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Upper Chromatic Number of $n-$Dimensional Cubes

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In Computer Science, the hypercube is an important type of interconnected network of processors. Currently, there are many computers that are built using it, and many parallel algorithms that are devised to use this network.
In this article, we find the formula to calculate the upper chromatic number of any n dimensional cube, or a hypercube, such that every cycle C_4 in the cube is colored with at most three colors. We prove that the upper chromatic number of any n-dimensional cube, Q_n with $n \geq 3$, such that each C_4 in the cube is colored with at most three colors is given by: $\overline{\chi} = 1 + 2^{n-1}$. This can also be understood as coloring a mixed hypergraph $H = (X, C, D)$ – in this case, the cube – where each C_4 is a $C - edge$ and the $D-$family is an empty set.

Introduction

A graph is a figure made up of points, called vertices, and lines, called edges, that connect exactly two points. A hypergraph is an extended form of graph where edges, called *hyperedges*, are subsets of the vertex set that can contain more than two vertices. A mixed hypergraph, denoted by $H = (X, C, D)$ is a hypergraph where X is the vertex set, C is the family of hyperedges called C-edges, and D is the family of hyperedges called D-edges. Two vertices of a graph are called adjacent or neighbors if they are connected by an edge. The neighborhood of vertex x in graph G is the set of all vertices in G that are adjacent to v.

Coloring of a graph is simply the labeling of its vertices, generally by positive integers. A proper coloring of a graph is the labeling of vertices of the graph in which the adjacent vertices must have different colors. A proper λ-coloring of a mixed hypergraph $H = (X, C, D)$ is a mapping $c : X \to \{1, 2, ..., \lambda\}$ such that every C-edge has at least two vertices with a common color and every D-edge has at least two vertices with a different color (Voloshin, 2009). Let $G = (X, E)$ be a graph, and i be the number of used colors. If each of i colors is used for the proper coloring of G, then such coloring is called the strict i-coloring. The maximum i for which there exists a strict i-coloring of a mixed hypergraph is called the upper chromatic number of that mixed hypergraph. It is denoted by $\bar{\chi}(H)$.

In a hypercube network, the number of processors, N, is in terms of power of 2, $N = 2^m$. The integer m is the number of processors any processor has two-way connections with. A processor P_i in this network is linked to other processors with indices whose binary representations differ from the binary representation of P_i in exactly one bit (Rosen, 2012).

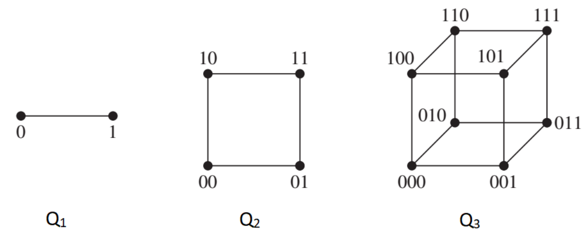

Figure 1. The n-cube Q_n, n = 1,2,3, (Harary, Hayes, & Wu, 1988)

An n-dimensional cube, or n-cube, denoted by Q_n, is defined as a graph that has vertices representing the 2^n bit strings of length n. Any two vertices are considered adjacent if and only if the bit strings that they represent differ in exactly one bit position (Rosen, 2012). A hypercube is a geometric figure which is analogous to a cube in three-dimensions. We may also denote Q_3, Q_4, Q_5, Q_6, . . . as 3D, 4D, 5D, 6D-cubes, . . ., respectively. A cycle C_n is defined as a connected graph on n vertices where each vertex has degree 2. The degree of a vertex is the number of vertices adjacent to it, or the number of its neighbors in the graph. So, a C_4 is a cycle made up of 4 vertices, all of which are connected to exactly two other vertices in the graph.

Throughout the article, when we mention an n-size cube, we will assume $n \in \mathbb{Z}$, $n \geq 3$. Also, when we say an **optimal coloring** of an n-dimensional cube, we mean it to be a proper coloring using maximum number of colors possible. Also, when we mention proper coloring for a hypercube, we understand that the coloring follows the restriction that each C_4 in the cube is colored with at most 3 colors.

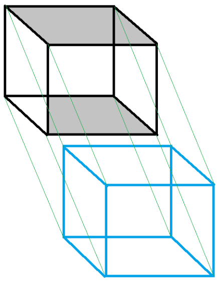

Figure 2. Q_4's with their respective major Q_3's in different orientations. In the cubes, the upper C_4 and the lower C_4 are shadowed respectively.

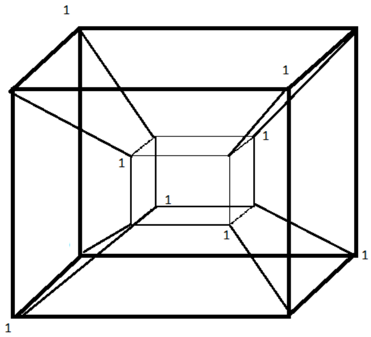

Figure 3. Q_4 with its major Q_3s labeled with a common color $\mathbf{k} = 1$.

Description

Any cube of size $n > 3$ can be thought of as a cube made up of multiple Q_3's. Such arrangements of Q_3's can be constructed in many ways. However, for the sake of uniformity, we imagine any Q_{n+1} as the combination of a Q_n and a copy of the Q_n, denoted by Q'_n, whose corresponding vertices are adjacent to each other. We define such n-cubes whose combination forms a larger $(n + 1)$-cube as **major cubes**.

For example, a Q_3 has only one major cube, which is itself. A Q_4 has two major cubes, which are the two Q_3s that are combined to form that particular Q_4. Similarly, a Q_5 has two major Q_4's, a Q_6 has two major Q_5's, and so on.

We are trying to find the maximum number of colors within the given constraint, and within the proper-coloring conventions of a mixed-hypergraph. We know that for any C_4 in a given cube, we need to have at least one repeated color (or two vertices with the same color). To maximize the overall number of colors used, we repeat a single color for all C_4s in the cube. Such color which is to be repeated throughout all C_4s of a cube can be defined as **common color**, denoted by **k**.

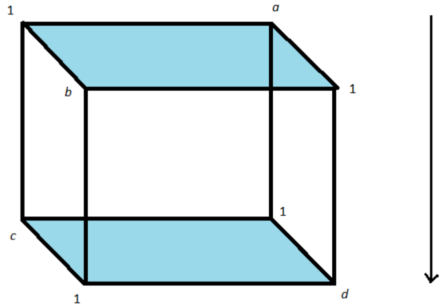

Figure 4. Q_3 with $\mathbf{k} = 1$ and four available vertices a, b, c, and d.

The Algorithm

The following steps will help us set-up the optimum coloring of a Q_n:

INPUT: Q_n without labelling.

OUTPUT: Q_n after the common color is applied.

1. Start by coloring the diagonal vertices of the upper C_4 of a **major cube** of the cube with a **common color 'k'** and go to the next C_4 in the vertically downward direction (See Figure 2 for upper and lower C_4s and Figure 4 for the direction and positioning).
2. Then, use **k** to color the diagonal vertices of either the lower C_4 of the same or the upper C_4 of a different major cube – whichever comes first in the vertical direction – in alternating positions.
3. Similarly, color the diagonal vertices of the next closest C_4 in the same positions as the coloring in Step 1.
4. Repeat Step 2 and Step 3 for the alternating application of **k** as long as no C_4 of the n^{th} cube remains colorless.
5. End.

This ensures that every C_4 of the cube has two vertices colored with the **common color k**.

At this point, we have used only one color and have set-up the constraint in such way that each C_4 of the cube has at least one repeated color. Now, we can add as many different colors as the number of remaining uncolored vertices.

Notice that for a Q_3, we get 4 uncolored vertices after the constraint is taken care of (See Figure 4). As any Q_n is some combination of $3D$ major cubes, the function that we are looking for should look like: $1+4N$ where N, the number of Q_3s in the given Q_n, is some function in terms of n.

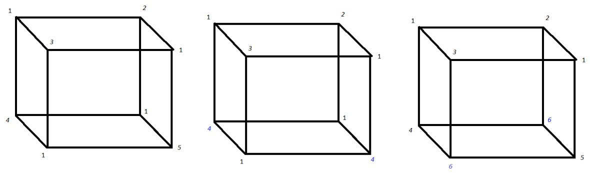

Figure 5. (a) On the left. (b) At the center. (c) On the right.

For all $n > 3$, Q_n has 2^{n-3} number of major Q_3's. With this, the maximum number of used colors becomes:

$$1 + 4(2^{n-3}) \Rightarrow 1 + 2^{n-1}$$

As Q_n has 2^n vertices, the color k colors half of the vertices, while the other half gets $2^{n/2}$ different colors. Also notice that the vertices colored with k form a maximal independent set (in any dimension).

How many common colors?

With the constraint that each C_4 in the cube should be colored by using at most 3 colors, the problem is to figure out how to maximize the number of colors for successful coloring. We are concerned with whether the concept of using a single **k** throughout the cube provides us with the maximum number of colors. The question is: could we repeat multiple colors throughout the cube instead of just one and achieve optimum coloring? Figure 5 shows the difference between repeating multiple colors and repeating a single color in a simple Q_3. Notice that the use of a single **k** (Figure 5(a)) optimizes the coloring and correctly gives the number of maximum colors in a $3D$-cube.

Figure 5(b) and 5(c) show that if we repeat more than one color, we either reduce the number of legitimate colors (as shown in Figure 5(b)) or we violate the constraint altogether (the cube in Figure 5(c) has C_4's colored with 4 different colors).

Theorem and Proof

Lemma 1. *An optimal coloring of a Q_n can be achieved only when all of its C_4's are colored with exactly three colors.*

Proof. The upper chromatic number of a C_4 is 4. However, under the given constraint, we know that no C_4 in Q_n can be colored with four different colors. Therefore, as per the proper coloring conventions of a mixed-hypergprah, the maximum number of colorings in Q_n can be achieved by coloring all C_4s in the cube with exactly three colors. □

Lemma 2. *If all the C_4s of a Q_n and its copy cube show proper coloring with exactly three colors, then all the C_4s of the corresponding Q_{n+1} formed by their combination also show proper coloring, given that the same **k** is repeated between the Q_n and the Q'_n in their diagonally alternating vertices.*

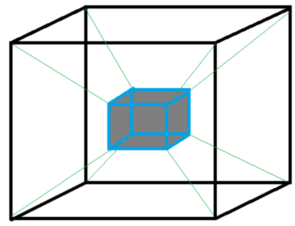

Figure 6. General representation of the Q_n. The outer cube represents the outermost major $3D$-cube of the Q_n, while the inner cube represents the layer of all the major $3D$-cubes from dimension 3 to dimension $n-1$.

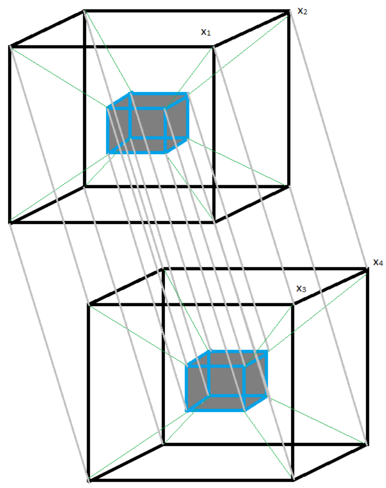

Figure 7. General representation of the Q_{n+1} made up of Q_n and Q'_n whose corresponding vertices are adjacent to each other.

Proof. We prove this conjecture by cases and by contradiction.

By contrary, let us assume that there exists a C_4 in Q_{n+1} that does not show proper coloring with exactly three colors. Let $\{x_1, x_2, x_3, x_4\}$ be the vertices of that arbitrary C_4.

If all of these vertices belong to the Q_n, or if all belong to Q'_n, then they are colored properly with exactly three colors. Notice that due to construction, it is impossible to have one vertex from the set in the cube and three other vertices in the copy. So, without loss of generality, let us assume that x_1 and x_2 belong to the n^{th} cube and x_3 and x_4 belong to the copy (See Figure 7)). Also, let us assume that for both cubes, $\mathbf{k} = 1$.

This leads to two cases:

Case 1: The C_4, $\{x_1\ x_2\ x_3\ x_4\}$, is colored with less than three colors:

Let us pick any C_4 from an optimally colored Q_{n+1} with

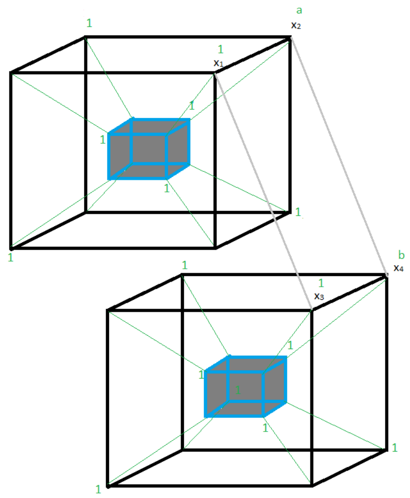

Figure 8. x_2 and x_4 are available for new colors a and b.

two vertices on Q_n and two vertices on its copy (as shown in Figure 8). Notice that we will always end up having two vertices, one from each cube, colored with the common color **k**, and the remaining two vertices, again one from each cube, colored with different new colors. This is because of the assumption that both Q_n and Q'_n are properly colored using the same **k** in their diagonally alternating vertices.

Now, let us try to color the vertex set $\{x_1\ x_2\ x_3\ x_4\}$ with less than three colors. Since the set already has two vertices colored with k, then the remaining two vertices cannot have different new colors. This means we either have to color all 4 vertices with k or have to apply same new color for the remaining two vertices. Either way, this implies that Q_n or Q'_n has at least one C_4 that is colored with less than three colors, which is a contradiction.

Case 2: The C_4, $\{x_1\ x_2\ x_3\ x_4\}$, is colored with more than three colors:

If all the vertices in the vertex set $\{x_1\ x_2\ x_3\ x_4\}$ are colored with different colors, then the set does not have a repeated common color **k**. This is a contradiction because if the Q_n and its copy cube show proper coloring with exactly three colors, and if they both have the same **k**, any C_4 picked from their connected configuration having two vertices on each cube will also have two vertices labeled with **k**.

Since both cases lead to contradictions, they must be false. Hence, the proposition is true. □

Theorem 3. *Q_n is a bipartite graph.*

Proof. The hyper-cube Q_n has 2^n vertices where each vertex corresponds to a binary string of length n. Any two vertices labeled by strings x and y are adjacent if and only if y can be obtained from x by changing exactly one bit. Let us partition

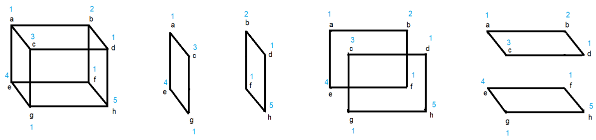

Figure 9. Coloring alternating diagonal vertices of Q_3 with a common color makes all of its C_4's colored with exactly three colors.

the vertices of Q_n in to two subsets: one with odd and another with even number of 1s in their binary string representation. This will divide the vertex set into two equal sized subsets. In such partition, no edges in the cube will have its vertices in the same subset, making Q_n a bipartite graph. □

Theorem 4. *Q_n is an n-regular graph.*

Proof. Let x be any arbitrary vertex of Q_n, and let the set N_x be defined as the neighborhood of x. Recall that in Q_n, each vertex corresponds to a binary string of length n. Therefore, any vertex in Q_n can be represented by a vector of length n, such that each element of the vector is a bit string. Let V_x be the binary vector of x. This implies that $|V_x| = n$. Consider a vertex y, such that $y \in N_x$. We know that the binary vectors of x and y must differ at exactly one position. Since $|V_x| = n$, this can happen in n different ways. Therefore, $|N_x| = n$, which means x is adjacent to n other vertices in Q_n, making Q_n an n-regular graph. □

Theorem 5. *The upper chromatic number of any n-dimensional cube, Q_n such that each C_4 in the cube is colored with at most three colors is given by:*

$$\overline{\chi} = 1 + 2^{n-1}, \quad \textit{for all } n \geq 3.$$

Proof. Lemma 1 shows that an optimal coloring can be achieved only when all of the C_4's of a cube are colored with exactly three colors. So, to prove the theorem, it is important to first verify whether we are coloring all the C_4's of a cube with exactly three colors. This is when the use of common color becomes key. In our initial set-up, we colored the alternating diagonal vertices of major Q_3's of a Q_n with **k**, and colored all the remaining vertices of the entire cube with different new colors. Such application of **k** ensures that all the C_4's of the cube have exactly two vertices available for coloring. And, as we map all those pairs of vertices in each C_4 with new colors, we achieve the coloring where all the C_4's of the cube are colored with exactly three colors–with **k** repeated twice in all of them (See Figure 9 and Figure 10).

Now, we prove the theorem by the method of Induction.

We should realize that each time we upgrade the configuration of a cube by increasing its dimension by 1, the number of major Q_3's in the subsequent larger Q_n doubles. For instance, a $5D$-cube has four major Q_3's, a $6D$-cube has eight major Q_3's, and so on(See Figure 11).

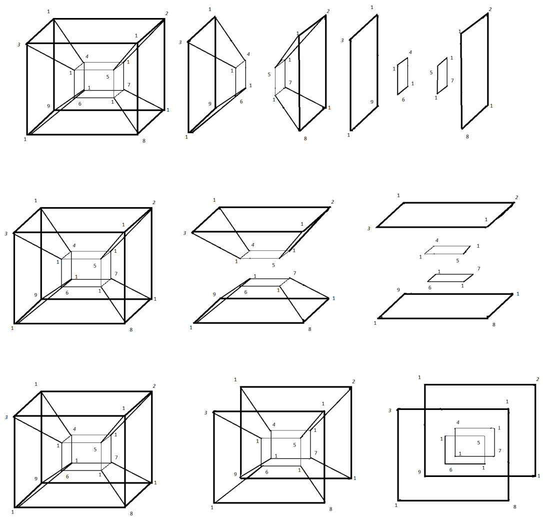

Figure 10. Similar results with the $4D$-cube.

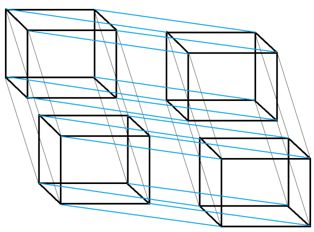

Figure 11. A $5D$-cube with its four distinct major Q_3's.

Let us assume that when $n = k$, when $k \geq 3$, the statement P_k: the upper chromatic number of a k^{th} dimensional cube, Q_k, within the given constraints, $\overline{\chi}_{k^{th}-cube} = 1 + 2^{k-1}$, is true. Now we try to prove that the theorem is valid for $n = k + 1$. This is equivalent to proving

$$P_{(k+1)} : \overline{\chi}_{(k+1)^{th}-cube} = 1 + 2^{(k+1)-1}.$$

Any Q_{k+1} is made up of a Q_k and Q'_k whose corresponding vertices are adjacent to each other. The Q'_k has the same common color as Q_k. This is because the Q_k and the Q'_k combine together to form a single Q_{k+1} and we do not want the Q_{k+1} to have two repeated colors (we only repeat one color throughout a cube). All the remaining vertices of Q'_k will be given different new colors in order to maximize the number of used colors so that the optimum coloring can be achieved. As the transition from k to $k+1$ doubles the number of major Q_3's in a cube, for the $(k + 1)^{\text{th}}$ cube, the upper chromatic number becomes:

$$\begin{aligned}\overline{\chi}_{(k+1)^{th}-cube} &= 1 + 2^{k-1} + 2^{k-1} \\ &= 1 + 2^{(k+1)-1}\end{aligned}$$

Therefore, $P_{(k+1)} : \overline{\chi}_{(k+1)^{th}-cube} = 1 + 2^{(k+1)-1}$ is true. So, by the Principle of Mathematical Induction, $\overline{\chi} = 1 + 2^{n-1}$, for all $n \geq 3$. □

In Figures 12-14 we verify the conjecture applies correctly in the simpler cubes.

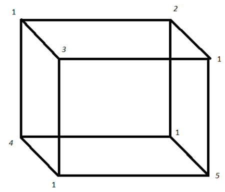

Figure 12. A $3D$-cube has an upper chromatic number of $1 + 4(2^{3-3}) = 5$ under the given constraints.

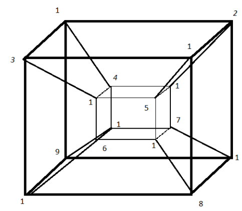

Figure 13. A $4D$-cube has an upper chromatic number of $1 + 4(2^{4-3}) = 9$ under the given constraints.

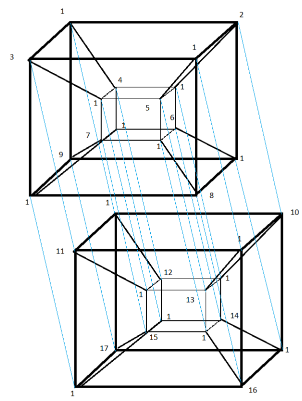

Figure 14. A $5D$-cube has an upper chromatic number of $1 + 4(2^{5-3}) = 17$ under the given constraints.

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