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ORIGINAL ARTICLE

On the Erdős-Sós Conjecture for $k = 9$

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Abstract

Let G be a graph with average degree greater than $k-2$. Erdős and Sós conjectured that G contains every tree on k vertices. The conjecture is known to be true for values of k up to 8. In this paper, we prove that the Erdős and Sós conjecture holds for $k = 9$.

Mathematics Subject Classification (2020). 05C35

Keywords. Erdős and Sós Conjecture, Extremal graph theory, Embedding trees into graphs

1. Introduction

The average degree of a graph G, denoted $\bar{d}(G)$, is $2|E(G)|/|V(G)|$. Erdős and Gallai [5] proved that if $\bar{d}(G) > k-2$, then G contains a path on k vertices. Subsequently, Erdős and Sós made the following conjecture:

Erdős-Sós Conjecture. *If G is a graph with $\bar{d}(G) > k-2$, then G contains every tree on k vertices.*

Various special cases of the conjecture have been proven. Many place restrictions on the graph G. The cases where the graph G has a number of vertices $k, k+1$, or $k+2$, were proved by Zhou [13], Slater, Teo, and Yap [8], and Woźniak [11], respectively. The cases where G has a number of vertices $k+3$ or $k+4$ were proved by Tiner [10], and Yuan and Zhang [12], respectively. Eaton and Tiner [4] proved the conjecture holds if a longest path in the graph G has at most $k+3$ vertices. In as early as 2003, Simonovits [1] announced a proof of the Erdős-Sós Conjecture for all sufficiently large values of k (joint work with Ajtai, Komlós, and Szemerédi).

Other cases that have been proven place restrictions on the class of trees. Sidorenko [7] proved the conjecture holds for every tree with a vertex having at least $\lceil \frac{k}{2} \rceil - 1$ leaf-neighbors. Eaton and Tiner [3] proved the following improvement:

Theorem 1.1. *If G is a graph with $\bar{d}(G) > k-2$, then G contains every tree on k vertices having a vertex with at least $\lceil \frac{k}{2} \rceil - 2$ leaf neighbors.*

A *spider* is a tree with one vertex of degree at least 3, called the *center*, and all others with degree at most 2. Fan, Hong, and Liu [2] proved the following:

Theorem 1.2. *If G is a graph with $\bar{d}(G) > k-2$, then G contains every spider on k vertices.*

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The *diameter* of a tree T, or $diam(T)$, is the number of edges on a longest path in T. McLennan [6] proved the following:

Theorem 1.3. *If G is a graph with $\bar{d}(G) > k-2$, then G contains every tree on k vertices that has diameter at most 4.*

A *double-broom* is a tree that contains a path $a_1, \ldots, a_r$, where each vertex not on the path is adjacent to either the vertex a_1 or a_r. Notice that a path is a double-broom. Tiner [9] proved the following:

Theorem 1.4. *If G is a graph with $\bar{d}(G) > k-2$, then G contains every double-broom on k vertices.*

Let G be a graph. For $A \subseteq V(G)$, the number of edges with at least one endpoint in A is $e^*_G(A)$, or simply $e^*(A)$. A proof of the following lemma is in [3]:

Lemma 1.5. *Let G be a graph with $\bar{d}(G) > k-2$. Let $W \subseteq V(G)$ and $G' = G - W$. If $e^*(W) \leq \frac{1}{2}(k-2)|W|$, then $\bar{d}(G') > k-2$.*

The minimum degree among all vertices in G is $\delta(G)$. For a natural number m, a graph G is *minimal with* $\bar{d}(G) > m$ if $\bar{d}(G') \leq m$ whenever G' is a proper subgraph of G. The following corollary follows from Lemma 1.5:

Corollary 1.6. *Let G be a graph that is minimal with $\bar{d}(G) > k-2$. If $W \subseteq V(G)$, then $e^*(W) > \frac{1}{2}|W|(k-2)$, and $\delta(G) \geq \lfloor \frac{k}{2} \rfloor$. Furthermore, for odd k, if $uv \in E(G)$, then either u or v has degree at least $\lfloor \frac{k}{2} \rfloor + 1$.*

If $ab \in E(G)$, then the vertex a *hits* b; otherwise, a *misses* b. Let $C, D \subseteq V(G)$. A vertex v *hits* C if there is a vertex c in the set C such that $vc \in E(G)$. The set D *hits* C if a vertex $d \in D$ hits C.

Eaton and Tiner [3] showed that the Erdős-Sós Conjecture holds for values of k at most 8. In this paper, we prove that the conjecture holds for $k = 9$.

For $k = 9$, the graph G in Corollary 1.6 has $\bar{d}(G) > 7$. This implies that $\delta(G) \geq 4$. Furthermore, for $u, v \in V(G)$, if the vertex u hits v, then either u or v has degree at least 5.

2. Proof of the main theorem

Theorem 2.1. *If G is a graph with $\bar{d}(G) > 7$, then G contains every tree on 9 vertices.*

Proof. If a subgraph G' of G that is minimal with $\bar{d}(G') > 7$ contains every tree on 9 vertices, then so does G. For this reason, we will simply assume that the graph G is minimal with $\bar{d}(G) > 7$. By Corollary 1.6, this implies that $\delta(G) \geq 4$, and if $uv \in E(G)$, then either u or v has degree at least 5.

Let T be a tree on 9 vertices, and notice that the diameter of T is at least 2 and at most 8. If $\text{diam}(T) \leq 4$, then the graph G contains T (by Theorem **??**).

Otherwise, $5 \leq \text{diam}(T) \leq 8$. If T has diameter 8, then T is a double-broom (more specifically a path), and G contains T (by Theorem 1.4). If T has diameter 7, then T is a spider, and G contains T (by Theorem 1.2).

Otherwise, $5 \leq \text{diam}(T) \leq 6$. We leave it to the readers to convince themselves that there are exactly eight trees of diameter 5, and exactly five trees of diameter 6, that are not already known to be in the graph G using Theorems 1.1 through 1.4. We label the 13 trees T_1 through T_{13}, where trees T_1 through T_8 have diameter 5, and trees T_9 through T_{13} have diameter 6. We will prove that each one of the trees T_1 through T_{13} is contained in G as a subgraph.

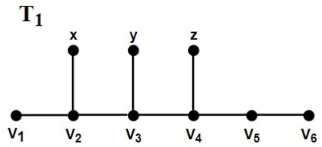
There are two main cases in our proof: Case 1 deals with trees of diameter 5, and Case 2 deals with trees of diameter 6. In the eight subcases of Case 1 (i.e., Cases 1.1 through

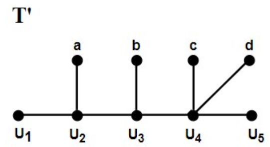
1.8), we prove that G contains the eight trees of diameter 5; and in the five subcases of Case 2 (i.e., Cases 2.1 through 2.5), we prove that G contains the five trees of diameter 6. In the title of each subcase, we define the tree T_i that will be proven to be in the graph G. Below each title is an image of the tree. In each proof, we state a tree T' that is already known to be in the graph G. We will then use T' and properties of the graph to prove that G contains T_i.

Case 2.2. Trees of diameter 5.

Let P be a path, where $P = v_1, v_2, v_3, v_4, v_5, v_6$. Since each of the trees T_1 through T_8 in this case has diameter 5, we will use the path P in the definition of each tree. The remaining three vertices used in each tree definition will be x, y, and z, and the three remaining edges will be stated in each case.

Case 2.2.1. $T_1 = P + \{v_2x, v_3y, v_4z\}$





Let $T' \subseteq G$ be the tree $u_1, u_2, u_3, u_4, u_5 + \{u_2a, u_3b, u_4c, u_4d\}$ in G. We know that T' is a subgraph of G since $\text{diam}(T') = 4$ (by Theorem **??**). Let $X = V(G) - V(T')$.

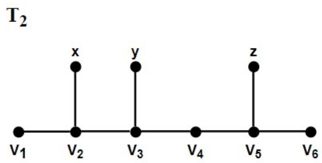
If two of the vertices in $\{u_5, c, d\}$ share an edge, then assume the vertex u_5 hits d. It follows that $u_1, u_2, u_3, u_4, u_5, d + \{u_2a, u_3b, u_4c\}$ is T_1 in G.

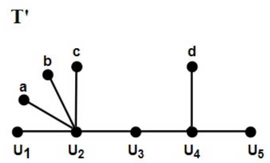
Otherwise, no two vertices in $\{u_5, c, d\}$ share an edge. If $\{u_5, c, d\}$ hits X, then assume the vertex u_5 hits $x \in X$. It follows that $u_1, u_2, u_3, u_4, u_5, x + \{u_2a, u_3b, u_4c\}$ is T_1 in G.

Otherwise, $\{u_5, c, d\}$ misses X. If $\{u_5, c, d\}$ hits $\{u_1, a\}$, then assume the vertex d hits u_1. Thus, $u_5, u_4, u_3, u_2, u_1, d + \{u_4c, u_3b, u_2a\}$ is T_1 in G.

Otherwise, $\{u_5, c, d\}$ misses $\{u_1, a\}$. Since $\delta(G) \geq 4$, this implies that the neighborhood of each vertex in $\{u_5, c, d\}$ is $\{u_2, u_3, u_4, b\}$, and $u_1, u_2, u_3, u_4, u_5, b + \{u_2a, u_3c, u_4d\}$ is T_1 in G.

Case 2.2.2. $T_2 = P + \{v_2x, v_3y, v_5z\}$





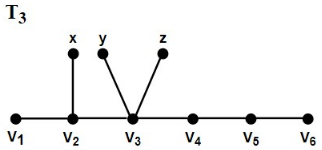
Let $T' \subseteq G$ be the tree $u_1, u_2, u_3, u_4, u_5 + \{u_2a, u_2b, u_2c, u_4d\}$ in G. We know that T' is a subgraph of G since $\text{diam}(T') = 4$ (by Theorem **??**). Let $X = V(G) - V(T')$.

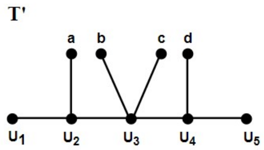
If a vertex in $\{u_1, a, b, c\}$ hits at least two vertices in $X \cup \{u_1, a, b, c\}$, then assume the vertex u_1 hits both a and b. It follows that $a, u_1, u_2, u_3, u_4, u_5 + \{u_1b, u_2c, u_4d\}$ is T_2 in G.

Otherwise, no vertex in $\{u_1, a, b, c\}$ hits two vertices in $X \cup \{u_1, a, b, c\}$. Since $\delta(G) \geq 4$, this implies that each vertex in $\{u_1, a, b, c\}$ hits at least two vertices in $\{u_3, u_4, u_5, d\}$. If two vertices in $\{u_1, a, b, c\}$ hit the vertex u_3, then assume a and b hit u_3. It follows that $u_5, u_4, u_3, a, u_2, u_1 + \{u_4d, u_3b, u_2c\}$ is T_2 in G.

Otherwise, at most one vertex in $\{u_1, a, b, c\}$ hits u_3. Assume $\{u_1, b, c\}$ misses u_3. Thus, each vertex in $\{u_1, b, c\}$ hits at least two vertices in $\{u_4, u_5, d\}$. This implies that either two vertices in $\{u_1, b, c\}$ hit u_5 or two vertices in $\{u_1, b, c\}$ hit d; assume that vertices b and c both hit u_5. It follows that $c, u_5, u_4, u_3, u_2, u_1, +\{u_5b, u_4d, u_2a\}$ is T_2 in G.

Case 2.2.3. $T_3 = P + \{v_2x, v_3y, v_3z\}$





Let $T' \subseteq G$ be the tree $u_1, u_2, u_3, u_4, u_5 + \{u_2a, u_3b, u_3c, u_4d\}$ in G. We know that T' is a subgraph of G since $\text{diam}(T') = 4$ (by Theorem **??**). Let $X = V(G) - V(T')$.

If $\{u_1, a, u_5, d\}$ hits X, then assume the vertex u_5 hits $x \in X$. It follows that $u_1, u_2, u_3, u_4, u_5, x+ \{u_2a, u_3b, u_3c\}$ is T_3 in G.

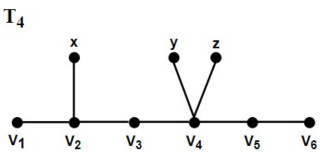
Otherwise, $\{u_1, a, u_5, d\}$ misses X. If u_1 hits a, or if u_5 hits d, then assume u_5 hits d. It follows that $u_1, u_2, u_3, u_4, u_5, d + \{u_2a, u_3b, u_3c\}$ is T_3 in G.

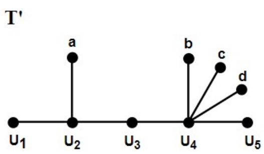
Otherwise, u_1 misses a, and u_5 misses d. If $\{u_1, a, u_5, d\}$ hits the vertex u_3, then assume the vertex d hits u_3. It follows that $u_1, u_2, u_3, d, u_4, u_5 + \{u_2a, u_3b, u_3c\}$ is T_3 in G.

Otherwise, $\{u_1, a, u_5, d\}$ misses the vertex u_3. If $\{u_1, a\}$ hits the vertex u_4, or if $\{u_5, d\}$ hits the vertex u_2, then assume that the vertex u_5 hits u_2. It follows that $c, u_3, u_2, u_5, u_4, d+ \{u_3b, u_2a, u_2u_1\}$ is T_3 in G.

Otherwise, $\{u_1, a\}$ misses the vertex u_4, and $\{u_5, d\}$ misses the vertex u_2. This implies that $N(u_1), N(a) \subseteq \{u_2, b, c, d, u_5\}$, and $N(u_5), N(d) \subseteq \{u_4, b, c, u_1, a\}$. Since $\delta(G) \geq 4$, we see that the vertex u_1 hits either d or u_5; assume that u_1 hits u_5. Since either vertex u_1 or u_5 has degree at least 5 (by Corollary 1.6), assume that $d(u_1) \geq 5$. This implies that $N(u_1) = \{u_2, b, c, d, u_5\}$, and $u_3, u_2, u_1, u_5, u_4, d + \{u_2a, u_1b, u_1c\}$ is T_3 in G.

Case 2.2.4. $T_4 = P + \{v_2x, v_4y, v_4z\}$





Let $T' \subseteq G$ be the tree $u_1, u_2, u_3, u_4, u_5 + \{u_2a, u_4b, u_4c, u_4d\}$ in G. We know that T' is a subgraph of G since $\text{diam}(T') = 4$ (by Theorem **??**). Let $X = V(G) - V(T')$.

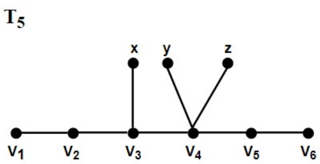
If two vertices in $\{u_5, b, c, d\}$ share an edge, then assume u_5 hits d. It follows that $u_1, u_2, u_3, u_4, u_5, d + \{u_2a, u_4b, u_4c\}$ is T_4 in G.

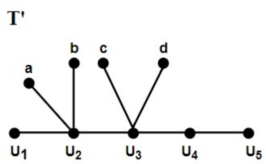
Otherwise no two vertices in $\{u_5, b, c, d\}$ share an edge. If $\{u_5, b, c, d\}$ hits X, then assume the vertex u_5 hits $x \in X$. It follows that $u_1, u_2, u_3, u_4, u_5, x + \{u_2a, u_4b, u_4c\}$ is T_4 in G.

Otherwise, $\{u_5, b, c, d\}$ misses X. Since $\delta(G) \geq 4$, this implies that each vertex in $\{u_5, b, c, d\}$ hits at least four vertices in $\{u_1, u_2, u_3, u_4, a\}$. If $\{u_5, b, c, d\}$ hits the vertex u_2, then assume that the vertex b hits u_2. Since the vertex d hits at least one of the vertices in $\{u_1, a\}$, assume d hits u_1. It follows that $u_5, u_4, u_3, u_2, u_1, d + \{u_4c, u_2a, u_2b\}$ is T_4 in G.

Otherwise, no vertex in $\{u_5, b, c, d\}$ hits u_2. This implies that the neighborhood of each vertex in $\{u_5, b, c, d\}$ is $\{u_1, u_3, u_4, a\}$, and $u_3, u_4, u_5, u_1, u_2, a + \{u_4b, u_1c, u_1d\}$ is T_4 in G.

Case 2.2.5. $T_5 = P + \{v_3x, v_4y, v_4z\}$





Let $T' \subseteq G$ be the tree $u_1, u_2, u_3, u_4, u_5 + \{u_2a, u_2b, u_3c, u_3d\}$ in G. We know that T' is a subgraph of G since $\mathrm{diam}(T') = 4$ (by Theorem **??**). Let $X = V(G) - V(T')$.

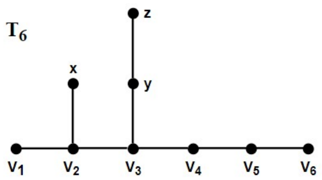
If two of the vertices in $\{u_1, a, b\}$ share an edge, then assume the vertex u_1 hits a. It follows that $a, u_1, u_2, u_3, u_4, u_5 + \{u_2b, u_3c, u_3d\}$ is T_5 in G.

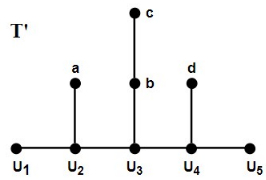
Otherwise, no two vertices in $\{u_1, a, b\}$ share an edge. If $\{u_1, a, b\}$ hits X, then assume the vertex u_1 hits $x \in X$. It follows that $x, u_1, u_2, u_3, u_4, u_5 + \{u_2b, u_3c, u_3d\}$ is T_5 in G.

Otherwise, $\{u_1, a, b\}$ misses X. If $\{u_1, a, b\}$ hits $\{c, d\}$, then assume that u_1 hits c. It follows that $u_5, u_4, u_3, u_2, u_1, c + \{u_3d, u_2a, u_2b\}$ is T_5 in G.

Otherwise, $\{u_1, a, b\}$ misses $\{c, d\}$. Since $\delta(G) \geq 4$, this implies that the neighborhood of each vertex in $\{u_1, a, b\}$ is $\{u_2, u_3, u_4, u_5\}$, and $u_5, u_1, u_2, u_3, u_4, b + \{u_2a, u_3c, u_3d\}$ is T_5 in G.

Case 2.2.6. $T_6 = P + \{v_2x, v_3y, yz\}$





Let $T' \subseteq G$ be the tree $u_1, u_2, u_3, u_4, u_5 + \{u_2a, u_3b, bc, u_4d\}$ in G. We know that T' is a subgraph of G since $\mathrm{diam}(T') = 4$ (by Theorem **??**). Let $X = V(G) - V(T')$.

If one of $\{u_1, a, u_5, d\}$ hits X, then assume u_5 hits $x \in X$. It follows that $u_1, u_2, u_3, u_4, u_5, x + \{u_2a, u_3b, bc\}$ is T_6 in G.

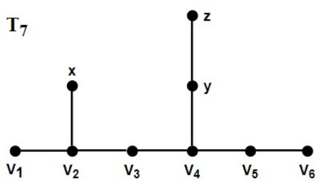
Otherwise, $\{u_1, a, u_5, d\}$ misses X. If the vertex u_1 hits a, or if u_5 hits d, then assume u_5 hits d. It follows that $u_1, u_2, u_3, u_4, u_5, d + \{u_2a, u_3b, bc\}$ is T_6 in G.

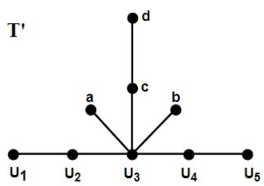
Otherwise, u_1 misses a, and u_5 misses d. If $\{u_1, a, d, u_5\}$ hits u_3, then assume the vertex d hits u_3. Thus, $u_1, u_2, u_3, d, u_4, u_5 + \{u_2a, u_3b, bc\}$ is T_6 in G.

Otherwise, $\{u_1, a, d, u_5\}$ misses u_3. If $\{u_1, a, d, u_5\}$ hits the vertex c, then assume that the vertex d hits c. It follows that $u_1, u_2, u_3, b, c, d + \{u_2a, u_3u_4, u_4u_5\}$ is T_6 in G.

Otherwise, $\{u_1, a, d, u_5\}$ misses c. This implies that $N(u_5), N(d) \subseteq \{u_1, u_2, u_4, a, b\}$, and $N(u_1), N(a) \subseteq \{u_2, u_4, u_5, b, d\}$. Since $\delta(G) \geq 4$, we see that the vertex u_5 hits $\{u_1, a\}$; assume that u_5 hits u_1. Since either vertex u_1 or u_5 has degree at least 5 (by Corollary 1.6), assume that $d(u_5) \geq 5$. It follows that $N(u_5) = \{u_1, u_2, u_4, a, b\}$, and $u_3, u_4, u_5, u_1, u_2, a + \{u_4d, u_5b, bc\}$ is T_6 in G.

Case 2.2.7. $T_7 = P + \{v_2x, v_4y, yz\}$





Let $T' \subseteq G$ be the tree $u_1, u_2, u_3, u_4, u_5 + \{u_3a, u_3b, u_3c, cd\}$ in G. We know that T' is a subgraph of G since $\mathrm{diam}(T') = 4$ (by Theorem **??**). Let $X = V(G) - V(T')$.

If the vertex u_1 hits u_3, then (u_1, u_2, u_3) is a 3-cycle in G. Since either vertex u_1 or u_2 has degree at least 5 (by Corollary 1.6), assume that $d(u_1) \geq 5$. Similarly, assume that each vertex in $\{u_5, d\}$ that hits u_3 has degree at least 5.

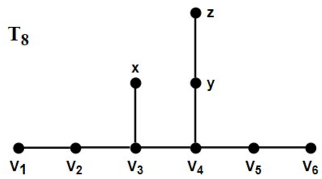
By the previous paragraph, we see that each vertex in $\{u_1, u_5, d\}$ hits at least four vertices in $G - u_3$. If a vertex in $\{u_1, u_5, d\}$ hits two vertices in $\{a, b\} \cup X$, then assume that u_1 hits both a and b. It follows that $a, u_1, u_2, u_3, u_4, u_5 + \{u_1b, u_3c, cd\}$ is T_7 in G.

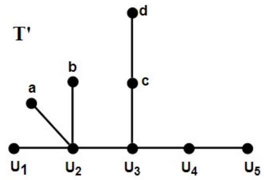
Otherwise, each vertex in $\{u_1, u_5, d\}$ hits at most one vertex in $\{a, b\} \cup X$. Since $\delta(G) \geq 4$, this implies that the vertex u_1 hits at least two vertices in $\{u_4, u_5, c, d\}$, the vertex u_5 hits at least two vertices in $\{u_1, u_2, c, d\}$, and the vertex d hits at least two vertices in $\{u_1, u_2, u_4, u_5\}$.

If the vertices in $\{u_1, u_5, d\}$ share at least two edges, then assume that the vertex u_1 hits both u_5 and d. It follows that $b, u_3, u_2, u_1, u_5, u_4 + \{u_3a, u_1d, dc\}$ is T_7 in G.

Otherwise, the vertices in $\{u_1, u_5, d\}$ share at most one edge. Thus, one of the vertices in $\{u_1, u_5, d\}$ misses the other two; assume that the vertex d misses $\{u_1, u_5\}$. It follows that the vertex d hits both u_2 and u_4, and $a, u_3, c, d, u_2, u_1 + \{u_3b, du_4, u_4u_5\}$ is T_7 in G.

Case 2.2.8. $T_8 = P + \{v_3x, v_4y, yz\}$





Let $T' \subseteq G$ be the tree $u_1, u_2, u_3, u_4, u_5 + \{u_2a, u_2b, u_3c, cd\}$ in G. We know that T' is a subgraph of G since $\mathrm{diam}(T') = 4$ (by Theorem **??**). Let $X = V(G) - V(T')$.

If two of the vertices in $\{u_1, a, b\}$ share an edge, then assume the vertex u_1 hits a. It follows that $a, u_1, u_2, u_3, u_4, u_5 + \{u_2b, u_3c, cd\}$ is T_8 in G.

Otherwise, no two vertices in $\{u_1, a, b\}$ share an edge. If $\{u_1, a, b\}$ hits X, then assume the vertex u_1 hits $x \in X$. It follows that $x, u_1, u_2, u_3, u_4, u_5 + \{u_2b, u_3c, cd\}$ is T_8 in G.

Otherwise, $\{u_1, a, b\}$ misses X. Since $\delta(G) \geq 4$, we see that each vertex in $\{u_1, a, b\}$ hits the vertex u_2 as well as at least three vertices in $\{u_3, u_4, u_5, c, d\}$. If a vertex in $\{u_1, a, b\}$ hits c and a different vertex in $\{u_1, a, b\}$ hits d, then assume that u_1 hits d, and b hits c. It follows that $d, u_1, u_2, u_3, u_4, u_5 + \{u_2a, u_3c, cb\}$ is T_8 in G.

Otherwise, if a vertex in $\{u_1, a, b\}$ hits c, then the other two vertices in $\{u_1, a, b\}$ miss d. If a vertex in $\{u_1, a, b\}$ hits u_4 and a different vertex in $\{u_1, a, b\}$ hits u_5, then assume that a hits u_4, and u_1 hits u_5. It follows that $u_5, u_1, u_2, u_3, u_4, a + \{u_2b, u_3c, cd\}$ is T_8 in G.

Otherwise, if a vertex in $\{u_1, a, b\}$ hits u_4, then the other two vertices in $\{u_1, a, b\}$ miss u_5. If a vertex in $\{u_1, a, b\}$ hits either both c and d, or both u_4 and u_5, then assume that the vertex b hits both c and d. This implies that $\{u_1, a\}$ misses $\{c, d\}$, and $N(u_1) = N(a) = \{u_2, u_3, u_4, u_5\}$, a contradiction (since the vertex a hits u_4, and the vertex u_1 hits u_5).

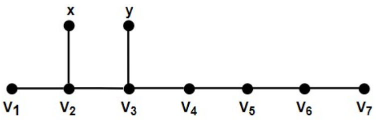
Otherwise, no vertex in $\{u_1, a, b\}$ hits either both c and d, or both u_4 and u_5. Since $\delta(G) \geq 4$, we see that each vertex in $\{u_1, a, b\}$ has precisely the same neighborhood. Specifically, all three vertices in $\{u_1, a, b\}$ hit both vertices u_2 and u_3, exactly one vertex in $\{c, d\}$, and exactly one vertex in $\{u_4, u_5\}$. Assume that each vertex in $\{u_1, a, b\}$ hits both c and u_5 (if each vertex in $\{u_1, a, b\}$ hits d instead of c, or u_4 instead of u_5, then the proof is similar). It follows that $N(u_1) = N(a) = N(b) = \{u_2, u_3, c, u_5\}$, and $u_3, u_1, u_2, b, u_5, u_4 + \{u_2a, bc, cd\}$ is T_8 in G.

Case 2.3. Trees of diameter 6.

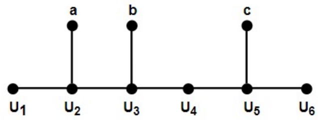
Let P be a path, where $P = v_1, v_2, v_3, v_4, v_5, v_6, v_7$. Since each of the trees T_9 through T_{13} in this case has diameter 6, we will use the path P in the definition of each tree. The remaining two vertices used in the tree definitions will be x and y, and the two remaining edges will be stated in each case.

Case 2.3.1. $T_9 = P + \{v_2x, v_3y\}$

T_9



T'



Let $T' \subseteq G$ be the tree $u_1, u_2, u_3, u_4, u_5, u_6 + \{u_2a, u_3b, u_5c\}$ in G. We know that T' is a subgraph of G because $T' = T_2$ (see Case 1.2.). Let $X = V(G) - V(T')$.

If $\{u_6, c\}$ hits X, then assume the vertex u_6 hits $x \in X$. Thus, u_1, u_2, u_3, $u_4, u_5, u_6, x + \{u_2a, u_3b\}$ is T_9 in G.

Otherwise, $\{u_6, c\}$ misses X. If the vertex u_6 hits c, then u_1, u_2, u_3, u_4, $u_5, u_6, c + \{u_2a, u_3b\}$ is T_9 in G.

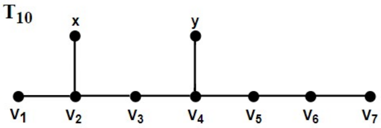
Otherwise, u_6 misses c. If $\{u_6, c\}$ hits u_4, then assume the vertex c hits u_4. It follows that $u_1, u_2, u_3, u_4, c, u_5, u_6 + \{u_2a, u_3b\}$ is T_9 in G.

Otherwise, $\{u_6, c\}$ misses u_4. If $\{u_6, c\}$ hits $\{u_1, a\}$, then assume the vertex c hits u_1. Thus, $u_4, u_3, u_2, u_1, c, u_5, u_6 + \{u_3b, u_2a\}$ is T_9 in G.

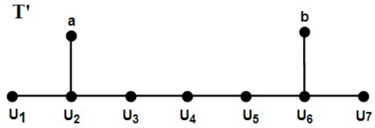
Otherwise, $\{u_6, c\}$ misses $\{u_1, a\}$. Since $\delta(G) \geq 4$, it follows that the neighborhood of each vertex u_6 and c is $\{u_2, u_3, b, u_5\}$, and u_1, u_2, u_3, u_4, u_5, $u_6, b + \{u_2a, u_3c\}$ is T_9 in G.

Case 2.3.2. $T_{10} = P + \{v_2x, v_4y\}$

T_{10}



T'



Let $T' \subseteq G$ be the tree $u_1, u_2, u_3, u_4, u_5, u_6, u_7 + \{u_2a, u_6b\}$ in G. We know that T' is a subgraph of G since T' is a double-broom (by Theorem 1.4). Let $X = V(G) - V(T')$.

If the vertex u_4 hits a vertex $x \in X$, then $u_1, u_2, u_3, u_4, u_5, u_6, u_7 + \{u_2a, u_4x\}$ is T_6 in G.

Otherwise, u_4 misses X. If the vertex u_4 hits $\{u_1, a, u_7, b\}$, then assume that u_4 hits b. Thus, $u_1, u_2, u_3, u_4, u_5, u_6, u_7 + \{u_2a, u_4b\}$ is T_6 in G.

Otherwise, u_4 misses $\{u_1, a, u_7, b\}$. Since $\delta(G) \geq 4$, we see that $N(u_4) = \{u_2, u_3, u_5, u_6\}$. If $\{u_1, a, u_7, b\}$ hits X, then assume that the vertex u_7 hits $x \in X$. It follows that $u_1, u_2, u_3, u_4, u_6, u_7, x + \{u_2a, u_4u_5\}$ is T_6 in G.

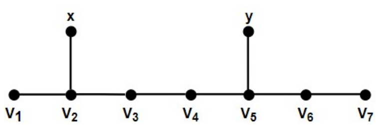
Otherwise, $\{u_1, a, u_7, b\}$ misses X. If the vertex u_1 hits a, or the vertex u_7 hits b, then assume that u_7 hits b. It follows that $u_1, u_2, u_3, u_4, u_6, u_7, b + \{u_2a, u_4u_5\}$ is T_6 in G.

Otherwise, the vertex u_1 misses a, and the vertex u_7 misses b. If $\{u_1, a\}$ hits u_6, or $\{u_7, b\}$ hits u_2, then assume that the vertex a hits u_6. It follows that $u_7, u_6, a, u_2, u_3, u_4, u_5 + \{u_6b, u_2u_1\}$ is T_6 in G.

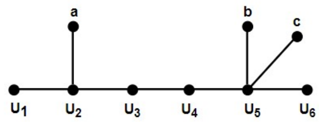
Otherwise, $\{u_1, a\}$ misses u_6, and $\{u_7, b\}$ misses u_2. Notice that $N(a)$, $N(u_1) \subseteq \{u_2, u_3, u_5, u_7, b\}$, and $N(b), N(u_7) \subseteq \{u_1, a, u_3, u_5, u_6\}$. Since $\delta(G) \geq 4$, we see that the vertex a hits either vertex u_7 or b; assume that a hits b. Since either vertex a or b has degree at least 5 (by Corollary 1.6), assume that $d(a) \geq 5$. It follows that $N(a) = \{u_2, u_3, u_5, u_7, b\}$, and $u_1, u_2, u_4, u_6, u_7, a, b + \{u_2u_3, u_6u_5\}$ is T_6 in G.

Case 2.3.3. $T_{11} = P + \{v_2x, v_5y\}$

T_{11}



T'



Let $T' \subseteq G$ be the tree $u_1, u_2, u_3, u_4, u_5, u_6 + \{u_2a, u_5b, u_5c\}$ in G. We know that T' is a subgraph of G since T' is a double-broom (by Theorem 1.4). Let $X = V(G) - V(T')$.

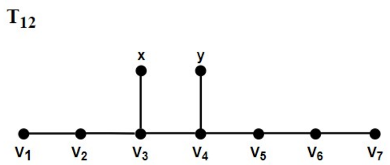
If two of the vertices in $\{u_6, b, c\}$ share an edge, then assume the vertex u_6 hits c. It follows that $u_1, u_2, u_3, u_4, u_5, u_6, c + \{u_2a, u_5b\}$ is T_{11} in G.

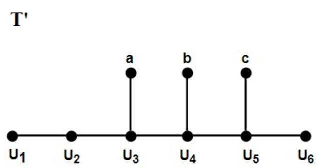
Otherwise, no two vertices in $\{u_6, b, c\}$ share an edge. If $\{u_6, b, c\}$ hits X, then assume the vertex u_6 hits $x \in X$. It follows that $u_1, u_2, u_3, u_4, u_5, u_6, x + \{u_2a, u_5b\}$ is T_{11} in G.

Otherwise, $\{u_6, b, c\}$ misses X. If $\{u_6, b, c\}$ hits $\{u_1, a\}$, then assume the vertex u_6 hits u_1. Thus, $b, u_5, u_4, u_3, u_2, u_1, u_6 + \{u_5c, u_2a\}$ is T_{11} in G.

Otherwise, $\{u_6, b, c\}$ misses $\{u_1, a\}$. Since $\delta(G) \geq 4$, we see that the neighborhood of each of the three vertices in $\{u_6, b, c\}$ is $\{u_2, u_3, u_4, u_5\}$, and $u_1, u_2, u_3, u_6, u_4, u_5, b + \{u_2a, u_4c\}$ is T_{11} in G.

Case 2.3.4. $T_{12} = P + \{v_3x, v_4y\}$





Let $T' \subseteq G$ be the tree $u_1, u_2, u_3, u_4, u_5, u_6 + \{u_3a, u_4b, u_5c\}$ in G. We know that T' is a subgraph of G because $T' = T_1$ (see Case 1.1.). Let $X = V(G) - V(T')$.

If $\{u_6, c\}$ hits X, then assume that the vertex u_6 hits $x \in X$. It follows that $u_1, u_2, u_3, u_4, u_5, u_6, x + \{u_3a, u_4b\}$ is T_{12} in G.

Otherwise, $\{u_6, c\}$ misses X. If the vertex u_6 hits c, then $u_1, u_2, u_3, u_4, u_5, u_6, c + \{u_3a, u_4b\}$ is T_{12} in G.

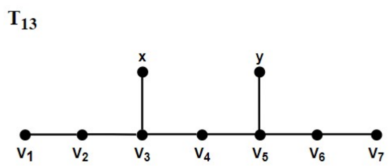
Otherwise, the vertex u_6 misses c. If $\{u_6, c\}$ hits u_1, then assume the vertex u_6 hits u_1. Thus, $c, u_5, u_4, u_3, u_2, u_1, u_6 + \{u_4b, u_3a\}$ is T_{12} in G.

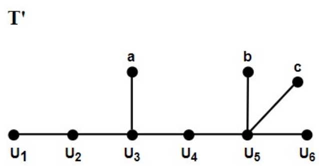
Otherwise, $\{u_6, c\}$ misses u_1. If $\{u_6, c\}$ hits u_4, then assume the vertex u_6 hits u_4. It follows that $u_1, u_2, u_3, u_4, u_6, u_5, c + \{u_3a, u_4b\}$ is T_{12} in G.

Otherwise, $\{u_6, c\}$ misses u_4. If $\{u_6, c\}$ hits a, then assume the vertex u_6 hits a. Thus, $a, u_6, u_5, u_4, u_3, u_2, u_1 + \{u_5c, u_4b\}$ is T_{12} in G.

Otherwise, $\{u_6, c\}$ misses a. Since $\delta(G) \geq 4$, we see that each vertex u_6 and c has neighborhood $\{u_2, u_3, b, u_5\}$, and $u_6, u_5, u_4, u_3, c, u_2, u_1 + \{u_4b, u_3a\}$ is T_{12} in G.

Case 2.3.5. $T_{13} = P + \{v_3x, v_5y\}$





Let $T' \subseteq G$ be the tree $u_1, u_2, u_3, u_4, u_5, u_6 + \{u_3a, u_5b, u_5c\}$ in G. We know that T' is a subgraph of G since T' has a vertex with three leaf neighbors (by Theorem 1.1). Let $X = V(G) - V(T')$.

If two of the vertices in $\{u_6, b, c\}$ share an edge, then assume the vertex u_6 hits c. It follows that $u_1, u_2, u_3, u_4, u_5, u_6, c + \{u_3a, u_5b\}$ is T_{13} in G.

Otherwise, no two vertices in $\{u_6, b, c\}$ share an edge. If $\{u_6, b, c\}$ hits X, then assume the vertex u_6 hits $x \in X$. It follows that $u_1, u_2, u_3, u_4, u_5, u_6, x + \{u_3a, u_5b\}$ is T_{13} in G.

Otherwise, $\{u_6, b, c\}$ misses X. If a vertex in $\{u_6, b, c\}$ hits the vertex u_3, and a different vertex in $\{u_6, b, c\}$ hits the vertex a, then assume that b hits u_3, and u_6 hits a. It follows that $u_1, u_2, u_3, u_4, u_5, u_6, a + \{u_3b, u_5c\}$ is T_{13} in G.

Otherwise, if a vertex in $\{u_6, b, c\}$ hits the vertex u_3, then the other two vertices in $\{u_6, b, c\}$ both miss the vertex a. If a vertex in $\{u_6, b, c\}$ hits the vertex u_2, and a different vertex in $\{u_6, b, c\}$ hits the vertex u_1, then assume that b hits u_2 and u_6 hits u_1. It follows that $b, u_2, u_3, u_4, u_5, u_6, u_1 + \{u_3a, u_5c\}$ is T_{13} in G.

Otherwise, if a vertex in $\{u_6, b, c\}$ hits the vertex u_2, then the other two vertices in $\{u_6, b, c\}$ both miss the vertex u_1. If a vertex in $\{u_6, b, c\}$ hits both u_1 and u_2, then assume that the vertex c hits both u_1 and u_2. It follows that $\{b, u_6\}$ misses $\{u_1, u_2\}$. Since $\delta(G) \geq 4$, this implies that $N(b) = N(u_6) = \{u_3, u_4, u_5, a\}$, a contradiction (since the vertex b hits u_3, and the vertex u_6 hits a).

Otherwise, no vertex in $\{u_6, b, c\}$ hits both u_1 and u_2. If a vertex in $\{u_6, b, c\}$ hits both u_3 and a, then assume that the vertex c hits both u_3 and a. It follows that $\{b, u_6\}$ misses $\{u_3, a\}$. This implies that $N(b) = N(u_6) = \{u_1, u_2, u_4, u_5\}$, a contradiction (since the vertex b hits u_2, and the vertex u_6 hits u_1).

Otherwise, no vertex in $\{u_6, b, c\}$ hits both u_3 and a. Since $\delta(G) \geq 4$, this implies that the three vertices in $\{u_6, b, c\}$ have precisely the same neighborhoods. Specifically, each vertex in $\{u_6, b, c\}$ hits both u_4 and u_5, as well as exactly one vertex from $\{u_3, a\}$, and exactly one vertex from $\{u_1, u_2\}$.

If each vertex in $\{u_6, b, c\}$ hits the vertex u_1, then $u_5, b, u_1, u_2, u_3, u_4, u_6 + \{u_1c, u_3a\}$ is T_{13} in G. Otherwise, each vertex in $\{u_6, b, c\}$ misses u_1. This implies that each vertex in $\{u_6, b, c\}$ hits u_2, and $a, u_3, u_2, c, u_5, u_6, u_4 + \{u_2u_1, u_5b\}$ is T_{13} in G. □

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