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ORIGINAL ARTICLE

A Cantor-like construction for the subsum set of an infinite series

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Abstract

The subsum set of a series $\sum_{n=1}^{\infty} a_n$ is the set of all numbers x such that $x = \sum_{n=1}^{\infty} c_n a_n$, where $\{c_n\}$ is a sequence consisting only of 0's and 1's. In this paper we describe a natural way to construct the subsum set of a series which is similar to the process used to construct the Cantor set. We then show how this construction leads to easy proofs of some known results, after which we present a few new results.

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1. Introduction

A paper by Johnson and Malin (P. Johnson and C. Malin, (2017)) defined the subsum of a series of non-negative terms and proved some interesting results. Recently several papers have been published on this topic. The main goal of this article is to describe a natural way to construct the set of all subsums of a series, and to show how this construction leads to two standard results. We will then describe our own new results. Finally we will acquaint readers with some of the recent developments in the field.

Formally, a number x is a subsum of the series $\sum_{n=1}^{\infty} a_n$ if $x = \sum_{n=1}^{\infty} c_n a_n$, where $\{c_n\}$ is a sequence consisting only of 0's and 1's. The subsum set of a series is the set of all possible subsums of the series. For example, a subsum of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ has the form $\sum_{n=1}^{\infty} \frac{c_n}{2^n}$, where $c_n = 0$ or 1 for each n. Hence each subsum has the form $0.c_1c_2c_3\ldots$ in binary notation. Since every real number in $[0, 1]$ can be written in this form, and the sum of $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is 1, it follows that the subsum set is $[0, 1]$. The series $\sum_{n=1}^{\infty} \frac{2}{3^n}$ also has sum 1, and its subsums have the form $\sum_{n=1}^{\infty} \frac{2c_n}{3^n}$ where $c_n = 0$ or 1 for each n. These are all the numbers in the interval $[0, 1]$ which can be written in base 3 notation without a 1. In fact, it is the well-known Cantor set.

In the main result of this paper, we show that the subsum set of a convergent series of positive terms can be obtained by a construction similar to the standard construction of the Cantor set. Some old results then follow easily as corollaries.

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2. The Cantor-like Construction

The first results on subsum sets were published by Kakeya (S. Kakeya, (1914)) We state them here.

Theorem 2.1. *Kakeya's Results Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms and let S be its subsum set.*

(i) *If $a_n \leq \sum_{i=n+1}^{\infty} a_i$ for all but finitely many n, then S is a finite union of closed and bounded intervals.*

(ii) *Furthermore, if $\{a_n\}$ is a non-increasing sequence and S is a finite union of closed and bounded intervals, then $a_n \leq \sum_{i=n+1}^{\infty} a_i$ for all but finitely many n.*

(iii) *If $a_n > \sum_{i=n+1}^{\infty} a_i$ for all but finitely many n, then S is homeomorphic to the Cantor set.*

In the usual construction of the Cantor set, we begin with the closed interval and remove the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$, leaving the closed set $F_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. From each interval of F_1 we remove the open middle third, leaving, $F_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$. By continuing this process, we get a nested sequence $\{F_n\}$ of compact sets. The Cantor set is defined to be $\cap_{n=1}^{\infty} F_n$.

We could also think of the construction like this: note that each interval F_n has length $\frac{1}{3^n}$ and that $\sum_{i=n+1}^{\infty} \frac{2}{3^i} = \frac{1}{3^n}$. Start with $[0, 1]$ and create two closed intervals of length $\sum_{i=2}^{\infty} \frac{2}{3^i} = \frac{1}{3}$, of which one has left endpoint 0 and the other has right endpoint 1. Then $F_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ is the union of these two intervals. Suppose that F_{n-1} has been defined as the union of closed intervals. Then we define F_n to be the union of all intervals of the form $\left[x, x + \sum_{i=n+1}^{\infty} \frac{2}{3^i}\right]$ and $\left[y - \sum_{i=n+1}^{\infty} \frac{2}{3^i}, y\right]$ for each interval $[x, y]$ of F_n. Then $\cap_{n=1}^{\infty} F_n$ is the Cantor set, and we have seen that it is also the subsum set of the series $\sum_{n=1}^{\infty} \frac{2}{3^n}$.

We will now describe a similar construction applied to a general convergent series with positive terms, and will show that the resulting set is the subsum set of the series.

Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms with sum s, and let $R_n = \sum_{i=n+1}^{\infty} a_i$. Define $F_1 = [0, R_1] \cup [s - R_1, s] = [0, R_1] \cup [a_1, s]$. Suppose that F_{n-1} has been defined as the union of closed intervals of length R_{n-1}. Then we define F_n to be the union of all intervals $[x, x + R_n]$ and $[y - R_n, y]$ for each interval $[x, y]$ in F_{n-1}. That is, each interval $[x, y]$ of F_{n-1} splits into the intervals $[x, x + R_n]$ and $[y - R_n, y]$. (Note that both $[x, x + R_n]$ and $[y - R_n, y]$ are contained in $[x, y]$ because $R_n < R_{n-1}$ and $[x, y]$ has length R_{n-1}). Then $\{F_n\}$ is a nested sequence of closed compact sets. By a standard result (S. Krantz, (2005) p. 75), $\bigcap_{n=1}^{\infty} F_n$ is nonempty. We will show that $\bigcap_{n=1}^{\infty} F_n$ is the subsum set of the series $\sum_{n=1}^{\infty} a_n$.

Our first lemma gives a precise description of the sequence $\{F_n\}$.

Lemma 2.2. $F_n = \bigcup_{(\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n})} [\sum_{\mathbf{i=1}}^{\mathbf{n}} c_i a_i, \quad \sum_{i=1}^{n} c_i a_i + R_n]$, *where $c_i = 0$ or 1.*

Proof. The proof is by induction on n. Setting $n = 1$ gives $F_1 = [0, R_1] \cup [a_1, a_1 + R_1] = [0, R_1] \cup [a_1, s]$, so the result is true for $n = 1$. Suppose it is true for F_n. Then from the construction we have just described, it follows that

$$F_{n+1} = \bigcup_{(\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n})} \left(\left[\sum_{\mathbf{i=1}}^{\mathbf{n}} c_i a_i, \ \sum_{i=1}^{n} c_i a_i + R_{n+1} \right] \cup \left[\sum_{\mathbf{i=1}}^{\mathbf{n}} c_i a_i + R_n - R_{n+1}, \ \sum_{i=1}^{n} c_i a_i + R_n \right] \right)$$

$$= \bigcup_{(\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n})} \left[\sum_{i=1}^{n} c_i a_i + 0.a_{n+1}, \ \sum_{i=1}^{n} c_i a_i + R_{n+1} \right] \cup \left[\sum_{i=1}^{n} c_i a_i + a_{n+1}, \sum_{i=1}^{n} c_i a_i + a_{n+1} + R_{n+1} \right]$$

$$= \bigcup_{(c_1, c_2, \ldots, c_n, c_{n+1})} \left[\sum_{i=1}^{n+1} c_i a_i, \sum_{i=1}^{n+1} c_i a_i + R_{n+1} \right]$$

which establishes the result. □

Lemma 2.3. *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with positive terms. Then x is a subsum of the series if and only if there exists a sequence $\{c_n\}$ of 0's and 1's such that for every n, $\sum_{i=1}^{n} c_i a_i \le x \le \sum_{i=1}^{n} c_i a_i + R_n$. In that case $x = \sum_{i=1}^{\infty} c_i a_i$.*

Proof. Suppose x is a subsum of the series. Then there exists a sequence $\{c_n\}$ of 0's and 1's such that $x = \sum_{n=1}^{\infty} c_i a_i$. It follows that for every n,

$$\sum_{i=1}^{n} c_i a_i \le x = \sum_{i=1}^{n} c_i a_i + \sum_{i=n+1}^{\infty} c_i a_i \le \sum_{i=1}^{n} c_i a_i + \sum_{i=n+1}^{\infty} a_i = \sum_{i=1}^{n} c_i a_i + R_n.$$

Now suppose there exists a sequence $\{c_n\}$ of 0's and 1's such that for every n,

$$\sum_{i=1}^{n} c_i a_i \le x \le \sum_{i=1}^{n} c_i a_i + R_n.$$

Taking the limit as $n \to \infty$, we see that

$$\sum_{i=1}^{\infty} c_i a_i \le x \le \sum_{i=1}^{\infty} c_i a_i + \lim_{n\to\infty} R_n.$$

Since $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} R_n = 0$. Hence $x = \sum_{i=1}^{\infty} c_i a_i$, and so x is a subsum. □

Finally we are ready to prove our theorem.

Theorem 2.4. *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms. Then the subsum set $S = \bigcap_{n=1}^{\infty} F_n$.*

Proof. Lemma 2.3 says that for each $x \in S$ there is a sequence $\{c_n\}$ of 0's and 1's such that for every n,

$$\sum_{i=1}^{n} c_i a_i \le x \le \sum_{i=1}^{n} c_i a_i + R_n.$$

It follows from Lemma 2.2 that $x \in \bigcap_{n=1}^{\infty} F_n$.

Now suppose $x \in \bigcap_{n=1}^{\infty} F_n$. We will define a sequence $\{c_n\}$ with the property described in Lemma 2.3. Let s be the sum of the series $\sum_{n=1}^{\infty} a_n$. Then $x \in F_1 = [0, R_1] \cup [a_1, s]$. If $x < a_1$, define $c_1 = 0$; if $x \ge a_1$, define $c_1 = 1$. In either case, $c_1 a_1 \le x \le c_1 a_1 + R_1$. Suppose we have defined $c_1, c_2, \ldots, c_n$ with the desired property. Then $x \in [\sum_{i=1}^{n} c_i a_i, \sum_{i=1}^{n} c_i a_i + R_n]$. F_{n+1} contains both of the intervals $[\sum_{i=1}^{n} c_i a_i, \sum_{i=1}^{n} c_i a_i + R_{n+1}]$ and $[\sum_{i=1}^{n} c_i a_i + a_{n+1}, \sum_{i=1}^{n} c_i a_i + R_n]$, and hence one of these intervals contains x. If $x < \sum_{i=1}^{n} c_i a_i + a_{n+1}$, define $c_{n+1} = 0$; if $x \ge \sum_{i=1}^{n} c_i a_i + a_{n+1}$, define $c_{n+1} = 1$. In either case,

$$\sum_{i=1}^{n+1} c_i a_i \le x \le \sum_{i=1}^{n+1} c_i a_i + R_{n+1}.$$

Thus a sequence with the property of Lemma 2 exists, and hence $x \in S$. □

3. Proof of Kakeya's results

Throughout this section, let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms and let S be its subsum set.

Corollary 3.1. *If $a_n \le \sum_{i=n+1}^{\infty} a_i$ for all but finitely many n, then S is a finite union of closed and bounded intervals.*

Proof. Suppose that $a_n \leq R_n = \sum_{i=n+1}^{\infty} a_i$ for all $n \geq N$. We claim that $F_n = F_N$ for all $n \geq N$. Let $n \geq N$. By Lemma 1, $F_n = \bigcup_{(\mathbf{c_1,c_2,\ldots,c_n})} [\sum_{\mathbf{i=1}}^{\mathbf{n}} c_i a_i, \ \ \sum_{i=1}^{n} c_i a_i + R_n]$. We get F_{n+1} by splitting each interval $[\sum_{\mathbf{i=1}}^{\mathbf{n}} c_i a_i, \ \ \sum_{i=1}^{n} c_i a_i + R_n]$ into the intervals $[\sum_{\mathbf{i=1}}^{\mathbf{n}} c_i a_i, \ \ \sum_{i=1}^{n} c_i a_i + R_{n+1}]$ and $[\sum_{\mathbf{i=1}}^{\mathbf{n}} c_i a_i + a_{n+1}, \ \sum_{i=1}^{n} c_i a_i + R_n]$. Since $a_{n+1} \leq R_{n+1}$, $\sum_{i=1}^{n} c_i a_i + a_{n+1} \leq \sum_{i=1}^{n+1} c_i a_i + R_{n+1}$, and therefore the union of the two intervals is

$$\left[\sum_{\mathbf{i=1}}^{\mathbf{n}} c_i a_i, \ \ \sum_{i=1}^{n} c_i a_i + R_n\right].$$

It follows that $F_{n+1} = F_n$ for each $n \geq N$, and hence $F_n = F_N$ for all $n \geq N$. Since $S = \bigcap_{n=1}^{\infty} F_n$ and $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$, $S = F_N$, which is the union of 2^N closed and bounded intervals.

□

Corollary 3.2. *If $\{a_n\}$ is a non-increasing sequence and S is a finite union of closed and bounded intervals, then $a_n \leq \sum_{i=n+1}^{\infty} a_i$ for all but finitely many n.*

Proof. Let $\{a_n\}$ be a non-increasing sequence and let S be a finite union of closed and bounded intervals. Now suppose that $a_n > \sum_{i=n+1}^{\infty} a_i$ for infinitely many n. We will show that 0 does not belong to an interval contained in S. Since $0 \in S$, this contradicts the hypothesis that S is a finite union of closed and bounded intervals.

Let $x \in S, x > 0$. Since $\lim_{n\to\infty} a_n = 0$, there is m such that $0 < a_m < x$. Since $a_n > R_n$ for infinitely many n, we can choose m such that both $a_{\mathrm{m}} < x$ and $a_m > R_m$ are true. Then F_m contains the disjoint intervals $[0, R_m]$ and $[a_m, a_m + R_m]$. Since there is a gap between R_m and a_m, there is no interval contained in S which contains both 0 and x. This completes the proof. □

Corollary 3.3. *S is homeomorphic to the Cantor set if $a_n > \sum_{i=n+1}^{\infty} a_i$ for all but finitely many n.*

Proof. A subset of a metric space is homeomorphic to the Cantor set if it is compact, perfect, and totally disconnected. (J. Hocking and G. Young, (1988), p. 100). Since S is the intersection of a family of nested closed and bounded intervals, it is compact. (S. Krantz (2005), p. 100). Recall that a set is perfect if it is nonempty and closed, and if every point is a limit point. Let $x \in S$, and let $\epsilon > 0$. Since $\lim_{n\to\infty} R_n = 0$, there is n such that $R_n < \frac{\epsilon}{2}$. Hence x lies in an interval of F_n with length less than $\frac{\epsilon}{2}$. This interval must be contained in the interval $(x-\epsilon, x+\epsilon)$. Hence $(x-\epsilon, x+\epsilon)$ contains both the left endpoint $\sum_{i=1}^{n} c_i a_i$ and the right endpoint $\sum_{i=1}^{n} c_i a_i + R_n$ of the interval of F_n, and both endpoints are in S. Therefore $(x-\epsilon, x+\epsilon)$ contains a point different from x, and hence x is a limit point of S.

To show that S is totally disconnected, it is sufficient to show that S does not contain an interval. Suppose that $a_n > \sum_{i=n+1}^{\infty} a_i$ for all $n \geq N$. Consider the series $\sum_{i=N}^{\infty} a_i$, and let its sum be t. We will show that the subsum set T of the series $\sum_{i=N}^{\infty} a_i$ is totally disconnected. The subsum set S of the original series $\sum_{i=1}^{\infty} a_i$ is the union of a finite number of translates of T. Since a finite union of totally disconnected subsets of the real line is totally disconnected, it will follow that S is totally disconnected.

Define $F_N = [0, R_N] \cup [a_N, t]$, and define F_n for $n > N$ as in the Cantor-like construction described in Section 2. Since $R_N < a_N$, the two intervals are disjoint. By Lemma 2.2 a component of F_n has the form $[\sum_{i=N}^{n} c_i a_i, \sum_{i=N}^{n} c_i a_i + R_n]$.

In F_{n+1}, it splits into the intervals $[\sum_{i=N}^{n} c_i a_i, \sum_{i=N}^{n} c_i a_i + R_{n+1}]$ and

$$\left[\sum_{i=N}^{n} c_i a_i + a_{n+1}, \sum_{i=N}^{n} c_i a_i + R_n\right],$$

which are disjoint, since $R_{n+1} < a_{n+1}$. It follows by induction that for each $n \geq N$, F_n consists of 2^{n-N+1} disjoint intervals, each of length R_n.

By Theorem 2.4, the subsum set T of the series $\sum_{i=N}^{\infty} a_i$ is equal to $\bigcap_{n=N}^{\infty} F_n$. Let $x, y \in T$. Since $\lim_{n\to\infty} R_n = 0$, there is some $n > N$ such that $R_n < |x - y|$. Then x and y must belong to different intervals in F_n. Since different intervals are disjoint, there is some point z strictly between x and y such that $z \notin F_n$, and hence $z \notin T$. Therefore T is totally disconnected. As mentioned earlier, it follows that the same is true for S.

□

4. Series similar to geometric series

We state and prove a theorem for such series.

Theorem 4.1. *Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms with sum s.*

(a) *If $\frac{1}{2} \leq \frac{a_{n+1}}{a_n} < 1$ for all n, then the subsum set $S = [0, s]$.*

(b) *If $\frac{a_{n+1}}{a_n} < r$ for all n, where r is a constant such that $0 < r < \frac{1}{2}$, then S is homeomorphic to the Cantor set.*

Proof. Proof of (a): Suppose $\frac{1}{2} \leq \frac{a_{n+1}}{a_n} < 1$ for all n. For each $n \in N$, it follows by induction that $a_{n+i} \geq \frac{1}{2^i} a_n$ for all $i \geq 1$. Therefore,

$$R_n = \sum_{i=n+1}^{\infty} a_i \geq \sum_{i=1}^{\infty} \frac{1}{2^i} a_n = a_n.$$

Now the proof of Corollary 3.1 goes through to show that $S = [0, s]$.

Proof of (b): If $\frac{a_{n+1}}{a_n} < r$ for all n, then $a_{n+i} < r^i a_n$ for all $i \geq 1$, and

$$R_n = \sum_{i=n+1}^{\infty} a_i < \sum_{i=1}^{\infty} r^i a_n = \frac{r}{1-r} a_n < a_n,$$

since $0 < r < \frac{1}{2}$. By Corollary 3.1, S is homeomorphic to the Cantor set.

□

Corollary 4.2. *Consider the geometric series $\sum_{n=1}^{\infty} ar^n$. If $\frac{1}{2} \leq r < 1$, the subsum set of the series is $\left[0, \frac{ar}{1-r}\right]$. If $0 < r < \frac{1}{2}$, the subsum set is homeomorphic to the Cantor set.*

Proof. This follows directly from Theorem 4.1. □

5. Representations of a subsum

For the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ the subsum $\frac{1}{2}$ can also be expressed as $\sum_{n=2}^{\infty} \frac{1}{2^n}$. A theorem of Menon (P. Menon (1948)) gives a necessary condition for the expression of a subsum to be unique. We restate this result, and give a proof which is essentially the same as Menon's argument. We then state and prove a result which is almost the converse of Menon's theorem.

Theorem 5.1. *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with positive terms. If $a_n > R_n$ for all n, every subsum has a unique representation.*

Proof. Suppose there is a subsum x that has two different representations $\sum_{n=1}^{\infty} c_{\mathrm{n}}\, a_n$ and $\sum_{n=1}^{\infty} d_{\mathrm{n}}\, a_n$. Let m be the smallest integer such that $c_m \neq d_m$. Without loss of generality, $c_m = 0$ and $d_m = 1$. Then

$$0.a_m + \sum_{i=m+1}^{\infty} c_i a_i = a_m + \sum_{i=m+1}^{\infty} d_i a_i.$$

Hence $a_m \leq \sum_{i=m+1}^{\infty} c_i a_i \leq \sum_{i=m+1}^{\infty} a_i = R_m$, which contradicts the hypothesis that $a_n > R_n$ for all n. □

Theorem 5.2. *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with positive and non-increasing terms, and with sum s. If*

(a) $a_1 \le R_1$ *and the subsum* $x \in [0, R_1] \cap [a_1, s]$, *or*
(b) $a_m \le R_m$ *for some* $m \ge 2$ *and the subsum*

$$x \in \left[\sum_{i=1}^{m-1} c_i a_i, \sum_{i=1}^{m-1} c_i a_i + R_m\right] \cap \left[\sum_{i=1}^{m-1} c_i a_i + a_m, \sum_{i=1}^{m-1} c_i a_i + R_{m-1}\right],$$

then x *can be expressed in at least two different ways*

Proof. We will prove the more general case (b), and leave (a) as an exercise for the reader. Suppose that $a_m \le R_m$ for some $m \ge 2$ and the subsum

$$x \in \left[\sum_{i=1}^{m-1} c_i a_i, \sum_{i=1}^{m-1} c_i a_i + R_m\right] \cap \left[\sum_{i=1}^{m-1} c_i a_i + a_m, \sum_{i=1}^{m-1} c_i a_i + R_{m-1}\right].$$

Since $x \in \left[\sum_{i=1}^{m-1} c_i a_i, \sum_{i=1}^{m-1} c_i a_i + R_m\right]$,

$$x \in \left[\sum_{i=1}^{m-1} c_i a_i, \sum_{i=1}^{m-1} c_i a_i + R_{m+1}\right] \cup \left[\sum_{i=1}^{m-1} c_i a_i + a_{m+1}, \sum_{i=1}^{m-1} c_i a_i + R_m\right].$$

Let $c_m = 0$, and define c_n when $n > m$ as in the proof of Theorem 2.4. (If $x < \sum_{i=1}^{m-1} c_i a_i + a_{m+1}$, let $c_{m+1} = 0$; if $x \ge \sum_{i=1}^{m-1} c_i a_i + a_{m+1}$, let $c_{m+1} = 1$, and so on). Note that

$$\sum_{i=1}^{m-1} c_i a_i + a_m \le x \le \sum_{i=1}^{m-1} c_i a_i + R_m.$$

Since the sequence (a_n) is non-increasing,

$$\sum_{i=1}^{m-1} c_i a_i + a_{m+1} \le \sum_{i=1}^{m-1} c_i a_i + a_m \le x.$$

Therefore,

$$x \in \left[\sum_{i=1}^{m-1} c_i a_i + a_{m+1}, \sum_{i=1}^{m-1} c_i a_i + R_m\right],$$

and hence we let $c_{m+1} = 1$. We get a sequence (c_n) such that $x = \sum_{i=1}^{\infty} c_i a_i$.

But it is also true that $x \in \left[\sum_{i=1}^{m-1} c_i a_i + a_m, \sum_{i=1}^{m-1} c_i a_i + R_{m-1}\right]$, and hence

$$x \in \left[\sum_{i=1}^{m-1} c_i a_i + a_m, \sum_{i=1}^{m-1} c_i a_i + a_m + R_{m+1}\right] \cup \left[\sum_{i=1}^{m-1} c_i a_i + a_m + a_{m+1}, \sum_{i=1}^{m-1} c_i a_i + R_{m-1}\right].$$

Let $c_m = 1$, and again define c_n when $n > m$ as in the proof of Theorem 2.4. We see that

$$x \le \sum_{i=1}^{m-1} c_i a_i + R_m = \sum_{i=1}^{m-1} c_i a_i + a_{m+1} + R_{m+1} \le \sum_{i=1}^{m-1} c_i a_i + a_m + R_{m+1}.$$

Hence $x \in \left[\sum_{i=1}^{m-1} c_i a_i + a_m, \sum_{i=1}^{m-1} c_i a_i + a_m + R_{m+1}\right]$. If $x < \sum_{i=1}^{m-1} c_i a_i + a_m + a_{m+1}$, let $c_{m+1} = 0$; if $x \ge \sum_{i=1}^{m-1} c_i a_i + a_m + a_{m+1}$, let $c_{m+1} = 1$. In either case we get a representation of x with $c_m = 1$, which differs from the representation found earlier in which $c_m = 0$. Therefore there are at least two different representations for x.

□

Example. Consider the series $\frac{3}{4} + \frac{1}{4} + \frac{3}{4^2} + \frac{1}{4^2} + \dots$. With our notation, $a_{2n-1} = \frac{3}{4^n}, a_{2n} = \frac{1}{4^n}$, $R_{2n-1} = \frac{7}{3}\left(\frac{1}{4^n}\right)$ and $R_{2n} = \frac{4}{3}\left(\frac{1}{4^n}\right)$. Note that $a_{2n-1} > R_{2n-1}$ and $a_{2n} < R_{2n}$ for all n. It is easy to verify that $\frac{1}{3} = \sum_{i=1}^{n-1} a_{2i} + R_{2n}$ for all n, so $\frac{1}{3}$ has infinitely many different representations as a subsum.

6. More Results on Subsums

Kakeya's results tell us the nature of the subsum set when $a_n \leq R_n$ for all but finitely many n, and when $a_n > R_n$ for all but finitely many n. They do not tell us what happens if both $a_n \leq R_n$ and $a_n > R_n$ for infinitely many n. According to (A. Bartoszweicz, M. Filipczak. and E. Szymonik (2014)), Kakeya conjectured that the subsum set was always either a finite union of closed and bounded intervals or a Cantor set, but this was shown to be false by (A. Weinstein and B. Shapiro (1980)). Further work by Nyman, Guthrie and Saenz (J. Guthrie and J. Nyman (1988), J. Nyman and R. Saenz (1997), J. Nyman and R. Saenz (2000)) established the following result.

Theorem 6.1. *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms and let S be its subsum set. Then exactly one of the following is true.*

(a) *S is a finite union of closed and bounded intervals.*
(b) *S is homeomorphic to the Cantor set.*
(c) *S is homeomorphic to the subsum set of the series $\sum_{n=1}^{\infty} b_n$, where $b_{2n-1} = \frac{3}{4^n}$ and $b_{2n} = \frac{2}{4^n}$.*

For (c) to occur it is necessary but not sufficient that there be infinitely many n for which $a_n \leq R_n$, and also infinitely many n for which $a_n > R_n$.

Guthrie and Nyman in (J. Guthrie and J. Nyman (1988)) show that the subsum set of the series $\sum_{n=1}^{\infty} b_n$ described in (c) contains the interval $\left[\frac{3}{4}, 1\right]$, and hence has nonempty interior. It is an example of a type of set called a M-Cantorval. Here is the formal definition, as found in (A. Bartoszweicz., M. Filipczak and E. Szymonik (2014)).

Definition 6.2. A set S is a M-Cantorval if:

(a) S is a non-empty compact subset of the real line.
(b) S is equal to the closure of its interior.
(c) Both endpoints of any component with nonempty interior are accumulation points of one-point components of S.

Bartoszweicz, Filipczak and Szymonik describe a family of series whose subsum sets are Cantorvals. A **multigeometric** series has the form

$$k_1 + k_2 + \ldots + k_m + k_1 q + k_2 q + \ldots + k_m q + k_1 q^2 + k_2 q^2 + \ldots + k_m q^2 + \ldots .$$

Denote it by $(k_1, k_2, \ldots, k_m; q)$. Here is the result from ((A. Bartoszweicz., M. Filipczak and E. Szymonik (2014)).

Theorem 6.3. *Let $k_1 \geq k_2 \geq \ldots \geq k_m$ be positive integers, and let $K = \sum_{i=1}^{m} k_i$. Suppose that the set $\{\sum_{i=1}^{m} c_i k_i : c_i = 0 \text{ or } 1\}$ contains the numbers $n_0, n_0+1, \ldots, n_0+n$ for some positive integers n_0 and n. Then if $\frac{1}{n+1} \leq q < \frac{k_m}{K+k_m}$, the subsum set of the series $(k_1, k_2, \ldots, k_m; q)$ is a Cantorval.*

The paper (J. Ferdinands and T.Ferdinands (2019)) generalizes this result by showing that the conclusion holds when the set $\{\sum_{i=1}^{m} c_i k_i : c_i = 0 \text{ or } 1\}$ contains the terms of an arithmetic progression. Nitecki in (Z. Nitecki, (2013)) gives a very nice survey of the topic of subsums of a series.

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