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Original Article

On a new method to find the roots of a cubic polynomial

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Abstract

In the paper a new method is developed to find the roots of a cubic polynomial. This is done by introducing a quadratic polynomial that has a root in common with the given cubic polynomial. In contrast to the well-known method by Cardano, which involves two cube roots, our method needs only one cube root.

Keywords: Cubic polynomial, depressed cubic polynomial, quadratic polynomial, roots, coefficients.

Mathematics Subject Classification (MSC2010): 12E05, 12E12.

1. Introduction

After the roots of a quadratic polynomial had been found in 2000 BC, it took a long time to determine the roots of a cubic polynomial. In the sixteenth century, Scipione del Ferro (1465–1526), who taught mathematics at the University of Bologna, solved the cubic equation $x^3 + ax = b$ without a quadratic term. But del Ferro did not publish his solution; instead before his death he revealed it to his student Antonio Fior. A mathematical contest was arranged between Fior and Tartaglia, another mathematician from Italy. Fior was asked to solve the cubic equation $x^3 + ax^2 = b$ without linear term, which he was not able to do. Since Tartaglia knew the solutions for both types of cubic equations, Fior lost to Tartaglia (see [1]).

In 1539, Cardano, another Italian mathematician, persuaded Tartaglia to disclose his solution by promising him that Cardano would not publish them without Tartaglia's consent. However, later Cardano had access to del Ferro's solution. Because of this he felt no obligation to keep his promise, and in 1545 he published Tartaglia's solution in his book 'Ars Magna'. Cardano's solution of the depressed cubic equation $x^3 + ax + b = 0$ involved replacing the variable x by the sum $u + v$ of two new variables u and v and then establishing a relation $uv = -a/3$ between u and v. With these manipulations the following solution was obtained (see [2]):

$$x = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} + \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\,.$$

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Back then complex numbers were not known, and whenever square roots of negative numbers were encountered, Cardano termed them as bad (or absurd) numbers. In 1572, another Italian mathematician, Rafael Bombelli, introduced the concept of a complex number. By making use of the relation $v = -a/3u$ in Cardano's method, i.e., $x = u - (a/3u)$, Bombelli showed that exactly three solutions $x_i = u_i - (a/3u_i)$ $(i = 1, 2, 3)$ can be obtained instead of nine combinations of solutions, namely, three cube roots of u^3 and three cube roots of v^3, so that there are nine possibilities for $x = u + v$. Bombelli also showed that by using $v = -a/3u$, there is no need to evaluate both cube roots in Cardano's solution, but one suffices (see [2]).

In 1683, the German mathematician Tschirnhaus introduced a polynomial transformation to eliminate all intermediate terms in a given polynomial, and thereby transforming it into a binomial whose roots can be found easily (see [5]). By using a Tschirnhaus transformation, the roots of cubic and quartic polynomials were determined, while the roots of a quintic polynomial could not be found by the same methods (see [3], [5], and [6]). When the simplest form of the Tschirnhaus transformation $x = u - (a/3)$ is applied to a general cubic polynomial $x^3 + ax^2 + bx + c$, the quadratic term in the transformed polynomial is eliminated, and thus a depressed cubic polynomial $u^3 + Au + B$ is obtained. Hence, without loss of generality, it is enough to find the roots of a depressed cubic polynomial.

Lagrange (1771), an Italian born mathematician, who had settled in France, attempted to derive a universal method to find the roots of polynomials by means of corresponding resolvent polynomials. For a cubic polynomial, the resolvent is a quadratic polynomial, while for a quartic polynomial it is a cubic polynomial. However, for a quintic polynomial, the resolvent is a sextic polynomial which made it impossible to find the roots. Later, the Norwegian mathematician Abel (1826), and the French mathematician Galois (1832) proved that polynomial equations of degree higher than four cannot be solved in radicals (see [2] and [3]).

Lagrange was able to identify certain intermediate parameters relating the solutions and the coefficients of the given polynomial by using appropriate roots of unity. For the depressed cubic $x^3 + ax + b$ the intermediate parameters u and v can be eexpressed in terms of its root r_1, r_2, and r_3 as follows:

$$u = r_1 + wr_2 + w^2x_3\,,$$

$$v = r_1 + w^2r_2 + wr_3\,,$$

where $w = (-1+\sqrt{3}\,i)/2$ is a primitive cube root of unity. Using the identity $r_1+r_2+r_3 = 0$ along with the above two identities, Lagrange obtained the following expressions for the roots r_1, r_2, and r_3 in terms of u and v:

$$r_1 = \frac{1}{3}(u+v)\,, \quad r_2 = \frac{1}{3}(w^2u + wv)\,, \quad r_3 = \frac{1}{3}(wu + w^2v)\,.$$

By observing that the given cubic polynomial can be written in terms of its roots r_1, r_2, and r_3 as

$$x^3 + ax + b = x^3 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3\,,$$

and by comparing the coefficients, one obtains the following relations between the coefficients a and b and the intermediate parameters u and v:

$$uv = -3a\,, \quad u^3 + v^3 = -27b\,.$$

Eliminating one of the parameters (say v) from the latter yields Lagrange's quadratic resolvent as a polynomial in u^3:

$$u^6 + 27bu^3 - 27a^3 = 0\,.$$

From Lagrange's resolvent one can determine first u and then subsequently v. Finally, knowing u and v enables one to obtain the roots r_1, r_2, and r_3. Notice that by applying an appropriate change of variables, Lagrange's resolvent can be transformed into the quadratic polynomial encountered in Cardano's method (see [2] and [4]).

It is interesting to note that even though the quadratic polynomial introduced in (2) of the next section is not related to Lagrange's resolvent, the quadratic polynomial in (9) of Section 2 can be transformed into Lagrange's resolvent by a suitable change of variables. This is due to the fact that irrespective of the method chosen to find the roots of a cubic polynomial by using some intermediate parameters which relate the solutions to the coefficients, one encounters a quadratic polynomial in one of the intermediate parameters, which in fact is Lagrange's resolvent or a quadratic polynomial that can be obtained by a change of variables from Lagrange's polynomial. The roots of this polynomial must be determined before the roots of the given cubic polynomial can be found.

For example, in the paper [6], which uses the Tschirnhaus transformation, the roots of Lagrange's resolvent inherent in identity (7) of [6] need to be determined before theroots of the cubic polynomial can be found. In [7] Dickson transforms the given cubic equation $x^3+px+q=0$ into a quartic equation by multiplying it with a factor $x-c$, where c is a free parameter. Then he derives a quadratic equation $c^2p+3cq-(p^2/3)=0$ in the variable c, which by a proper change of variables can be transformed into Lagrange's resolvent (see [7, (3)]). Observe that c is an intermediate parameter which relates the solution $x=a-(c/2)$ to the coefficients p and q. Here a is another intermediate parameter (see (1) in [7]). Therefore, the quadratic equation in c must be solved before one can determine the solutions of the given cubic equation. Similarly, in [8], when solving the cubic equation $x^3+ax+b=0$ by Tschirnhaus' method, we encounter Lagrange's resolvent in the form of the polynomial $4ac^2-6bc-(a^2/3)$. In order to see how this is Lagrange's resolvent, we let $c=kz$ in $4ac^2-6bc-(a^2/3)=0$, and then we set $k=-1/18a$, which yields $z^2+27bz-27a^3=0$. The intermediate parameter c relates the solution $x=c+d$ to the coefficients a and b, where d is another intermediate parameter (see [8]).

The present paper develops a new method for determining the roots of a cubic polynomial. In this method, the given cubic polynomial shares a common root with a quadratic polynomial, where the latter has two coefficients as free parameters. So the common root satisfies a quadratic and a cubic equation. Elimination of the common root from these two equations results in an expression in the two unknown coefficients of the quadratic polynomial. Since we need one more equation to determine the two free parameters, another appropriate equation is introduced. This leads to the determination of the two free parameters and enables us to find the three roots of the given cubic polynomial. The method is explained in detail in the next section, and an example illustrating the method can be found in the last section.

In the method presented in this paper, we obtain a cubic binomial in the variable $d+(2a/3)$, where d is a free parameter in a quadratic polynomial (see (2) and (11) below). This is in a manner similar to Tschirnhaus' method, where a cubic binomial y^3+B in the variable y is obtained by applying a quadratic Tschirnhaus transformation $y=x^2+cx+d$ to the depressed cubic polynomial x^3+ax+b (see [5] and [6]). Unfortunately, Tschirnhaus' method yields six possible solutions of a given cubic equation, which is due to the fact that for each root of the polynomial $y^3+B=0$ there are two values for x (see the transformation given above). However among these six values of x, three are not roots of the given depressed cubic polynomial, and until very recently, there was no way other than by trial and error to identify the roots among the six values for x. In a recent paper [8], a given cubic equation is solved by applying a Tschirnhaus transformation, but without obtaining the extra solutions.

2. Main result

Consider a depressed cubic polynomial

$$p(x) = x^3 + ax + b\,, \tag{2.1}$$

where the coefficients a and b are complex numbers. In addition, we can assume that they satisfy $ab \neq 0$. Our aim is to determine the roots of $p(x)$. Let us consider a quadratic polynomial,

$$q(x) = x^2 - cx - d, \tag{2.2}$$

where the coefficients c and d are free parameters. Suppose that the cubic polynomial in (1) and the quadratic polynomial in (2) have a common root. Then such a common root r must satisfy the equations $p(r) = 0$ and $q(r) = 0$, i.e.,

$$r^3 + ar + b = 0\,, \tag{2.3}$$

$$r^2 - cr - d = 0\,. \tag{2.4}$$

By first, eliminating r^3 from (3) and (4), and then eliminating r^2 by using (4), we obtain that

$$(c^2 + d + a)r = -(cd + b)\,. \tag{2.5}$$

If $c^2 + d + a \neq 0$, we get an expression for r in terms of c and d:

$$r = -\frac{cd + b}{c^2 + d + a}\,. \tag{2.6}$$

The remaining case of $c^2 + d + a = 0$ is discussed in the next section.

We now eliminate r from (4) by using (6), and thus we obtain an equation for the two free parameters c and d:

$$d^3 + 2ad^2 + a^2d - bc^3 + ac^2d - 3bcd - abc - b^2 = 0\,. \tag{2.7}$$

In order to be able to determine c and d, we need one more equation. By rearranging (7)

$$\left(d + \frac{2a}{3}\right)^3 + \left(ac^2 - 3bc - \frac{a^2}{3}\right)d - bc^3 - abc - \left(b^2 + \frac{8a^3}{27}\right) = 0\,, \tag{2.8}$$

and by setting the coefficient of d equal to zero, we deduce that

$$ac^2 - 3bc - \frac{a^2}{3} = 0\,. \tag{2.9}$$

Observe that (9) is a quadratic equation in c. As by assumption $a \neq 0$, the solutions of (9) are

$$c = \frac{1}{2}\left(\frac{3b}{a} \pm \sqrt{\left(\frac{3b}{a}\right)^2 + \frac{4a}{3}}\right)\,. \tag{2.10}$$

Furthermore, it follows from (8) that $bc^3 + abc + [b^2 + (8a^3/27)]$ is a perfect cube, namely,

$$\left(d + \frac{2a}{3}\right)^3 = f^3\,, \tag{2.11}$$

where f^3 is given by

$$f^3 = bc^3 + abc + \left(b^2 + \frac{8a^3}{27}\right)\,. \tag{2.12}$$

Factoring (11) yields three values for $d + (2a/3)$,

$$d + \frac{2a}{3} = f,\quad fw,\quad fw^2\,,$$

where $w = (-1 + \sqrt{3}\, i)/2$ and the quantity f is the principal cube root of f^3. Recall that the principal cube root of a complex number is the cube root with the largest magnitude of the real part. Consequently, the three values of d are

$$d = f - \frac{2a}{3}, \;\; fw - \frac{2a}{3}, \;\; fw^2 - \frac{2a}{3}\,. \tag{2.13}$$

Observe that w and w^2 are the two primitive cube roots of unity.

Finally, when these three values for the free parameter d are used in (6), we obtain the three roots r_1, r_2, and r_3 of the given cubic polynomial in (1). Note that there is only one cube root needed to determine f from f^3. This is in contrast to Cardano's method that involves two cube roots.

3. The special case $c^2 + d + a = 0$

In this special case we can not use (6) to find the roots of cubic polynomial $p(x)$. So instead, we consider the identity (5). Since the left-hand-side of (5) is zero, we have that $cd + b = 0$. But by assumption $b \neq 0$, and thus we can use $d = -b/c$ in $c^2 + d + a = 0$ to obtain $c^3 + ac - b = 0$. This shows that $-c$ is one root of the given cubic polynomial $p(x)$. Consequently the term $x + c$ is a linear factor of $p(x)$, and by division we obtain the following quadratic factor:

$$x^3 + ax + b = (x + c)[x^2 - cx + (c^2 + a)]\,.$$

Notice that $c^2 + a = -d$ implies that

$$x^3 + ax + b = (x + c)(x^2 - cx - d) = (x + c)q(x)\,,$$

which means that in this case both roots of the quadratic polynomial $q(x)$ in (2) are roots of the cubic polynomial $p(x)$ in (1). Consequently, we obtain the following three roots of the given cubic polynomial:

$$r_1 = -c, \;\; r_{2,3} = \frac{1}{2}\left(c \pm \sqrt{-3c^2 - 4a}\right).$$

4. An example

We conclude the paper by an example to illustrate the proposed method. Consider the cubic polynomial

$$p(x) = x^3 - 9x - 12\,.$$

Suppose that $p(x)$ and a quadratic polynomial $q(x) = x^2 - cx - d$ have a common root r. From (10) we obtain two values for the free parameter c, namely, $c = 1$ and $c = 3$, and any of the two values can be chosen.

Let $c = 1$. Then we obtain from (12) that $f^3 = 24$, and therefore the principal cube root of f^3 is $2\sqrt[3]{3}$. Moreover, from (13) we get the following three values for the free parameter d:

$$d_1 = 6 + 2\sqrt[3]{3} \;\text{ and }\; d_{2,3} = (6 - \sqrt[3]{3}) \pm \sqrt{3}\sqrt[3]{3}i\,.$$

By using each of these values for d in (6) we obtain the following three roots of the given cubic polynomial:

$$r_1 = \sqrt[3]{3} + (\sqrt[3]{3})^2 \;\text{ and }\; r_{2,3} = -\frac{[3 + 2\sqrt[3]{3} + (\sqrt[3]{3})^2] \pm \sqrt{3}\sqrt[3]{3}i}{1 + \sqrt[3]{3} + (\sqrt[3]{3})^2}\,,$$

Note that in the above calculation the term $1 + \sqrt[3]{3} + (\sqrt[3]{3})^2$ can be expressed as

$$1 + \sqrt[3]{3} + (\sqrt[3]{3})^2 = \frac{1 - (\sqrt[3]{3})^3}{1 - \sqrt[3]{3}} = \frac{-2}{1 - \sqrt[3]{3}}\,.$$

Hence the above expressions for r_2 and r_3 can be simplified as follows:

$$r_{2,3} = -\frac{1}{2}\left[\sqrt[3]{3} + (\sqrt[3]{3})^2\right] \pm \frac{\sqrt{3}}{2}\left[\sqrt[3]{3} - (\sqrt[3]{3})^2\right] i\,.$$

When we use the other value $c = 3$, we obtain that $f^3 = -72$, and thus $f = -2(\sqrt[3]{3})^2$. By virtue of (13), we get that $d_1 = 6 - 2(\sqrt[3]{3})^2$, and therefore we deduce from (6) that one root is $r_1 = \sqrt[3]{3} + (\sqrt[3]{3})^2$. The determination of the remaining two roots is left as an exercise to the interested reader.

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