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ORIGINAL ARTICLE

Squared Rectangles and Elementary Proof of Uniqueness

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Abstract

We present an elementary proof of the uniqueness of algebraic solutions for each dissection of a rectangle into squares. The known proof converts the problem into an electrical network problem and uses Kirchhoff's Laws. We simply use linear algebra.

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1. Introduction

A *squared rectangle (or square)* is a tiling of a rectangle (or square) by smaller squares. People are mostly interested in the search for squared squares that are *perfect* meaning all square tiles are of different sizes. [2–6, 8].

Suppose we want to find a squared rectangle of 8 squares. The size or the aspect ratio of the rectangle is not specified. The ratio will be determined by the way the rectangle is dissected. One technique to find such tiling is to sketch a dissection and to solve horizontal and vertical compatibility equations that equate different ways of computing the lengths of common sides. As an example, we draw a dissection of a rectangle into rectangles, which are presumed to be squares, as in Figure 1 . Then assign a few variables for the lengths of squares and find the lengths of other squares in terms of those variables. Comparing total horizontal and total vertical lengths, we get $5x + 2y = x + 3y + z$ and $4x + 4y - z = 3x + 2y + z$. Solving them gives smallest integer solutions shown on the right in Figure 1. The solutions could be multiplied by a scaling factor to make a bigger or smaller rectangle.

A natural question to ask is whether the system of horizontal and vertical compatibility linear equations would suffice to produce a unique solution (up to a scaling factor). In fact, it is a theorem by Brooks, Smith, Stone, and Tutte [2] that they always give a unique solution algebraically.

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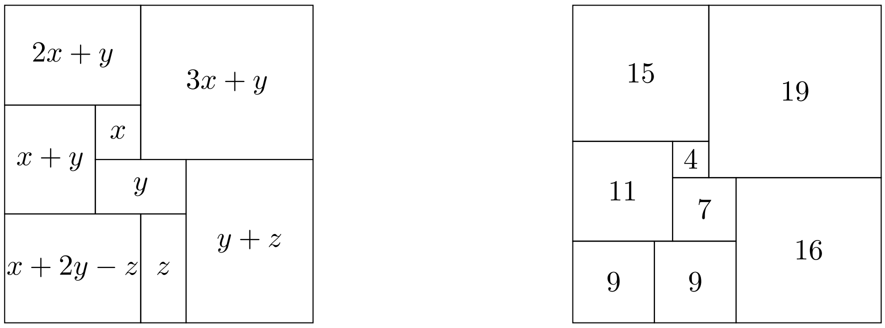
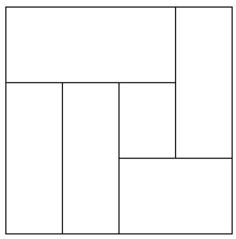
Figure 1. Finding a Squared Rectangle from a Dissection



Theorem 1.1 (Brooks, Smith, Stone, and Tutte [2]). *Any geometrically feasible dissection of a rectangle is completely determined up to a scaling factor by its vertical and horizontal compatibility equations.*

We remark that the requirement of a geometrically feasible dissection stems from the fact that the equations for some dissections, for example,



produce a negative solution, which implies that tiling by squares in such an arrangement is not possible.

The proof of the theorem depends on electrical network theory. The tiling is viewed as square plates of metal connected on edges with electricity flowing from top to bottom. The electrical flow in each square is proportional to the length of the square. And Kirchhoff's theorem, a theorem in graph theory, under the assumption of the law of conservation of current, which are realized by the compatibility equations, implies the flow in each square is uniquely determined.

In this paper we present an alternative proof of the theorem using elementary linear algebra rather than graph theory. Specifically, we prove that a particular collection of vertical and horizontal compatibility equations that are chosen systematically have a unique solution. The number of equations we choose will be the same as the number of squares.

2. Choosing Equations

We may assume the base of the rectangle is 1 and prove the uniqueness strictly without regarding scaling factors. Let n be the number of squares tiling the rectangle and for $1 \leq j \leq n$, let s_j be the j-th square tile and f_j its length. Make levels (horizontal strips) as in Figure 2 separated by horizontal edges of the tiles. The levels are ordered from top to bottom from level 1 to level m. We say a square s_j *overlaps* level i if the interior of s_j intersects level i. For example, s_3 overlaps levels 1 and 2 in Figure 2.

We make one equation for each level taking the total sum of the lengths of squares *overlapping* the level. We call these equations the *horizontal equations*. For example, in

Figure 2. A Dissection Example of a Rectangle

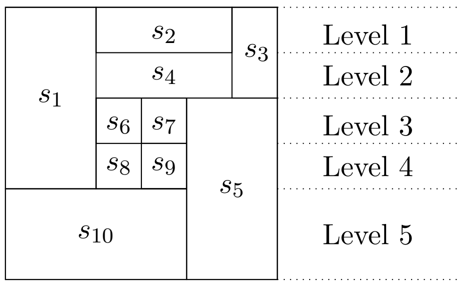


Figure 2, the horizontal equations are as follows.

$$\begin{aligned} f_1 + f_2 + f_3 &= 1 \quad \text{(Level 1)} \\ f_1 + f_4 + f_3 &= 1 \quad \text{(Level 2)} \\ f_1 + f_6 + f_7 + f_5 &= 1 \quad \text{(Level 3)} \\ f_1 + f_8 + f_9 + f_5 &= 1 \quad \text{(Level 4)} \\ f_{10} + f_5 &= 1 \quad \text{(Level 5)} \end{aligned} \tag{2.1}$$

For the vertical compatibility equations, first we categorize the squares into groups. For each $1 \le k \le m$, let G_k be the set of squares overlapping level k but not overlapping levels less than k. In other words, it is the set of squares whose upper edges are aligned with the upper boundary of level k. We call G_k the k-th *group.* We may assume the squares are sequenced in such a way that they are numbered in consecutive order along with the groups. For example, in Figure 2,

$$G_1 = \{s_1, s_2, s_3\}, \quad G_2 = \{s_4\}, \quad G_3 = \{s_5, s_6, s_7\}, \quad G_4 = \{s_8, s_9\}, \quad G_5 = \{s_{10}\}.$$

If two squares belong to the same group, then the vertical distances from the upper edges of the squares to the bottom side of the whole rectangle must be equal. To measure those distances, we follow the largest squares of the groups downward to the bottom. For example, in Figure 2, the distance from s_3 to the bottom could be expressed in many ways, but we pick $f_3 + f_5$ rather than $f_3 + f_7 + f_8 + f_{10}$ or any other expressions. Comparing the vertical distances for s_1, s_2, and s_3 in G_1, we get two equations $f_1 + f_{10} = f_2 + f_4 + f_5 = f_3 + f_5$. Likewise, for each group G_k, we get $|G_k| - 1$ equations. We call them the *vertical equations.*

$$\begin{aligned} f_1 + f_{10} = f_2 + f_4 + f_5 &= f_3 + f_5 && \text{(Group 1)} \\ f_5 = f_6 + f_8 + f_{10} &= f_7 + f_8 + f_{10} && \text{(Group 3)} \\ f_8 + f_{10} &= f_9 + f_{10} && \text{(Group 4)} \end{aligned}$$

The number of horizontal and vertical equations together is exactly the total number of squares. We will prove that the system of linear equations obtained in this way always has a unique solution.

3. Proving Uniqueness of Solutions

In each group, we pick a largest square. If there are multiple squares of the same size, we pick one of them once and for all. When we refer to the largest square of a group, we mean the chosen one in the group. Figure 2 will be used throughout this section as an example. In this example, we will choose s_1, s_4, s_5, s_8, and s_{10} for the largest squares. Let x_k be the length of the largest square in G_k. In the example,

$$x_1 = f_1, \quad x_2 = f_4, \quad x_3 = f_5, \quad x_4 = f_8, \quad x_5 = f_{10}.$$

We reserve the use of the index i for levels, j for squares, and k for groups. The following lemma implies the lengths of the largest squares completely determine the lengths of all squares.

Lemma 3.1. *Each f_j can be expressed as a linear combination of x_k, $1 \le k \le m$, by solving the vertical equations.*

Proof. The vertical equations have been constructed specifically for this purpose. The rigorous construction of the vertical equations and the proof of the lemma follows.

First we define a *chain* C_j of each square s_j, $1 \le j \le n$, recursively from the bottom to top as a set of squares to be used in the distance formula. If $s_j \in G_m$, that is, if the square is at the bottom of the rectangle, define $C_j = \{s_j\}$. Otherwise, the lower edge of s_j is aligned with the upper edge of one of the largest squares, say in group G_k. Define $C_j = \{s_j\} \cup C_{j'}$ where $C_{j'}$ is the chain of the largest square of the group G_k. In other words, to make each chain, after the beginning square we move down following the largest squares recursively to reach the bottom. Each chain consists of the largest squares of some groups in addition to the beginning square s_j, which may or may not be a largest square.

For example, the following are the chains for Figure 2 where the largest squares are underlined.

Group 1: $C_1 = \{\underline{s_1}, \underline{s_{10}}\}$, $\quad C_2 = \{s_2, \underline{s_4}, \underline{s_5}\}$, $\quad C_3 = \{s_3, \underline{s_5}\}$
Group 2: $C_4 = \{\underline{s_4}, \underline{s_5}\}$
Group 3: $C_5 = \{\underline{s_5}\}$, $\quad C_6 = \{s_6, \underline{s_8}, \underline{s_{10}}\}$, $\quad C_7 = \{s_7, \underline{s_8}, \underline{s_{10}}\}$
Group 4: $C_8 = \{\underline{s_8}, \underline{s_{10}}\}$, $\quad C_9 = \{s_9, \underline{s_{10}}\}$
Group 5: $C_{10} = \{\underline{s_{10}}\}$

The sum of the lengths of all squares in each chain C_j is the vertical distance d_j from the upper edge of the square s_j to the bottom of the whole rectangle.

$$d_j = \sum_{s_p \in C_j} f_p.$$

For the chains in the same group, the vertical distances are equal to each other. Let $s_{j'}$ be the largest square in a group. For any other square s_j in the same group,

$$d_j = \sum_{s_p \in C_j} f_p = \sum_{s_q \in C_{j'}} f_q = d_{j'}. \tag{3.1}$$

For example, in Group 1 above, we have the following equations comparing d_2 and d_3 to d_1.

$$f_2 + f_4 + f_5 = f_1 + f_{10}, \qquad f_3 + f_5 = f_1 + f_{10}.$$

Expressing these equations in terms of x_k's, we obtain

$$f_2 + x_1 + x_3 = x_1 + x_5, \qquad f_3 + x_3 = x_1 + x_5.$$

This can be done in general since C_j consists of the largest squares except for one, that is, s_j, and $C_{j'}$ consists of the largest squares only. We arrive at equations of the form

$$f_j + \sum x_u = \sum x_v \tag{3.2}$$

where the sums are over the groups which the largest squares in the chains belong to. Moving $\sum x_u$ to the other side, we express f_j as a linear combination of x_k's. □

Let $F = (f_1 \quad f_2 \quad \cdots \quad f_n)^\top$ and $X = (x_1 \quad x_2 \quad \cdots \quad x_m)^\top$. Then

$$F = BX \tag{3.3}$$

where B is the coefficient matrix from Lemma 3.1, which is an $n \times m$ matrix. Note that B is a matrix of 0, 1, and -1's. For example, Figure 2 gives the following equations and matrices.

$$\begin{aligned}
\text{Group 1:} \quad & f_1 = x_1 \\
& f_2 = x_1 - x_2 - x_3 + x_5 \\
& f_3 = x_1 - x_3 + x_5 \\
\text{Group 2:} \quad & f_4 = x_2 \\
\text{Group 3:} \quad & f_5 = x_3 \\
& f_6 = x_3 - x_4 - x_5 \\
& f_7 = x_3 - x_4 - x_5 \\
\text{Group 4:} \quad & f_8 = x_4 \\
& f_9 = x_4 \\
\text{Group 5:} \quad & f_{10} = x_5
\end{aligned}$$

$$F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \\ f_{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ \hdashline 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ \hdashline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = BX$$

The first part of the next lemma is obvious.

Lemma 3.2. *Each entry of B is one of the following values.*

$$\begin{cases} B_{jk} = 1 & \text{if } x_k \text{ belongs to only the right side of Equation (3.2).} \\ B_{jk} = -1 & \text{if } x_k \text{ belongs to only the left side of Equation (3.2).} \\ B_{jk} = 0 & \text{if } x_k \text{ belongs to both sides or neither side of Equation (3.2).} \end{cases}$$

The matrix B is group-wise upper triangular, that is, $B_{jk} = 0$ if s_j belongs to a group $> k$. If s_j belongs to group k, then $B_{jk} = 1$.

Proof. If s_j belongs to a group $> k$, the sums in Equation (3.2) do not involve x_k as the chains are constructed toward the bottom of the rectangle. Next, if s_j belongs to group k, then either s_j is the largest square, in which case, $f_j = x_k$, or the right side of Equation (3.2) begins with x_k and the left side has f_j and x_u with $u > k$. □

Next we compose the horizontal equations with the vertical equations. There is one horizontal equation for each level where the sum of the lengths of all squares overlapping the level equals 1. So the number of horizontal equations is m. Let E be the $m \times n$ coefficient matrix for those equations so that

$$EF = \mathbf{1} \tag{3.4}$$

where $\mathbf{1} = (1 \quad 1 \quad \cdots \quad 1)^\top$. For example, Equations (2.1) give the following matrix.

$$E = \left[\begin{array}{ccc:c:ccc:cc:c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{3.5}$$

The matrix E tells which squares overlap which levels.

Define $H = EB$. Then H is an $m \times m$ square matrix. Combining equations (3.3) and (3.4),

$$HX = \mathbf{1} \tag{3.6}$$

This is a system of linear equations for the largest squares. We will show that H is a nondegenerate square matrix, proving that X is uniquely determined. Then Equation (3.3) will determine F, or the lengths f_j of all squares.

To that end, we define a new square matrix A as a submatrix of E consisting of the columns corresponding to the largest squares. For example, the matrix E in Equation (3.5) gives the following A. It consists of columns $1, 4, 5, 8$ and 10.

$$A = \left[\begin{array}{c:c:c:c:c} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

By definition, the entries in A are characterized as follows.

Lemma 3.3. *Each entry of the matrix A is either 0 or 1.*

$$\begin{cases} A_{ik} = 1 & \textit{if the largest square in group } k \textit{ overlaps level } i \\ A_{ik} = 0 & \textit{if the largest square in group } k \textit{ does not overlap level } i \end{cases}$$

Consequently, A is lower triangular and its diagonal entries are 1.

In Proposition 3.5, we will prove that the horizontal coefficient matrix E and the vertical coefficient matrix B are closely related, that is, $E = AB^\top$. We note the following properties resulting from Lemmas 3.2 and 3.3 to analyze the entries of $AB^\top$.

Lemma 3.4. *Three possible values of $A_{ik}B_{jk}$ are* 1*,* -1*, and* 0*.*

- $A_{ik}B_{jk} = 1$ *if* $A_{ik} = 1$ *and* $B_{jk} = 1$*, in which case, the largest square in group k overlaps level i and x_k appears on the right side only of Equation (3.2).*
- $A_{ik}B_{jk} = -1$ *if* $A_{ik} = 1$ *and* $B_{jk} = -1$*, in which case, the largest square in group k overlaps level i and x_k appears on the left side only of Equation (3.2).*
- $A_{ik}B_{jk} = 0$ *if* $A_{ik} = 0$ *or* $B_{jk} = 0$*, in which case, either the largest square in group k does not overlap level i, or x_k appears on both or neither sides of Equation (3.2).*

The next proposition is the key observation leading to the proof of the theorem.

Proposition 3.5. *With the definitions above,* $E = AB^\top$*.*

Proof. We compare each entry E_{ij} with $(AB^\top)_{ij} = \sum_{k=1}^m A_{ik}B_{jk}$. By Lemma 3.3, A is lower triangular, that is, $A_{ik} = 0$ for $k > i$, thus

$$\sum_{k=1}^m A_{ik}B_{jk} = \sum_{k=1}^i A_{ik}B_{jk}.$$

There are three possible positions of the square s_j relative to level i. The first case is when s_j does not overlap level i but overlaps some levels $> i$. The second is when s_j does not overlap level i but overlaps some levels $< i$. In those cases, $E_{ij} = 0$. The last case is when s_j overlaps level i and $E_{ij} = 1$. In Figure 2, relative to level 3, s_8, s_9, s_{10} are of the first case, s_2, s_3, s_4 of the second, and s_1, s_5, s_6, s_7 of the third case.

In the first case s_j belongs to a group $> i$. Thus, $B_{jk} = 0$ for $k \le i$ by Lemma 3.2. Therefore, $\sum_{k=1}^i A_{ik}B_{jk} = 0 = E_{ij}$.

In the second case, we prove the terms in $\sum_{k=1}^i A_{ik}B_{jk}$ are either all zeros or all zeros but two, which are 1 and -1. Because s_j is positioned above level i geometrically, the chain C_j in Equation (3.1) should contain exactly one square overlapping level i, and so does $C_{j'}$. And those squares overlapping the level give one term each on each side of Equation (3.2). Because s_j does not overlap level i, those terms cannot be f_j but they are x's. Let x_u (on the left) and x_v (on the right) be those terms. By Lemma 3.4, $A_{ik}B_{jk}$ is possibly nonzero only for $k = u$ and $k = v$.

$$\sum_{k=1}^{i} A_{ik}B_{jk} = \sum_{k\in\{u,v\}} A_{ik}B_{jk}.$$

In the case $u = v$, $A_{iu}B_{ju} = A_{iv}B_{jv} = 0$ because $B_{ju} = B_{jv} = 0$ by canceling of terms $x_u = x_v$ in Equation (3.2). In the other case where $u \neq v$, $A_{iu}B_{ju} = -1$ and $A_{iv}B_{jv} = 1$.

In the third case, $E_{ij} = 1$ and we prove the terms in $\sum_{k=1}^{i} A_{ik}B_{jk}$ are all zeros but one term, which is 1. This case is similar to the second case except that in Equation (3.2) the terms corresponding to the squares overlapping level i are f_j on the left and x_v for some v on the right. Therefore, $\sum_{k=1}^{i} A_{ik}B_{jk} = A_{iv}B_{jv} = 1$ by Lemma 3.4. □

The next proposition completes the proof of the uniqueness theorem.

Proposition 3.6. *The equation $HX = \mathbf{1}$ has a unique solution.*

Proof. Since $H = EB = AB^\top B$ by Proposition 3.5 and $\det A = 1$ by Lemma 3.3, it is enough to prove $B^\top B$ is nondegenerate. The quadratic form associated with the symmetric matrix $B^\top B$ is

$$X^\top(B^\top B)X = (BX)^\top(BX) = \sum_{j=1}^{n}(B_jX)^2$$

where B_j is the j-th row of matrix B. By equations (3.3), the last sum equals $\sum_{j=1}^{n} f_j^2$, which expresses the area of the whole rectangle by definition of f_j. Therefore, the quadratic form is positive definite. □

We presented a proof by elementary means of the uniqueness of the solution of a dissection of a rectangle by squares. The matrix E, which describes the configuration of the dissection, determines other matrices A (by definition), B (by Proposition 3.5), and $H = EB$. If one wants to enumerate all possible dissections with a specified number of squares, one way would be to enumerate all possible matrices E.

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