Problems Set 29

Alabama Journal of Mathematics (Spring/Fall 2010)

1. Deduce whether each of the five statements below is true or false:
   (a) All five of these statements are false.
   (b) All five of these statements are true.
   (c) At least one of these statements are false.
   (d) At least one of these statements are true.
   (e) If either of the first two statements is true, then all others are true.

2. The Fibonacci numbers are defined by $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Prove by mathematical induction that

$$F(n) \times F(n+1) = \sum_{0 \leq k \leq n} F(k)^2 \quad \text{for all } n \geq 0.$$

3. Solve this system of congruence equations:

$$4x + 6y \equiv 7 (\text{mod } 11)$$
$$9x + 3y \equiv 10 (\text{mod } 11)$$

4. toss two N-sided dice. Let R denote the value on the red die, whose faces are labelled $0, 1, 2, \ldots, N-1$. Let G denote the value on the green die, whose faces are $1, 2, 3, \ldots, N$. Determine these probabilities:
   (a) Prob $(R < G)$
   (b) Prob $(R = G)$
   (c) Prob $(R > G)$

5. Solve the recurrence $a_0 = 1$, $a_n = 2a_{n-1}$, and let

$$G(x) = \sum_k a_k \frac{x^k}{k!}.$$

   Then evaluate $G(3)$.

6. Define a "perfect number" to be a positive integer that equals the sum of its *proper* divisors. For example, 6 and 28 are perfect because $6 = 1+2+3$ and $28 = 1 + 2 + 4 + 7 + 14$. Prove each of these clames:

(a) The sum of the reciprocals of *all* divisors of a perfect number must equal 2. (For example, $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=2$.)

(b) If 2^n-1 is prime, then n is prime and $2^{n-1}(2^n-1)$ is perfect.

7. Bag 1 contains 9 red and 12 blue marbles, and bag 2 contains 3 red and 8 blue marbles. First we choose a bag, with the probability of choosing each bag proportional to the number of marbles it contains. Next two marbles are drawn at random from the chosen bag. If both marbles are red, what is the conditional probability that we have chosen bag 1?

8. (Submitted by Peter Johnson and Caleb Petrie, Auburn University) Let $\lceil\ \ \rceil$ denote the "round up" or "ceiling" function. For instance, $\lceil 2.1\rceil = 3$, and $\lceil -2.1\rceil = -2$. Suppose that $n>0$ and m are integers. Find, as an expression in n and m, the largest possible value of $\sum_{\imath=1}^{n}\lceil a_i\rceil$, if $a_1,\ldots,a_n$ are required to be real numbers adding up to m. And, of course, show that your answer is correct.

9. (Submitted by Jonathan Waite)

(a) Suppose that k and n are relatively prime, and $a_1,a_2,\ldots,a_k$ are integers such that $a_1+a_2+\cdots+a_k=n$. Prove that the k cyclic permutations of the sequence $a_1,a_2,\ldots,a_k$ are distinct.

(b) Suppose that k and n are not relatively prime. Show that there exist non-negative integers $a_1,a_2,\ldots,a_k$ such that $a_1+a_2+\cdots+a_k=n$ and the k cyclic permutations of the sequence $a_1,a_2,\ldots,a_k$ are not distinct.

Solution to problem number 2 from Spring 2007

Problem:

If $w+x+y+z=12$ and $w^2+x^2+y^2+z^2=48$, determine the largest possible value of w.

Solution:

Let

$$w=f(x,y,z)=12-x-y-z \quad \text{(objective function)}$$

and

$$g(w,x,y,z)=w^2+x^2+y^2+z^2 \quad \text{(constraint)}.$$

Scratch:

$$\begin{aligned}
w^2 &= (12-x-y-z)^2 \\
&= (12-x-y-z)(12-x-y-z) \\
&= 14412x-12y-12z-12x+x^2+xy+xz-12y+xy+y^2 \\
&\quad +yz-12z+xz+yz+z^2 \\
&= 144+x^2+y^2+z^2+2xy+2xz+2yz-24x-24y-24z
\end{aligned}$$

This implies that

$$\begin{aligned}
g(x) &= 144+2x^2+2y^2+2z^2+2xy+2xz+2yz-24x-24y-24z \\
&= 2(72+x^2+y^2+z^2+xy+xz+yz-12x-12y-12z)
\end{aligned}$$

Also,

$$\nabla(f(x,y,z)) = -\vec{i}-\vec{j}-\vec{k}$$

and

$$\nabla(g(x,y,z)) = 2\left((2x+y+z12)\vec{i}+(x+2y+z12)\vec{j}+(x+y+2z12)\vec{k}\right)$$

Using the method of LaGrange Multipliers:

For $g(x,y,z) \neq 0$, there exists some real number λ such that $\nabla(f(x,y,z)) = \lambda\nabla(g(x,y,z))$.

Thus, we are looking for a value of lambda such that the system of equations

$$\begin{aligned}
f_x(x,y,z) &= \lambda g_x(x,y,z) \\
f_y(x,y,z) &= \lambda g_y(x,y,z) \\
f_z(x,y,z) &= \lambda g_z(x,y,z) \\
g(x,y,z) &= 48
\end{aligned}$$

The first three lines above are equivalent to

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 11 \\ 11 \end{pmatrix}$$

This leads us to see that $x=y=z$.

We now solve using the "$g(x,y,z) = 48$" equation, using our newfound knowledge that $x=y=z$. So

$$48+x^2+y^2+z^2+xy+xz+yz-12x-12y-12z=0$$

Because $x=y=z$, $y^2=x^2=xy$, etc., the equation above simplifies to

$$48+6x^2 36x=0.$$

So $x = 2$ or $x = 4$.

If $x = 4$, then $x = y = z$ implies that $y = 4$ and $z = 4$. In that case, applied to the original equations, $w = 0$.

If $x = 2$, then $x = y = z$ implies that $y = 2$ and $z = 2$. In that case, applied to the original equations, $w = 6$.

Both these scenarios comply with our constraint, therefore 0 is the minimum for w and 6 is the maximum for w in this problem.

Since we wanted the largest possible value, and $w = 6$ is the maximum for the problem, then $w = 6$ is the largest possible value of w in the given problem.

Submitted by Aaron Murray.