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ORIGINAL ARTICLE

Robert Recorde's number theory: A representation of diametral numbers

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Abstract

Robert Recorde (c.1512-1558) has long been esteemed for his skillful presentation of basic arithmetic and algebra in his most well-known works, *The Grounde of Artes* and *The Whetstone of Witte*. But Recorde's extensive treatment of number theory in *The Whetstone of Witte* has received little attention. In this paper, we will discuss Recorde's presentation of diametral numbers, and contrast his presentation with that of his source, Michael Stifel. We will also discuss the connection between diametral numbers and congruent numbers.

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1. Introduction

Academics, particularly mathematicians, have long regarded Robert Recorde as a preeminent scholar from the Tudor age. As with many of the intellectual lights of that age, Recorde's works and interests were not restricted to a particularly narrow field of endeavor. He wrote on a variety of mathematical and scientific topics, as well as working in physiology and medicine. Yet it was in the field of mathematics that he made his most lasting contributions. Recent scholarship attests to the relevance of Recorde's contributions to the early modern intellectual world. All of Recorde's surviving manuscripts are presently available in print.

Of particular interest is Recorde's last, and most sophisticated, presentation of mathematics in the work *The Whetstone of Witte*. Modern scholars commend the work as the first English language presentation of rudimentary algebra. But what is often neglected is the extensive chapter (72 pages) that deals with number theory.

If one regards number theory as "the queen of mathematics," as Gauss dubbed it, then Recorde's early role as an advocate for number theory has a unique place in the world of Early Modern Mathematics. Recorde recognized the importance of number theory's emphasis on numerical patterns, which can help hone the pattern recognition skills that are essential to survive and thrive in the world around us. Pattern recognition is in constant use for categorizing and classifying our surroundings, and for making predictions that can help us decide which restaurant will be less crowded during happy hour, or whether "fight"

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or "flight" is more appropriate in a dangerous situation. "Our brains receive signals from the eyes and process them into 'Tiger!' or 'Wasp!', prompting appropriate evasive action; this indicates that our senses have in part evolved for the purpose of detecting patterns." [4, p. 28] In his number theory chapter, Recorde gives his readers ample opportunities to hone their pattern recognition skills with his treatment of diametral numbers, which we will discuss in this paper. We will contrast Recorde's treatment with that of his source, Michael Stifel, and we will briefly discuss the connection between diametral numbers and congruent numbers.

2. Diametral Numbers

Recorde defines a "*diametralle nomber*" as

> a number as hath twoo partes of that nature: that if thei bee multiplied together, thei will make the saied *diametralle nomber*: And the squares of those twoo partes, beeyng added together, will make a square nomber also: whose roote is the *diameter* to that *diametralle nomber*. [3, p. 40].

Recorde then gives 12 as an example of a diametral number, since 12 is the product of 3 and 4, the sum of the squares of which is 25, a perfect square. He gives the following diagram to illustrate:

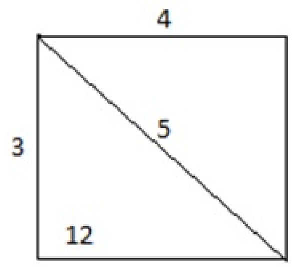

Figure 1. Recorde's diametral number illustration

We can see that diametral numbers are obtained by multiplying the two smaller numbers of a Pythagorean triple. Note also that what Recorde calls the *diameter* is simply the third number of a Pythagorean triple.

Stifel's book was originally written in Latin as *Arithmetica Integra* and translated into German as *Vollständiger Lehrgang der Arithmetik*. He gives the same introductory example as Recorde before explaining several interesting properties of diametral numbers, which are expanded upon extensively by Recorde.

3. Properties of Diametral Numbers

Recorde begins by explaining that Pythagorean triples should have the form odd-even-odd or even-even-even, when the numbers are listed in increasing order, e.g. $3-4-5$. Next, he points out that diametral numbers are always even. He also explains that all odd integers greater than one and all even integers greater than four can be the smallest number of a Pythagorean triple. Recorde incorrectly asserts that if the second number of a Pythagorean triple is even, then it must be divisible by 4. He later gives an extensive table containing Pythagorean triples and their corresponding diametral numbers, and $24-70-74$, which is listed in the table, is clearly a counterexample for his assertion, although it is worth noting that this counterexample is not a primitive Pythagorean triple.

Recorde's final observation is this: "If the lesser side bee an odde nomber, then ordinarily the square of it is iuste equalle with that that amounteth by the addition of the *diameter*, to the greater nomber." [3, p. 46]. He gives $3-4-5$ as an example, noting that $3^2 = 4+5$.

Recorde's assertion proves to be correct in general, at least for the primitive Pythagorean triples that he lists. As we will discuss more thoroughly later, the primitive Pythagorean triples with an odd smallest number have the form

$$2k + 1 - 2k^2 + 2k - 2k^2 + 2k + 1$$

where k is a natural number. We see that

$$(2k + 1)^2 = (2k^2 + 2k) + (2k^2 + 2k + 1),$$

verifying Recorde's assertion.

Recorde uses this last observation as a way to find diametral numbers of this type:

> When any odde number is propounded: as the lesser side of a *diametralle nomber*, . . . multiplie that proponed nomber by it selfe, and it will make a square nomber, and will be an odde nomber: so that of it you shall finde no iuste halfe. Therefore take you those two nombers, that are nexte unto the halfe of it: The lesser shall alwaies bee an euen nomber, and shall be the seconde side of the *diametralle nomber*: The other nomber whiche is the greater shall alwaies be an odde nomber: and shall bee the *diameter* of that nomber whiche you desire [3, p. 47].

Recorde then has the Master present the Scholar with several examples, e.g. $7^2 = 49$, half of which is 24.5. The two numbers nearest this are 24 and 25, making $7 - 24 - 25$ the sought-after Pythagorean triple, and 7ů24 $= 168$ the corresponding diametral number. The precocious Scholar replies, "Touching this I nede no more instruction: the thyng is so manifeste." [3, p. 48]. He tests his understanding by presenting a couple of examples of his own.

Next, Recorde points out how one "lesser side" may generate multiple diametral numbers. For example, $9 - 40 - 41 - 360$ and $9 - 12 - 15 - 308$, or $15 - 112 - 113 - 1680$, $15 - 20 - 25 - 300$, and $15 - 36 - 39 - 540$. Note that here we are giving the Pythagorean triple, followed by the corresponding diametral number. Related to this, Recorde mentions that "there is no *diametralle nomber*, that can haue any more *diameters* then one. Yet maie one nomber bee the *diameter* to diuerse other." [3, p. 58]. As an example, he gives $7 - 24 - 25 - 168$ and $15 - 20 - 25 - 300$, where 25 is the diameter of both diametral numbers, 168 and 300. This observation is also in Stifel's work:

> Es ist unmöglich, dass eine Diametralzahl mehr als einen Diameter hat, wie es ebenso unmöglich ist, dass eine Quadratzahl mehr als eine Quadratwurzel hat. Es ist aber möglich, dass ein Diameter der Diameter mehrerer Diametralzahlen ist [5, p. 38].

This translates as:

> It is impossible that a diametral number has more than one diameter, just as it is impossible for a square number to have more than one square root. But it is possible that one diameter is the diameter of several diametral numbers.

He presents the illuminating diagram in Figure 2, where we can see $39 - 52 - 65 - 2028$ and $25 - 60 - 65 - 1500$.

The first assertion of both Recorde and Stifel proves false, as we see from the examples of $12-35-37-420$ and $20-21-29-420$, where the diametral number 420 corresponds to two different diameters, 37 and 29. This is understandable, however, because the second of these is missing from the lists of both men.

Next the Master instructs the Scholar in a method to find diametral numbers of a second type, when the lesser number is even:

> You shall make it square, as you did in the other nombers, that wer odde: And of that square you shall take twoo quarters, whiche you shall alter in

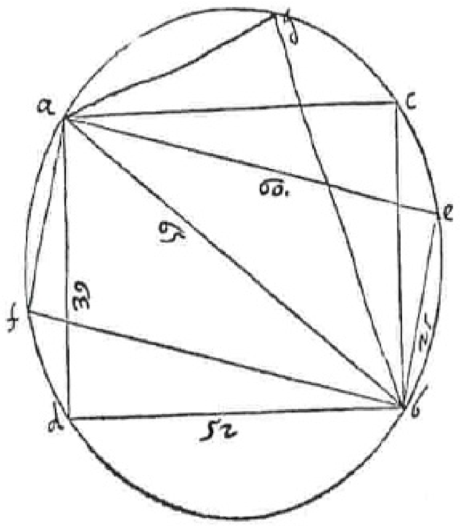

Figure 2. One diameter corresponding to two diametral numbers.

> soche sorte, that you shall abate .1. from the one quarter, and put it to the other quarter. [3, p. 54].

As an example, Recorde starts with 8, which he squares to get 64. Then he quarters 64 to get 16, and thus 15 and 17 complete the Pythagorean triple of $8 - 15 - 17$. The corresponding diametral number is 120.

At this point, Recorde presents a quite extensive, although not comprehensive, table, listing Pythagorean triples and their corresponding diametral numbers, from $3-4-5-12$ to $40 - 399 - 401 - 15960$. There are several diametral numbers missing from the table, although most of the ones missed have neither of the two forms discussed by Recorde. Recorde did not miss any diametral numbers of the first form, but $30 - 224 - 226 - 6720$ has the second form and is missing from his list. Recorde instructs his readers to study the table thoroughly: "Whereby you maie gather not onely the true understandyng of the former rules: But also in theim you maie see other notable conclusions: and straunge workes of the natures of nombers." [3, p. 56].

Now Recorde presents more properties of diametral numbers, including that diametral numbers are divisible by 12, end in one of the digits 0, 2, or 8, and are never perfect squares. All but the first of these are mentioned in Stifel's work as well. Then the Scholar wants to know, "How shall I knowe, when a nomber is proponed, whether it be a diametralle nomber, or not?" The Master replies, "In that thyng I finde a tediouse trauell, by any rules, in those that write of it. But I will ease you of moche paine therein." [3, p. 58]. He advises to first verify that the number satisfies the aforementioned properties of diametral numbers (last digit 0, 2, or 8; divisible by 12; and not a perfect square). Then line up the divisors of the number in pairs. For example, to determine if 120 is a diametral number, list its divisors like so:

2	3	4	5	6	8	10
60	40	30	24	20	15	12

Then he gives the following guidelines to use when testing the divisors:

> If the lesser nomber bee an odde nomber, the square of it must contain double to that greater nomber (that is coupled with it) and one more. And if the lesser be an euen nomber (of them twoo that you would examine)

> then must the square of it containe the greater nomber (that standeth by it) .4. tymes, and .4. more. [3, p. 59].

Using the divisors of 120 to illustrate, we see that $2^2 \neq 4(60) + 4$, $3^2 \neq 2(40) + 1$, $4^2 \neq 4(30) + 4$, $5^2 \neq 2(24) + 1$, and $6^2 \neq 4(20) + 4$, but $8^2 = 4(15) + 4$. So 120 is a diametral number, and the corresponding Pythagorean triple is $8 - 15 - 17$. This method can still be cumbersome for numbers with many divisors, so Recorde gives a shortcut:

> Wherefore if your nomber bee soche a one, as hath many partes, you maie chose one by gesse, which you thinke will go nigh to serue your purpose: and if you finde it to smalle, then set theim doune onely that bee greater then it, til you finde one other iuste: and then haue you your purpose... But and if the parte whiche you tooke by gesse, be to great, you shall refuse all partes aboue it, and take onely lesser partes, til you finde a iuste parte for your purpose: or els one that is to litle. And if in descendynge orderly, you finde no iuste parte, before you come to one that is to litle, then is your nomber no diametralle nomber [3, p. 62].

To illustrate this technique, let's use 2780. Even though 16 is not a divisor of 2780, we can still use it as a benchmark to get an idea of possible divisors to investigate. Note that $2780 = 16 \cdot 173.75$. While $16^2 = 256$, we see that $4(173.75) + 4 = 699$, so clearly 16 is too small. When we observe also that $2(173.75) + 1 = 348.5$, we see that we can disregard both even and odd divisors below 16. To get an upper bound, we will use $2780 = 25 \cdot 111.2$. While $25^2 = 625$, we have $2(111.2) + 1 = 223.4$, so 25 is too large. Seeing that $4(111.2) + 4 = 448.8$ allows us to disregard both even and odd divisors above 25. The only divisor of 2780 that lies between our two bounds is 20. Since $2780 = 20 \cdot 139$, computing $20^2 = 400$ and $4(139) + 4 = 560$ allows us to conclude that 2780 is not a diametral number, according to Recorde's methodology.

Recorde has the Scholar attempt to determine whether or not 43200 is a diametral number, only to be interrupted by the Master:

> I will ease you of your paines in that. For bicause here is more to be considered. You remember that I tolde you before, in makyng of *diametralle nombers*, how that some nombers doe followe the rules of other, of whiche thei be compounde... all soche *diametralle nombers*, must bee excluded from these rules, whiche bee made peculiarly for nombers that haue their owne proper formes, and depende not of other [3, p. 63].

In other words, the rules given by Recorde apply only to diametral numbers that are obtained from primitive Pythagorean triples. Recorde does give these helpful guidelines:

> If any nomber ende in ciphers, abate euen ciphers, as often as you can... and if the reste be a *diametralle nomber*, so was the first. And therefore in this laste example .432. is a *diametralle nomber*, as well as .43200. Also if any nomber beeyng diuided by any square nomber, doe make a *diametralle nomber* in the *quotiente*, then was the first nomber a *diametralle nomber* also [3, p. 65].

In other words, it may be helpful to reduce a number modulo squares before trying to determine if it is a diametral number.

Recorde gives one final characteristic of diametral numbers. He has previously instructed his readers with a basic knowledge of ratios and proportions, and he asks them to draw upon it here:

> The twoo sides of all *diametralle nombers*, haue soche a proportion together, as here you see expressed in some one of these formes: if thei bee continued as here thei be began.
>
> The firste order.

$$\frac{3}{4} : \frac{5}{12} : \frac{7}{24} : \frac{9}{40} : \frac{11}{60} : \frac{13}{84} : \frac{15}{112} : \frac{17}{144} : \frac{19}{180} : \frac{21}{220} : \frac{23}{264} : \frac{25}{312} : \frac{27}{360} : \frac{29}{420} :$$

$$\frac{31}{480} : \frac{33}{544} : \frac{35}{612} : \frac{37}{684} : \frac{39}{760} : \text{etc.}$$

The seconde order.

$$\frac{8}{15} : \frac{12}{35} : \frac{16}{63} : \frac{20}{99} : \frac{24}{143} : \frac{28}{195} : \frac{32}{255} : \frac{36}{323} : \frac{40}{399} : \frac{48}{575} : \frac{52}{2704} : \text{etc.}$$

Note that each list has a typo. In the first list, $\frac{27}{360}$ should be $\frac{27}{364}$ and in the second list, $\frac{52}{2704}$ should be $\frac{52}{675}$.

To see how to use these lists, let's look at 480. Because 480 has a pair of divisors, namely 16 and 30, whose ratio reduces to $\frac{8}{15}$, which is in the second list of ratios, we can conclude that 480 is a diametral number. It is here that Recorde reveals Stifel as his source:

> *Stifelius* doeth set them so, that the numerator standeth for the seconde, or greater side: and the denominator for the firste nomber, or lesser side. And for the more delectable contemplation, to behold their forme of progression, he setteth doune as many whole nombers, as the fraction will giue [3, p. 64].

Recorde then presents the lists as Stifel gave them in Arithmetica Integra:

The firste order.

$$1\frac{1}{3} : 2\frac{2}{5} : 3\frac{3}{7} : 4\frac{4}{9} : 5\frac{5}{11} : 6\frac{6}{13} : 7\frac{7}{15} : \text{etc.}$$

The seconde order.

$$1\frac{7}{8} : 2\frac{11}{12} : 3\frac{15}{16} : 4\frac{19}{20} : 5\frac{23}{24} : 6\frac{27}{28} : 7\frac{31}{32} : \text{etc.}$$

In a 1912 paper, Ernst Meyer categorizes diametral numbers, as presented by Stifel, as follows: "Ein Produkt $a \cdot b$ ist dann und nur dann eine Diametralzahl, wenn sich verhält entweder $a : b = 2\alpha{+}1 : 2\alpha^2{+}2\alpha$ oder $a : b = 4\alpha^2{+}8\alpha{+}3 : 4\alpha{+}4 = (2\alpha{+}1)(2\alpha{+}3) : 4(\alpha{+}1)$." [2, p. 281]. This translates as, "A product $a{\cdot}b$ is then and only then a diametral number if either $a : b = 2\alpha{+}1 : 2\alpha^2{+}2\alpha$ or $a : b = 4\alpha^2{+}8\alpha{+}3 : 4\alpha{+}4 = (2\alpha{+}1)(2\alpha{+}3) : 4(\alpha{+}1)$." These ratios correspond to the first order and the second order, respectively, of the ratios of the sides of a right triangle given by both Stifel and Recorde. However, as Meyer points out, "Die Beschränkung, daß nur solche Zahlen Diametralzahlen sind, ist unrichtig." This translates as, "The restriction that only such numbers are diametral is incorrect." Here are some diametral numbers, preceded by their corresponding Pythagorean triples, which were missed by both Stifel and Recorde: $20-21-29-420$, $28-45-53-1260$, $33-56-65-1848$, $36-77-85-2772$, $39-80-89-3120$, $44-117-125-5148$, and $48-55-73-2640$.

Meyer catalogues several ways of obtaining Pythagorean triples, and thus diametral numbers. He presents $a : b = m^2 - n^2 : 2mn$ as the optimal choice [Cf. 1, p., 170]. The above missing numbers can be found by using the following values of m and n, respectively: $m = 5, n = 2; m = 7, n = 2; m = 7, n = 4; m = 9, n = 2; m = 8, n = 5; m = 11, n = 2$; and $m = 8, n = 3$. All examples presented by Stifel and Recorde can also be obtained by these formulas. For example, $3-4-5-12$ corresponds to $m = 2, n = 1$.

4. Congruent Numbers and Diametral Numbers

Before we conclude, let's discuss congruent numbers and their connection to diametral numbers. A natural number k is called congruent if there exists a rational number x such that $x^2 + k$ and $x^2 - k$ are both squares of rational numbers. For example, Leonardo Pisano, circa 1220, found 5 to be a congruent number by noting that $(\frac{41}{12})^2 + 5 = (\frac{49}{12})^2$ and $(\frac{41}{12})^2 - 5 = (\frac{31}{12})^2$ [1, p. 460]. This definition of a congruent number is equivalent to the following: A natural number k is a *congruent number* if it is the area of a right triangle with rational side lengths.

Theorem 4.1. The two definitions of congruent numbers as stated above are equivalent.

Proof. ($\Rightarrow$) Assume that k is a natural number satisfying the first definition. So there exists a rational number x such that $x^2 + k = y^2$ and $x^2 - k = z^2$, for rational numbers y and z. Since $k \neq 0$, we know that $y \neq z$. Without loss of generality, assume that $y > z$. Then take the sides of a triangle to be $y + z$, $y - z$, and $2x$. These sides are most certainly rational, and

$$(y+z)^2 + (y-z)^2 = 2y^2 + 2z^2 = 2x^2 + 2k + 2x^2 - 2k = 4x^2 = (2x)^2$$

so the triangle is a right triangle. The area of the triangle is

$$\frac{1}{2}(y+z)(y-z) = \frac{y^2 - z^2}{2} - \frac{x^2 + k - (x^2 - k)}{2} = k$$

so k satisfies the second definition of a congruent number.

($\Leftarrow$) Now assume that k is a natural number satisfying the second definition. So k is the area of a right triangle with rational sides. That is, there exists a triangle with sides a, b, and c, such that a, b, and c are rational, $a^2 + b^2 = c^2$, and $k = \frac{1}{2}ab$. Note that a cannot equal b. If $a = b$, then $a^2 + b^2 = c^2$ becomes $2a^2 = c^2$, and $2 = \frac{c^2}{a^2}$ leads to $\sqrt{2} = \frac{c}{a}$, a contradiction. Now, without loss of generality, assume $a > b$. Then

$$\left(\frac{c}{2}\right)^2 - k = \frac{c^2}{4} - \frac{1}{2}ab = \frac{a^2 + b^2 - 2ab}{4} = \left(\frac{a-b}{2}\right)^2,$$

and

$$\left(\frac{c}{2}\right)^2 + k = \frac{c^2}{4} + \frac{1}{2}ab = \frac{a^2 + b^2 + 2ab}{4} = \left(\frac{a+b}{2}\right)^2$$

.

So $x^2 + k$ and $x^2 - k$ are both squares of rational numbers, for $x = \frac{c}{2}$. Thus k satisfies the first definition of a congruent number, and our proof is complete.

Since a diametral number is obtained by multiplying the lengths of the legs of a right triangle, and congruent numbers correspond to the areas of right triangles, we see that we can divide diametral numbers by 2 to obtain congruent numbers, e.g. the diametral number 12 corresponds to the congruent number 6. Note that this process cannot always be reversed to obtain diametral numbers from congruent numbers, since the triangles corresponding to congruent numbers can have rational side lengths, but the triangles giving diametral numbers must have positive integer side lengths. For example, 5 is a congruent number, as noted above, but 10 is not a diametral number.

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