Alabama Journal of Mathematics
43 (2019)
ISSN 2373-0404

On a Problem of Hedetniemi

Justin Haenel
University of Vermont
justin.haenel@uvm.edu

Peter Johnson
Auburn University
johnspd@auburn.edu

The question of for which connected, finite simple graphs the collection of longest paths in the graph has a system of distinct vertex representatives is considered. Similar consideration is given to the same question for cycles.

Introduction

Steve Hedetniemi, in casual conversation, created an entire menagerie of sometimes, but not always, interesting questions in graph theory.

His first question: for which simple graphs does the family of open neighborhoods have a "system of distinct representatives"? Such a "system" for a graph G would be a one-to-one function

$$\phi : V(G) \to V(G)$$

such that for each $v \in V(G)$,

$$\phi(v) \in N_G(v) = \{u \in V(G) | uv \in E(G)\}.$$

[For notation, see West (2001) or any graph theory textbook].

In this case, the "family of open neighborhoods" is an indexed collection, $\{N_G(v) | v \in V(G)\}$: as in P. Hall's (1935) original formulation, the sets of which we want a system of distinct representatives are not necessarily distinct sets.

The original question had a reasonably interesting answer (Hedetniemi, Holliday, & Johnson, 2018): there is such a system of distinct representatives of the open neighborhoods if and only if the graph has a spanning subgraph every component of which is either an edge or a cycle. Notice that if we replace "open" by "closed" in the original question, the answer is trivial: every graph has such a system of distinct representatives. Each vertex can be taken as the representative of its own closed neighborhood.

Hedetniemi went on to propose replacing the indexed family of open neighborhoods in a graph, in his original question, by other collections of structures in the graph. For instance, one can ask, for which graphs does the collection of maximum matchings in the graph have a system of distinct edge representatives?

It may well be that no concise answer to such a question is possible– it may be that the distinct-representatives-of-open-neighborhoods problem is anomalous among such problems. Nonetheless, useful discoveries may result from the inquiry.

This research was supported by the NSF grant no. 1560257.

In this paper, we are interested in distinct vertex representatives of the longest paths in connected graphs. Later, we extend this consideration to cycles.

Denote the path on n vertices by P_n, and denote the cycle on n vertices by C_n. For other graph theory definitions and notations, we refer the reader to West (2001).

A maximum path in a graph G is a subgraph isomorphic to P_n for some n such that no subgraph of G is isomorphic to P_k for any $k > n$. Maximum cycles are defined similarly. In what follows, all graphs are finite, connected, and simple – no loops nor multiple edges are allowed. The order of a graph G is $|V(G)|$, the number of vertices in G.

In the following section we show that an obvious necessary condition for the existence of a system of distinct vertex representatives of the family of longest paths in a graph — that the number of longest paths is no greater than the number of vertices in the union of those paths — is not sufficient. Then we derive from Hall's (1935) marriage theorem a sufficient condition for the existence of a system of distinct vertex representatives of the longest paths in a graph. We end the section with three examples, the second of which shows that the sufficient condition derived from Hall's theorem is not necessary.

We then note analogs for maximum cycles of the results for the longest paths, and pose a problem.

SDR-P-good

For a graph G, denote the set of maximum paths of G by $P(G)$. Denote by $V^*(P(G))$ the union of the vertex sets of the maximum paths of G; in other words, $V^*(P(G))$ is the set of all vertices that have the good fortune of lying on a maximum path in G. G is SDR-P-good if $P(G)$ has a system of distinct vertex representatives (SDR). An SDR for $P(G)$ is a 1-1 function

$$\Phi : P(G) \to \text{V(G)}$$

such that

$$\Phi(p) \in V(p), \forall p \in P(G).$$

Note that such a function maps $P(G)$ into $V^*(P(G))$.

Proposition 1. *If G is SDR-P-good, then*

$$|P(G)| \leq |V^*(P(G))|.$$

We prove the contrapositive of the statement.

Proof. From the definition, any SDR is a 1-1 function from $P(G)$ into $V^*(P(G))$, and clearly no such function from $P(G)$ could be 1-1 if $|P(G)| > |V^*(P(G))|$. □

The immediate question is if $|P(G)| \leq |V^*(P(G))|$ is sufficient for G to be SDR-P-good.

Proposition 2. $|P(G)| \leq |V^*(P(G))|$ *is not sufficient for G to be SDR-P-good.*

Proof. Consider the example provided in Figure 1. The longest paths in the graph shown are P_{13}'s. There are $2 \cdot 15 + 2 = 32$ such paths , i.e., $|P(G)| = 32$, while $|V(G)| = |V^*(P(G))| = 33$. The "dangling" P_5 is a subpath of only two maximum paths. Therefore only two of those 5 vertices can serve in a collection of representatives of the maximum paths in this graph. Consequently, the number of vertices in a set of distinct representatives of a maximal subcollection of $P(G)$ is no greater than $15+11+2+2 = 30 < 32$. Thus there is no system of distinct vertex representatives of the paths in $P(G)$, for this graph G. □

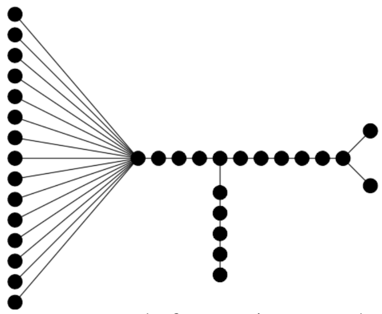

Figure 1. Counter Example for Maximum Paths

For more examples, replace the P_{11} in the "spine" of the example above by P_{2k+1}, the "fan" of 15 vertices by a fan of $2k + 5$ vertices, and the P_5 "hanging path" by a P_k, shrewdly placed so that it is a subpath of only 2 paths of order $2k + 3$ in the graph G thus obtained. The maximum paths in G are of order $2k + 3$ and there are

$$|P(G)| = 2(2k + 5) + 2 = 4k + 12$$

of them. Meanwhile,

$$|V^*(P(G))| = |V(G)| = 5k + 8.$$

Since at most 2 paths in $P(G)$ can be represented by vertices of the hanging P_k, no more than $4k + 10$ paths in $P(G)$ can be supplied with a system of distinct representatives.

The point is, we can produce connected graphs G with $|V^*(G)| - |P(G)|$ arbitrarily large that are not SDR-P-good. However, these graphs are trees, and

$$\frac{|V^*(P(G))|}{|P(G)|} < \frac{5}{4}$$

for each such G. These observations raise two questions.

1. If G is 2-connected and $|P(G)| \leq |V^*(P(G))|$, does it necessarily follow that G is SDR-P-good? (A *cut-vertex* in a connected graph G is a vertex $v \in V(G)$ such that $G - v$ is not connected. A connected graph is *2-connected* if and only if it has no cut-vertex. Obviously the graphs just described are not 2-connected.)
2. Is
$$\left\{ \frac{|V^*(P(G))|}{|P(G)|} \mid \text{G is connected and not SDR-}P\text{-good} \right\}$$
bounded? If so, what is the least upper bound of this set?

Hall's Theorem

In a graph G, if $v \in V(G)$ the open neighbor set of v in G will be denoted $N_G(v)$. If $S \subseteq V(G)$,

$$N_G(S) = \cup_{u \in S} N_G(u).$$

A *matching* in a graph G is an *independent* subset of $E(G)$; this means that no two different edges in the set share a vertex. A matching $M \subseteq E(G)$ *saturates* a set $S \subseteq V(G)$ if and only if each vertex $v \in S$ is incident to some edge $e \in M$.

The following version of Hall's (1935) theorem on systems of distinct representatives can be found in West (2001) or almost any graph theory textbook.

Theorem 3 (Hall's Theorem)**.** *For a bipartite graph G with bipartition A, B, a necessary and sufficient condition for the existence of a matching in G which saturates A is as follows: For every $S \subseteq A$,*

$$|N_G(S)| \geq |S|$$

In any graph G, the degree of a vertex $v \in V(G)$ in G will be denoted $d_G(v)$.

Corollary 4. *Suppose that H is a bipartite graph with bipartition A, B, and*

$$\min_{a \in A} d_H(a) \geq \max_{b \in B} d_H(b) > 0.$$

Then there is a matching in H which saturates A.

Proof. Let $\delta_A = \min_{a\in A} d_H(a)$ and $\Delta_B = \max_{b\in B} d_H(b)$.

Suppose that $S \subseteq A$. Let $E(S, N_H(S))$ denote the set of edges in H that have one end in S. Counting the edges in this set by counting the edge ends in S, we have

$$\begin{aligned}\delta_A|S| &\leq \sum_{a\in S} d_H(a) = |E(S, N_H(S))| \\ &\leq \sum_{b\in N_H(S)} d_H(b) \leq \Delta_B|N_H(S)| \\ &\leq \delta_A|N_H(S)|,\end{aligned}$$

by the assumption that $\delta_A \geq \Delta_B$. By the assumption that $\Delta_B > 0$, we have that $|S| \leq |N_H(S)|$. Since $S \subseteq A$ was arbitrary, the conclusion follows from Theorem 3. □

Corollary 5. *Suppose that G is a graph in which the maximum paths have order q. If every vertex of G lies on no more than q maximum paths then G is SDR-P-good.*

Proof. Let H be the bipartite graph with bipartition

$$A = P(G),\ B = V(G),$$

with $a \in A$, $b \in B$ adjacent if and only if the vertex b is on the path a, in G. By our assumption,

$$d_H(a) = q \geq d_H(b)$$

for every $a \in A$, $b \in B$. By Corollary 4, there is a matching M in H which saturates A. For $ab \in M$ take $b \in V(G)$ as the representative of $a \in P(G)$. The result is an SDR for $P(G)$. □

The following is a companion to Corollary 5.

Proposition 6. *Suppose that every maximum path in P(G) has order q and that every vertex in $V^*(P(G))$ lies on at least q maximum paths in G. Suppose that at least one vertex of G lies on more than q maximum paths in G. Then G is not SDR-P-good.*

Proof. Make a bipartite graph H with bipartition $A = P(G)$, $B = V^*(P(G))$, and $p \in P(G)$, $v \in V^*(P(G))$ adjacent in H if and only if $v \in V(p)$ – i.e., v is a vertex on the path p. Counting $E(H)$ in two different ways, we have that

$$\begin{aligned}|E(H)| &= \sum_{p\in A} d_H(p) = q|P(G)| \\ &= \sum_{v\in B} d_H(v) > q|V^*(P(G))|\end{aligned}$$

whence $|P(G)| > |V^*(P(G))|$. □

Example 1. *The maximum paths on a cycle C_n are all of the form $C_n - e$, e an edge. There are n of these, and each vertex is on every one of them – these paths are Hamilton paths. Therefore, since the order of the maximum paths is n, by Corollary 5 C_n is SDR-P-good. Of course, this is fairly obvious.*

Example 2. *The tree T in Figure 2 has 6 longest paths, each of order 5. Vertices x, y, z lie on all 6 of these paths Here is an SDR for $P(T)$:*

$$\begin{aligned}\phi(u_1xyzv_1) &= u_1, & \phi(u_1xyzv_2) &= x,\\ \phi(u_2xyzv_1) &= u_2, & \phi(u_2xyzv_2) &= y,\\ \phi(u_3xyzv_1) &= u_3, & \phi(u_3xyzv_2) &= z.\end{aligned}$$

This example shows that the sufficient condition for SDR-P-goodness given in Corollary 5 is not necessary.

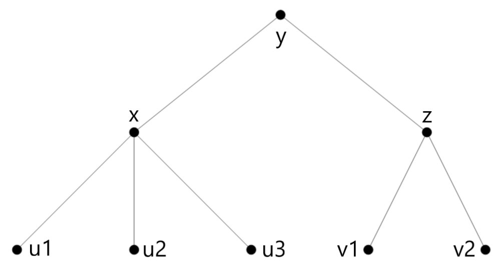

Figure 2. An SDR-P-good graph not satisfying the hypothesis of Corollary 5.

Example 3. *Since there is a Hamilton path in the Petersen graph P_e (Fig. 3), the order of each maximum path in P_e is 10. We will see that P_e is not SDR-P-good.*

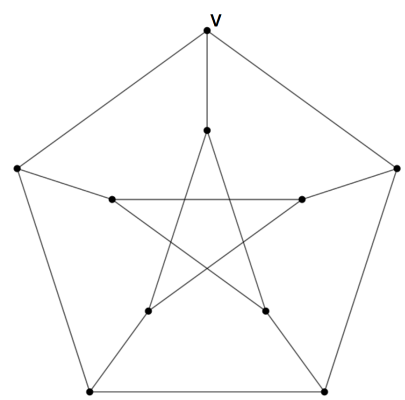

Figure 3. The Petersen graph

Proposition 6 may come in handy. Since the Petersen graph is vertex-transitive, different vertices lie on the same number of maximum paths in P_e. A little work shows that the vertex v is an end vertex of at least 12 different Hamilton paths in P_e. Case closed: P_e is not SDR-P-good.

Of course it was not necessary to invoke Proposition 6 to achieve this conclusion, since, by the claims above,

$$|P(G)| > 12 > 10 = |V^*(P(G))|.$$

Extension to Maximum Cycles

SDR-C-good

We define SDR-C-good, $C(G)$, and $V^*(C(G))$ analogously to the definitions of SDR-P-good, $P(G)$, and $V^*(C(G))$.

A natural question is if

$$|V^*(C(G))| \geq |C(G)| \Rightarrow \text{G is SDR-}C\text{-good}$$

Theorem 7. *If G is SDR-C-good, then*

$$|C(G)| \leq |V^*(C(G))|,$$

but this inequality is not sufficient for G to be SDR-C-good.

Proof. The necessary condition is obvious. Now, consider Figure 4. □

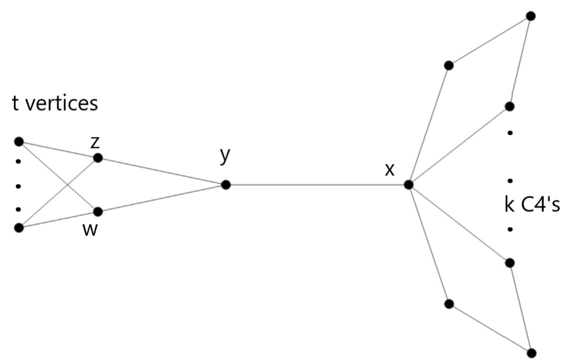

Figure 4. Counter Example for Maximum Cycles

In Figure 4, x is in k C_4's which are otherwise vertex-disjoint, while y, z, and w are all in each of t distinct C_4's. There are also $\binom{t}{2}$ C_4's, each with vertex set consisting of z, w, and two of the vertices other than y to which z and w are adjacent.

If $\binom{t}{2} + t > t + 3$, i.e., if $t > 3$, then there is no way to represent the $\binom{t}{2} + t$ C_4's on the left in Figure 4 by a system of distinct vertex representatives, and so the graph G in this figure is SDR-C-bad. On the other hand,

$$|V^*(C(G))| = 3k + t + 4 \text{ and } |C(G)| = k + t + \binom{t}{2},$$

so we can make

$$\frac{|V^*(C(G))|}{|C(G)|}$$

arbitrarily large.

We do have a question remaining, since obviously x and y are cut-vertices of G: If a graph G is 2-connected, and

$$|C(G)| \leq |V^*(C(G))|,$$

is G necessarily SDR-C-good?

Hall's theorem leads to analogs of Corollary 5 and Proposition 6 for maximum cycles.

Proposition 8. *Suppose that G is a graph in which maximum cycles have order q.*

1. *If each vertex of G lies on no more than q maximum cycles in G, then G is SDR-C-good.*
2. *If each vertex of G lies on at least q maximum cycles, and at least one vertex of G lies on more than q maximum cycles, then G is SDR-C-bad.*

Note: We wish to commend and thank the referee, whose diligence and constructive suggestions led to a considerable improvement of the paper.

References

Hall, P. (1935). On representatives of subsets. *Journal of the London Mathematical Society*, *s1-10*(1), 26-30. doi: 10.1112/jlms/s1-10.37.26

Hedetniemi, S., Holliday, S., & Johnson, P. (2018). Neighborhood representatives. *Congressus Numerantium*, *231*, 117-119.

West, D. B. (2001). *Introduction to graph theory* (2nd ed.). Pearson.