collinear points, we know that with respect to the Euclidean distance, three medians always meet at one point. This point is called the centroid, or barycenter, of the given triangle.

In this section, we first show that any interior point of an interval in the plane can be the barycenter with respect to some distance. As a consequence, we show that any interior point of a Euclidean triangle can serve as the barycenter of its three vertices with respect to a certain distance.

3.1. Barycenter of Two Points

In this section, we generalize the known result from the paper [1] to the two-dimensional case $\mathbb{R}^2$

Theorem 3.2. *Let $A(x_A, y_A)$ and $B(x_B, y_B)$ be two points such that $0 < x_A < x_B$ and $0 < y_A < y_B$. Let $\Omega = \{(x,y) \,|\, x \in (x_A, x_B),\, y \in (y_A, y_B)\}$. Then, for every point $C \in \Omega$, there exists a distance d such that C belongs to a geodesic connecting A and B, and C is the barycenter of A and B.*

Proof. Recall that in [1], it was proved that for any $c \in (a,b) \subset (0,\infty)$, there exists $s_0 \in \mathbb{R}$ such that

$$c = \left(\frac{a^{s_0} + b^{s_0}}{2}\right)^{1/s_0}.$$

When $c = \sqrt{ab}$, we understand this as the limit:

$$c = \lim_{s \to 0} \left(\frac{a^s + b^s}{2}\right)^{1/s}.$$

Applying this fact to (x_A, x_B) and (y_A, y_B), we see that for any $C \in \Omega$, there exist $s_1, s_2 \in \mathbb{R}$ such that

$$\begin{cases} x_C = \left(\dfrac{x_A^{s_1} + x_B^{s_1}}{2}\right)^{1/s_1}, \\ y_C = \left(\dfrac{y_A^{s_2} + y_B^{s_2}}{2}\right)^{1/s_2}. \end{cases}$$

By Theorem 2.1, the geodesic that connects A and B with respect to d_G is

$$\begin{cases} x(\lambda) = ((1-\lambda)x_A^{s_1} + \lambda x_B^{s_1})^{1/s_1}, \\ y(\lambda) = ((1-\lambda)y_A^{s_2} + \lambda y_B^{s_2})^{1/s_2}, \end{cases} \qquad \lambda \in [0,1].$$

Let $\lambda = \dfrac{1}{2}$. We obtain

$$\begin{cases} x = \left(\dfrac{x_A^{s_1} + x_B^{s_1}}{2}\right)^{1/s_1}, \\ y = \left(\dfrac{y_A^{s_2} + y_B^{s_2}}{2}\right)^{1/s_2}. \end{cases}$$

Therefore, C lies on the geodesic connecting the two points A and B with respect to d_G.

Finally, we show that C is a midpoint of A and B (in the d_G metric). Indeed, we have

$$\begin{aligned} d_G(A,C) &= \sqrt{(x_C^{s_1} - x_A^{s_1})^2 + (y_C^{s_2} - y_A^{s_2})^2} \\ &= \sqrt{\left(\frac{x_B^{s_1} - x_A^{s_1}}{2}\right)^2 + \left(\frac{y_B^{s_2} - y_A^{s_2}}{2}\right)^2}, \end{aligned}$$

$$\begin{aligned} d_G(B,C) &= \sqrt{(x_C^{s_1} - x_B^{s_1})^2 + (y_C^{s_2} - y_B^{s_2})^2} \\ &= \sqrt{\left(\frac{x_A^{s_1} - x_B^{s_1}}{2}\right)^2 + \left(\frac{y_A^{s_2} - y_B^{s_2}}{2}\right)^2}. \end{aligned}$$

It is easy to see that

$$d_G(A,C) = d_G(B,C),$$

which means C is a barycenter of A and B. □

3.2. Barycenter of Three Points

We start this section by stating the following lemma.

Lemma 3.3. *Given positive numbers a_1, a_2, a_3, then for every c satisfying*

$$\min\{a_1, a_2, a_3\} < c < \max\{a_1, a_2, a_3\},$$

there exists a number s such that

$$c = \left(\frac{a_1^s + a_2^s + a_3^s}{3}\right)^{1/s}.$$

Proof. Define the function

$$g(s) = \begin{cases} \left(\dfrac{a_1^s + a_2^s + a_3^s}{3}\right)^{1/s}, & s \neq 0, \\ \sqrt[3]{a_1 a_2 a_3}, & s = 0. \end{cases}$$

It is easy to see that $g(s)$ is continuous on $\mathbb{R} \setminus \{0\}$. The continuity at $s = 0$ follows from the fact that

$$\lim_{s \to 0} \left(\frac{a_1^s + a_2^s + a_3^s}{3} \right)^{1/s} = \sqrt[3]{a_1 a_2 a_3}.$$

Furthermore,

$$\lim_{s \to -\infty} g(s) = \min\{a_1, a_2, a_3\},$$
$$\lim_{s \to \infty} g(s) = \max\{a_1, a_2, a_3\}.$$

By the Intermediate Value Theorem, for every

$$\min\{a_1, a_2, a_3\} < c < \max\{a_1, a_2, a_3\},$$

there exists a number s such that

$$c = \left(\frac{a_1^s + a_2^s + a_3^s}{3} \right)^{1/s}.$$

□

This lemma extends a theorem given in [1] in the setting of three positive real numbers. Building on this, we arrive at the following result, which we find particularly interesting.

Theorem 3.4. *Assume $A(x_A, y_A)$, $B(x_B, y_B)$, and $C(x_C, y_C)$ are non-collinear points with positive coordinates. Then, for every point $M(x_M, y_M)$ inside the triangle ABC, there exists a metric d such that M is the barycenter of A, B, and C.*

Proof. By Lemma 3.3, for $M(x_M, y_M)$ inside the triangle ABC, there exist $s_1, s_2 \in \mathbb{R}$ such that

$$\begin{cases} x_M = \left(\dfrac{x_A^{s_1} + x_B^{s_1} + x_C^{s_1}}{3} \right)^{1/s_1}, \\ y_M = \left(\dfrac{y_A^{s_2} + y_B^{s_2} + y_C^{s_2}}{3} \right)^{1/s_2}. \end{cases}$$

By Theorem 2.1, we can describe the geodesics that join A to B, B to C, and A to C by the following parametric equations:

$$(AB) : \begin{cases} x(\lambda) = \left((1-\lambda) x_A^{s_1} + \lambda x_B^{s_1}\right)^{1/s_1}, \\ y(\lambda) = \left((1-\lambda) y_A^{s_2} + \lambda y_B^{s_2}\right)^{1/s_2}, \end{cases}$$

$$(BC) : \begin{cases} x(\lambda) = \left((1-\lambda) x_B^{s_1} + \lambda x_C^{s_1}\right)^{1/s_1}, \\ y(\lambda) = \left((1-\lambda) y_B^{s_2} + \lambda y_C^{s_2}\right)^{1/s_2}, \end{cases}$$

$$(AC) : \begin{cases} x(\lambda) = \left((1-\lambda) x_A^{s_1} + \lambda x_C^{s_1}\right)^{1/s_1}, \\ y(\lambda) = \left((1-\lambda) y_A^{s_2} + \lambda y_C^{s_2}\right)^{1/s_2}, \end{cases}$$

where $\lambda \in [0, 1]$.

The function d_G defined by

$$d_G(A, B) = \sqrt{(x_A^{s_1} - x_B^{s_1})^2 + (y_A^{s_2} - y_B^{s_2})^2}$$

is a metric.

Following the same logic as in the two-point case, the three medians intersect at M when $\lambda = \dfrac{1}{3}$. Finally, to verify that M satisfies Definition 3.1, consider the function:

$$\begin{aligned} f(x,y) =& d_G^2((x,y),A) + d_G^2((x,y),B) + d_G^2((x,y),C) \\ =&(x^{s_1} - x_A^{s_1})^2 + (y^{s_2} - y_A^{s_2})^2 \\ +&(x^{s_1} - x_B^{s_1})^2 + (y^{s_2} - y_B^{s_2})^2 \\ +&(x^{s_1} - x_C^{s_1})^2 + (y^{s_2} - y_C^{s_2})^2. \end{aligned}$$

The gradient of $f(x,y)$ is:

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2s_1 x^{s_1-1}\left(3x^{s_1} - x_A^{s_1} - x_B^{s_1} - x_C^{s_1}\right), \\ \dfrac{\partial f}{\partial y} = 2s_2 y^{s_2-1}\left(3y^{s_2} - y_A^{s_2} - y_B^{s_2} - y_C^{s_2}\right). \end{cases}$$

The critical point occurs at

$$\begin{cases} x^{s_1} = \dfrac{x_A^{s_1} + x_B^{s_1} + x_C^{s_1}}{3}, \\ y^{s_2} = \dfrac{y_A^{s_2} + y_B^{s_2} + y_C^{s_2}}{3}. \end{cases}$$

Hence,

$$\begin{cases} x = \left(\dfrac{x_A^{s_1} + x_B^{s_1} + x_C^{s_1}}{3}\right)^{1/s_1}, \\ y = \left(\dfrac{y_A^{s_2} + y_B^{s_2} + y_C^{s_2}}{3}\right)^{1/s_2}. \end{cases}$$

Thus, M satisfies Definition 3.1; that is, M is the barycenter of A, B, and C. □

4. Concluding remarks

The arguments used in the previous section can be generalized to $\mathbb{R}^n$ by using n numbers $s_1, s_2, \cdots, s_n$.

Theorem 4.1. *Let $A_i = (x_{i1}, x_{i2}, \cdots, x_{in})$ for $i = 1, 2, \cdots, n$. Let $x_0 = (x_{01}, x_{02}, \cdots, x_{0n})$ be such that, for each j,*

$$\min\{x_{ij}\} \le x_{0j} \le \max\{x_{ij}\}.$$

Then, there exist positive numbers $s_1, s_2, \cdots, s_n$ such that x_0 is the barycenter of A_i with respect to the following distance:

$$d_S(A,B) = \sqrt{(x_{A1} - x_{B1})^{s_1} + (x_{A2} - x_{B2})^{s_2} + \cdots + (x_{An} - x_{Bn})^{s_n}}.$$

Moreover,

$$x_{0i} = \left(\frac{x_{1i}^{s_i} + x_{2i}^{s_i} + \cdots + x_{ni}^{s_i}}{n}\right)^{1/s_i}, \quad i = 1, 2, \cdots, n.$$

In machine learning, quantum information theory, and other applied sciences, positive definite matrices often serve as data points. Finding the barycenter of a set of positive definite matrices with respect to a given divergence is a significant yet challenging problem. Unfortunately, for more than two positive definite matrices, there is no explicit formula for their barycenter.

In [5], for two positive definite matrices A and B, and for a symmetric Kubo-Ando matrix mean σ, Virosztek constructed a divergence on the set of positive definite matrices such that the matrix $A\sigma B$ becomes the barycenter of A and B. The symmetry condition is crucial in his construction. Unfortunately, without this condition, it remains unclear whether such a divergence can exist. The simplest case to consider is the weighted arithmetic mean $(1-\lambda)A+\lambda B$. If $\lambda \neq 1/2$, we do not know how to construct a distance such that the point $(1-\lambda)A+\lambda B$ is the barycenter of A and B.

Finally, given the simplicity of the mathematics presented in this paper, we believe that constructing a distance such that a given point becomes the barycenter of a data set holds significant potential for practical applications.

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