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Recurrence and Asymptotic Relations of a Generalized Elliptic-type Integral

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Abstract

Epstein-Hubbell 1963 elliptic-type integrals occur in radiation field problems. In this paper, we consider a generalization of the elliptic-type integrals introduced by Kalla and Tuan 1996 and developed by Salman 2006. We give some recurrence relations for this generalized special function. Some of these relations can be used for analytic continuation to different domains. Asymptotic relations of the generalized function are constructed at one of its singular points. These relations are derived by using some of the analytic continuation formulas of the related Gauss hypergeometric function.

Introduction

Elliptic integrals (Abramowitz and Stegun, 1972; Byrd and Friedman, 1971; Evans et al., 1993; Kaplan, 1950) occur in a number of physical problems. Epstein and Hubbell 1963 have treated the elliptic-type integrals

$$\Omega_j(k) = \int_0^{\pi} \frac{d\theta}{(1-k^2\cos\theta)^{j+\frac{1}{2}}}, \quad 0 \le k < 1, \quad j = 0, 1, 2, \ldots \tag{1}$$

Certain problems dealing with the computation of the radiation field off axis from a uniform circular disk according to an arbitrary angular distribution law (Berger and Lamkin, 1958; Hubbell et al., 1961), when treated with a Legendre polynomial expansion method, give rise to integrals of the form (1). For $j = 0, 1,\ \Omega_j(k)$ reduces to

$$\Omega_0(k) = \frac{\sqrt{2}\lambda}{k} K(\lambda), \quad \Omega_1(k) = \frac{\sqrt{2}\lambda}{k(1-k^2)} E(\lambda), \tag{2}$$

where $\lambda^2 = \dfrac{2k^2}{1-k^2}$, and $K(\lambda)$ and $E(\lambda)$ are the complete elliptic integrals of the first and second kinds, respectively (Lebedev, 1965, (7.10.11)) which are defined by

$$K(\lambda) = \int_0^{\pi/2} (1-\lambda^2 \sin^2\theta)^{-\frac{1}{2}} d\theta, \qquad 0 \le \lambda < 1, \tag{3}$$

$$E(\lambda) = \int_0^{\pi/2} (1 - \lambda^2 \sin^2\theta)^{\frac{1}{2}} d\theta, \qquad 0 \le \lambda < 1, \tag{4}$$

A historical survey for further developments and studies of these elliptic-type integrals and their generalizations can be found in (Salman, 2006).

Srivastava and Siddiqi 1995 studied the following unification of the elliptic-type integrals

$$\Lambda_{\lambda,\mu}^{(\alpha,\beta)}(\rho;\ k) = \int_0^{\pi} \frac{\cos^{2\alpha-1}(\theta/2) \sin^{2\beta-1}(\theta/2) d\theta}{(1-\rho\sin^2(\theta/2))^{\lambda}\,(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}, \tag{5}$$

$0 \le k < 1,\ |\rho| < 1,\ \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0,\ \lambda, \mu \in \mathbf{C}$.

Kalla and Tuan 1996 have considered the following generalized elliptic-type integral

$$\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;\ k) = \int_0^{\pi} \frac{\cos^{2\alpha-1}(\theta/2) \sin^{2\beta-1}(\theta/2) d\theta}{(1-\rho\sin^2(\theta/2))^{\lambda}\,(1+\delta\cos^2(\theta/2))^{\gamma}\,(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}, \tag{6}$$

$0 \le k < 1$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $|\rho| < 1$, $|\delta| < 1$, $\lambda, \gamma, \mu \in \mathbf{C}$, where it is tacitly assumed that ρ and δ can be complex numbers whenever λ and γ) are non-negative integers. They've introduced asymptotic expansions of (6) when k^2 is sufficiently close to 1.

Using trigonometric identities, we can show that the integral in (6) can be rewritten in the form

$$\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;\ k) = \frac{1}{(1+\delta)^{\gamma}(1-k^2)^{\mu+\frac{1}{2}}} \times \int_0^1 \frac{t^{\beta-1}(1-t)^{\alpha-1} dt}{(1-\rho t)^{\lambda} \left(1 - \frac{\delta\, t}{1+\delta}\right)^{\gamma} \left(1 - \frac{2k^2 t}{k^2-1}\right)^{\mu+\frac{1}{2}}}, \tag{7}$$

$0 \le k < 1,\ \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0,\ |\rho| < 1,\ \mathrm{Re}(\delta) > -\frac{1}{2},\ \lambda, \gamma, \mu \in \mathbf{C}$.

In (Salman, 2006), Salman has shown that the integral in (7) gives analytic function of ρ, δ and k in the domain where ρ is in the complex plane cut along $[1, \infty)$, δ is in the complex plane cut along $(-\infty, -1]$ and k^2 is in the complex plane cut along the intervals $(-\infty, -1]$ and $[1, \infty)$.

Two special functions are frequently used in this work. The first is the Gauss' hypergeometric functions ${}_2F_1$(Prudnikov et al., 1986, Vol. 3, (7.2.1.1))

$${}_2F_1(a, b;\ c;\ z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad |z| < 1. \tag{8}$$

and the second is the Laurecella's hypergeometric function of three variables $F_D^{(3)}$ (Prudnikov et al., 1986, (7.2.4.15))

$$F_D^{(3)}(a, b_1, b_2, b_3;\ c;\ z_1, z_2, z_3) = \sum_{l,m,n=0}^{\infty} \frac{(a)_{l+m+n} (b_1)_l (b_2)_m (b_3)_n}{(c)_{l+m+n} l! m! n!} z_1^l z_2^m z_3^n, \tag{9}$$

$|z_j| < 1, \quad j = 1, 2$ and 3.

The following formulas has been shown in (Salman, 2006), and we shall use them in developing some of our results.

$$\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;\ k)=\frac{B(\alpha,\beta)}{(1+\delta)^{\gamma}(1-k^2)^{\mu+1/2}}\times F_D^{(3)}\left(\beta,\lambda,\gamma,\mu+\tfrac{1}{2};\ \alpha+\beta;\ \rho,\frac{\delta}{1+\delta},\frac{2k^2}{k^2-1}\right), \tag{10}$$

with $k^2<\frac{1}{3},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ |\rho|<1,\ \mathrm{Re}(\delta)>-\frac{1}{2},\ \lambda,\gamma,\mu\in\mathbf{C}$,

$$\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;\ k)=\frac{B(\alpha,\beta)}{(1-\rho)^{\lambda}(1+k^2)^{\mu+1/2}}\times F_D^{(3)}\left(\alpha,\lambda,\gamma,\mu+\frac{1}{2};\ \alpha+\beta;\ \frac{\rho}{\rho-1},-\delta,\frac{2k^2}{1+k^2}\right), \tag{11}$$

with $k^2<1,\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\rho)<\frac{1}{2},\ |\delta|<1,\ \lambda,\gamma,\mu\in\mathbf{C}$,

$$\begin{aligned}\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;\ k)&=\frac{B(\alpha,\beta)}{(1-\rho)^{\lambda}(1+k^2)^{\mu+1/2}}\sum_{l,m=0}^{\infty}\frac{(\alpha)_{l+m}(\lambda)_l(\gamma)_m}{(\alpha+\beta)_{l+m}l!m!}\\&\times\left(\frac{\rho}{\rho-1}\right)^l(-\delta)^m\ {}_2F_1\left(\alpha+l+m,\mu+\tfrac{1}{2};\ \alpha+\beta+l+m;\ \frac{2k^2}{1+k^2}\right),\end{aligned} \tag{12}$$

with $k^2<1,\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\rho)<\frac{1}{2},\ |\delta|<1,\ \lambda,\gamma,\mu\in\mathbf{C}$,

$$\begin{aligned}\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;\ k)&=\frac{B(\alpha,\beta)}{(1-\rho)^{\lambda}(1-k^2)^{\mu+1/2}}\sum_{l,m=0}^{\infty}\frac{(\alpha)_{l+m}(\lambda)_l(\gamma)_m}{(\alpha+\beta)_{l+m}l!m!}\\&\times\left(\frac{\rho}{\rho-1}\right)^l(-\delta)^m\ {}_2F_1\left(\beta,\mu+\tfrac{1}{2};\ \alpha+\beta+l+m;\ \frac{2k^2}{k^2-1}\right),\end{aligned} \tag{13}$$

with $k^2<\frac{1}{3},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\rho)<\frac{1}{2},\ |\delta|<1,\ \lambda,\gamma,\mu\in\mathbf{C}$.

Recurrence Formulas

In this section, we give some recurrence relations of (6). Some of these relations can be used as analytic continuation formulas which somehow extend the domain of the function (6).

Analytic continuation formulas

If we replace k by ik in relation (11), we obtain

$$\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;\ ik)=\frac{B(\alpha,\beta)}{(1-\rho)^{\lambda}(1-k^2)^{\mu+1/2}}\times F_D^{(3)}\left(\alpha,\lambda,\gamma,\mu+\frac{1}{2};\ \alpha+\beta;\ \frac{\rho}{\rho-1},-\delta,\frac{2k^2}{k^2-1}\right), \tag{14}$$

with $k^2<\frac{1}{3},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\rho)<\frac{1}{2},\ |\delta|<1,\ \lambda,\gamma,\mu\in\mathbf{C}$.

Changing $\rho \to \dfrac{\rho}{\rho-1}$, $\delta \to \dfrac{-\delta}{1+\delta}$ and $\alpha \leftrightarrow \beta$ into (10) yields

$$\Lambda^{(\beta,\alpha)}_{(\lambda,\gamma,\mu)}\left(\frac{\rho}{\rho-1}, \frac{-\delta}{1+\delta};\ k\right) = \frac{B(\alpha,\beta)}{(1+\delta)^{-\gamma}(1-k^2)^{\mu+1/2}} \times F_D^{(3)}\left(\alpha, \lambda, \gamma, \mu+\tfrac{1}{2};\ \alpha+\beta;\ \frac{\rho}{\rho-1}, -\delta, \frac{2k^2}{k^2-1}\right), \tag{15}$$

with $k^2 < \frac{1}{3},\ \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0,\ \mathrm{Re}(\rho) < \frac{1}{2},\ |\delta| < 1,\ \lambda, \gamma, \mu \in \mathbf{C}$.

Comparing (14) and (15), we can easily get

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho, \delta;\ ik) = \frac{1}{(1-\rho)^{\lambda}(1+\delta)^{\gamma}} \Lambda^{(\beta,\alpha)}_{(\lambda,\gamma,\mu)}\left(\frac{\rho}{\rho-1}, \frac{-\delta}{1+\delta};\ k\right), \tag{16}$$

If we replace k by ik into (16) and use the fact that $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho, \delta;\ k)$ is an even function with respect to k, we obtain

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho, \delta;\ k) = \frac{1}{(1-\rho)^{\lambda}(1+\delta)^{\gamma}} \Lambda^{(\beta,\alpha)}_{(\lambda,\gamma,\mu)}\left(\frac{\rho}{\rho-1}, \frac{-\delta}{1+\delta};\ ik\right), \tag{17}$$

with $\mathrm{Re}(\rho) < \frac{1}{2},\ \mathrm{Re}(\delta) > -\frac{1}{2},\ 0 \le k < 1$.

Another relation can be obtained by rewriting (6) as

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho, \delta;\ k) = \frac{1}{(1-\rho)^{\lambda}(1+\delta)^{\gamma}} \times \int_0^{\pi} \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)d\theta}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}\left(1-\dfrac{\delta}{1+\delta}\sin^2(\theta/2)\right)^{\gamma}\left(1+\dfrac{\rho}{1-\rho}\cos^2(\theta/2)\right)^{\lambda}}, \tag{18}$$

with $\mathrm{Re}(\rho) < \frac{1}{2},\ \mathrm{Re}(\delta) > -\frac{1}{2},\ 0 \le k < 1$, and the comparison with (6) to get

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho, \delta;\ k) = \frac{1}{(1-\rho)^{\lambda}(1+\delta)^{\gamma}} \Lambda^{(\alpha,\beta)}_{(\gamma,\lambda,\mu)}\left(\frac{\delta}{1+\delta}, \frac{\rho}{1-\rho};\ k\right), \tag{19}$$

for $\mathrm{Re}(\rho) < \frac{1}{2},\ \mathrm{Re}(\delta) > -\frac{1}{2},\ 0 \le k < 1$.

Relations (17) and (19) can be used to give analytic continuations for the elliptic-type integral $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho, \delta;\ k)$, defined in (6), with respect to ρ and δ. It extends the regions of ρ and δ, which are $|\rho| < 1$ and $|\delta| < 1$, into the half-planes $\mathrm{Re}(\rho) < \frac{1}{2}$ and $\mathrm{Re}(\delta) > -\frac{1}{2}$.

Recurrence formulas

Due to the importance of the recurrence relations into the computational field of transcendental functions, we will derive some of these relations for the elliptic-type integral $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho, \delta;\ k)$.

To simplify the notations, we will drop some of the parameters when they remain unchanged in both sides of a recurrence relation. For instance, $\Lambda_{\mu+1}^{\beta+1}$ is understood as $\Lambda_{(\lambda,\gamma,\mu+1)}^{(\alpha,\beta+1)}(\rho,\delta;\ k)$.

Some of the recurrence relations can be obtained by using some trigonometric identities into the integrand of (6). Some simple relations of this type are,

$$\Lambda^{(\alpha,\beta)} = \Lambda^{(\alpha+1,\beta)} + \Lambda^{(\alpha,\beta+1)}, \tag{20}$$
$$\Lambda^{(\alpha,\beta)} = \Lambda^{(\alpha-1,\beta)} - \Lambda^{(\alpha-1,\beta+1)}, \tag{21}$$
$$\Lambda^{(\alpha,\beta)} = \Lambda^{(\alpha,\beta-1)} - \Lambda^{(\alpha+1,\beta-1)}, \tag{22}$$
$$\Lambda_{\lambda}^{\beta} = \Lambda_{\lambda+1}^{\beta} - \rho\Lambda_{\lambda+1}^{\beta+1}, \tag{23}$$
$$\Lambda_{\lambda}^{\alpha} = (1-\rho)\Lambda_{\lambda+1}^{\alpha} + \rho\Lambda_{\lambda+1}^{\alpha+1}, \tag{24}$$
$$\Lambda_{\mu}^{\alpha} = (1+k^2)\Lambda_{\mu+1}^{\alpha} - 2k^2\Lambda_{\mu+1}^{\alpha+1}, \tag{25}$$
$$\Lambda_{\mu}^{\beta} = (1-k^2)\Lambda_{\mu+1}^{\beta} + 2k^2\Lambda_{\mu+1}^{\beta+1}, \tag{26}$$
$$\Lambda_{\mu}^{(\alpha,\beta)} = \Lambda_{\mu+1}^{(\alpha,\beta)} - k^2\Lambda_{\mu+1}^{(\alpha+1,\beta)} + k^2\Lambda_{\mu+1}^{(\alpha,\beta+1)}, \tag{27}$$
$$\Lambda_{\gamma}^{\alpha} = \Lambda_{\gamma+1}^{\alpha} + \delta\Lambda_{\gamma+1}^{\alpha+1}, \tag{28}$$
$$\Lambda_{\gamma}^{\beta} = (1+\delta)\Lambda_{\gamma+1}^{\beta} - \delta\Lambda_{\gamma+1}^{\beta+1}. \tag{29}$$

Further recurrence relations can be derived if we use integration by parts in (6). Among the relations of this type we cite the following

$$\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)} = \frac{1}{\rho(\lambda-1)} \times\left((\alpha-1)\Lambda_{(\lambda-1,\gamma,\mu)}^{(\alpha-1,\beta)} - (\beta-1)\Lambda_{(\lambda-1,\gamma,\mu)}^{(\alpha,\beta-1)} + (2\mu+1)k^2\Lambda_{(\lambda-1,\gamma,\mu+1)}^{(\alpha,\beta)} - \gamma\delta\Lambda_{(\lambda-1,\gamma+1,\mu)}^{(\alpha,\beta)}\right), \tag{30}$$

for $\mathrm{Re}(\alpha) > 1, \mathrm{Re}(\beta) > 1, \lambda \neq 1, \rho \neq 0$,

$$\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)} = \frac{1}{\delta(\gamma-1)} \times\left((\alpha-1)\Lambda_{(\lambda,\gamma-1,\mu)}^{(\alpha-1,\beta)} - (\beta-1)\Lambda_{(\lambda,\gamma-1,\mu)}^{(\alpha,\beta-1)} + (2\mu+1)k^2\Lambda_{(\lambda,\gamma-1,\mu+1)}^{(\alpha,\beta)} - \lambda\rho\Lambda_{(\lambda+1,\gamma-1,\mu)}^{(\alpha,\beta)}\right), \tag{31}$$

for $\mathrm{Re}(\alpha) > 1, \mathrm{Re}(\beta) > 1, \gamma \neq 1, \delta \neq 0$,

$$\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)} = \frac{1}{k^2(2\mu-1)} \times\left((1-\alpha)\Lambda_{(\lambda,\gamma,\mu-1)}^{(\alpha-1,\beta)} + (\beta-1)\Lambda_{(\lambda,\gamma,\mu-1)}^{(\alpha,\beta-1)} + \lambda\rho\Lambda_{(\lambda+1,\gamma,\mu-1)}^{(\alpha,\beta)} + \gamma\delta\Lambda_{(\lambda,\gamma+1,\mu-1)}^{(\alpha,\beta)}\right), \tag{32}$$

for $\mathrm{Re}(\alpha) > 1, \mathrm{Re}(\beta) > 1, k \neq 1, \mu \neq \frac{1}{2}$,

$$\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)} = \frac{1}{\beta}\left((\alpha-1)\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha-1,\beta+1)} + (2\mu+1)k^2\Lambda_{(\lambda,\gamma,\mu+1)}^{(\alpha,\beta+1)} - \lambda\rho\Lambda_{(\lambda+1,\gamma,\mu)}^{(\alpha,\beta+1)} - \gamma\delta\Lambda_{(\lambda,\gamma+1,\mu)}^{(\alpha,\beta+1)}\right), \tag{33}$$

for $\mathrm{Re}(\alpha) > 1, \mathrm{Re}(\beta) > 0$, and

$$\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)} = \frac{1}{\alpha}\left((\beta-1)\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha+1,\beta-1)} - (2\mu+1)k^2\Lambda_{(\lambda,\gamma,\mu+1)}^{(\alpha+1,\beta)} + \lambda\rho\Lambda_{(\lambda+1,\gamma,\mu)}^{(\alpha+1,\beta)} + \gamma\delta\Lambda_{(\lambda,\gamma+1,\mu)}^{(\alpha+1,\beta)}\right), \tag{34}$$

for $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 1$.

Some of these recurrence relations, such as (33) and (34), can be particularly used to give analytic continuations for the integral $\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;\ k)$ with respect to α or β. For instance, if we suppose $\alpha+\beta > 1$, then, a repeated application of (33) can be used to evaluate of $\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;\ k)$ when $\mathrm{Re}(\beta) < 0$ and $\beta \neq -1, -2, \ldots$.

Asymptotic Relations

In this section, we will give some asymptotic relations of $\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;\ k)$ when k^2 is very close to unity. These relations are derived by using some of the analytic continuation formulas of the related Gauss hypergeometric function.

The first of these asymptotic relations can be obtained when we use (Lebedev, 1965, (9.5.7)) in (12),

$$\begin{aligned} &\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;\ k) = \\ &\quad \frac{1}{(1-\rho)^{\lambda}(1+k^2)^{\mu+1/2}} \sum_{l,m=0}^{\infty} \frac{(\lambda)_l(\gamma)_m}{l!m!}\left(\frac{\rho}{\rho-1}\right)^l(-\delta)^m \\ &\quad \times \left\{ \frac{\Gamma(\alpha+l+m)\Gamma(\beta-\mu-\frac{1}{2})}{\Gamma(\alpha+\beta+l+m-\mu-\frac{1}{2})}\ {}_2F_1\left(\alpha+l+m, \mu+\tfrac{1}{2};\ \mu-\beta+\tfrac{3}{2};\ \frac{1-k^2}{1+k^2}\right) \right. \\ &\quad + \left(\frac{1-k^2}{1+k^2}\right)^{\beta-\mu-\frac{1}{2}} \frac{\Gamma(\beta)\Gamma(\mu-\beta+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})} \\ &\quad \left. \times\ {}_2F_1\left(\beta, \alpha+\beta+l+m-\mu-\tfrac{1}{2};\ \beta-\mu+\tfrac{1}{2};\ \frac{1-k^2}{1+k^2}\right)\right\}, \end{aligned} \tag{35}$$

for $k^2 < 1$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\rho) < \frac{1}{2}$, $|\delta| < 1$, $\mu-\beta+\frac{1}{2} \neq 0, \pm1, \pm2, \ldots$, which gives an asymptotic representation for $\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;\ k)$ when k^2 is close to 1.

Relation (35) is no longer applicable if $\mu-\beta+\frac{1}{2}$ is an integer. First, if $\mu+\frac{1}{2} = \beta-q,\ q = 0, 1, 2, \ldots$, one can use (Lebedev, 1965, (9.7.5)) into (12) to obtain

$$\begin{aligned} &\Lambda_{(\lambda,\gamma,\beta-q-\frac{1}{2})}^{(\alpha,\beta)}(\rho,\delta;\ k) = \\ &\quad \frac{1}{(1-\rho)^{\lambda}(1+k^2)^{\beta-q}} \sum_{l,m=0}^{\infty} \frac{(\lambda)_l(\gamma)_m}{l!m!}\left(\frac{\rho}{\rho-1}\right)^l(-\delta)^m \\ &\quad \times \left\{ \frac{1}{(\alpha+l+m)_q} \sum_{n=0}^{q-1} \frac{(-1)^n(q-n-1)!(\alpha+l+m)_n(\beta-q)_n}{n!}\left(\frac{1-k^2}{1+k^2}\right)^n \right. \\ &\quad + (1-\beta)_q \sum_{n=0}^{\infty} \frac{(\alpha+l+m+q)_n(\beta)_n}{(q+n)!n!}[\psi(n+1)+\psi(q+n+1) \\ &\quad \left. - \psi(\alpha+l+m+q+n) - \psi(\beta+n) - \log\left(\frac{1-k^2}{1+k^2}\right)]\left(\frac{1-k^2}{1+k^2}\right)^{q+n}\right\}, \end{aligned} \tag{36}$$

for $k^2 < 1$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\rho) < \frac{1}{2}$, $|\delta| < 1$, $q = 0, 1, 2, \ldots$, $\beta - q \neq 0, -1, -2, \ldots$

Secondly, if $\mu + \frac{1}{2} = \beta + q,\ q = 0, 1, 2, \ldots$, then using (Lebedev, 1965, (9.7.5)) into (12) yields

$$\begin{aligned}
&\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\beta+q-\frac{1}{2})}(\rho, \delta;\ k) = \\
&\quad \frac{1}{(1-\rho)^{\lambda}(1-k^2)^{q}(1+k^2)^{\beta}} \sum_{l,m=0}^{\infty} \frac{(\lambda)_l(\gamma)_m}{l!m!} \left(\frac{\rho}{\rho-1}\right)^{l} (-\delta)^m \\
&\quad \times \left\{ \frac{1}{(\beta)_q} \sum_{n=0}^{q-1} \frac{(-1)^n (q-n-1)!(\alpha+l+m-q)_n(\beta)_n}{n!} \left(\frac{1-k^2}{1+k^2}\right)^{n} \right. \\
&\quad + (1-\alpha-l-m)_q \sum_{n=0}^{\infty} \frac{(\alpha+l+m)_n(\beta+q)_n}{(q+n)!n!} [\psi(n+1) + \psi(q+n+1) \\
&\quad \left. - \psi(\alpha+l+m+n) - \psi(\beta+q+n) - \log\left(\frac{1-k^2}{1+k^2}\right)] \left(\frac{1-k^2}{1+k^2}\right)^{q+n} \right\},
\end{aligned} \tag{37}$$

for $k^2 < 1$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\rho) < \frac{1}{2}$, $|\delta| < 1$, $q = 0, 1, 2, \ldots$, $\alpha - q \neq 0, -1, -2, \ldots$.

Further asymptotic formula can be obtained if we use (Lebedev, 1965, (9.5.9)) into (13) that is

$$\begin{aligned}
&\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho, \delta;\ k) = \\
&\quad \frac{1}{(1-\rho)^{\lambda}(1-k^2)^{\mu+1/2}} \sum_{l,m=0}^{\infty} \frac{(\lambda)_l(\gamma)_m}{l!m!} \left(\frac{\rho}{\rho-1}\right)^{l} (-\delta)^m \\
&\quad \times \left\{ \left(\frac{k^2-1}{2k^2}\right)^{\beta} \frac{\Gamma(\beta)\Gamma(\mu-\beta+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})}\ {}_2F_1\left(\beta, 1-\alpha-l-m;\ \beta-\mu+\tfrac{1}{2};\ \frac{k^2-1}{2k^2}\right) \right. \\
&\quad + \left(\frac{k^2-1}{2k^2}\right)^{\mu+\frac{1}{2}} \frac{\Gamma(\alpha+l+m)\Gamma(\beta-\mu-\frac{1}{2})}{\Gamma(\alpha+\beta+l+m-\mu-\frac{1}{2})} \\
&\quad \left. \times\ {}_2F_1\left(\mu+\tfrac{1}{2}, \mu-\alpha-\beta-l-m+\tfrac{3}{2};\ \mu-\beta+\tfrac{3}{2};\ \frac{k^2-1}{2k^2}\right) \right\},
\end{aligned} \tag{38}$$

for $\frac{1}{3} < k^2 < 1$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\rho) < \frac{1}{2}$, $|\delta| < 1$, $\beta - \mu - \frac{1}{2} \neq 0, \pm 1, \pm 2, \ldots$, which can be used when k^2 is close to 1.

Relation (38) is no longer applicable if $\mu - \beta + \frac{1}{2}$ is an integer. For the case $\mu + \frac{1}{2} = \beta + q,\ q = 0, 1, 2, \ldots$, using (Lebedev, 1965, (9.7.7)) into (13) gives

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\beta+q-\frac{1}{2})}(\rho,\delta;\ k) = \frac{(2k^2)^{-\beta}}{(1-\rho)^{\lambda}(1-k^2)^q}\sum_{l,m=0}^{\infty}\frac{(\lambda)_l(\gamma)_m}{l!m!}\left(\frac{\rho}{\rho-1}\right)^l(-\delta)^m \times\left\{\frac{1}{(\beta)_q}\sum_{n=0}^{q-1}\frac{(q-n-1)!(\beta)_n(1-\alpha-l-m)_n}{n!}\left(\frac{1-k^2}{2k^2}\right)^n + (-1)^q\sum_{n=0}^{\infty}\frac{(\beta+q)_n(1-\alpha-l-m)_{q+n}}{(q+n)!n!}[\psi(n+1)+\psi(q+n+1) - \psi(\beta+q+n)-\ \psi(\alpha+l+m-q-n)+\log\left(\frac{2k^2}{1-k^2}\right)]\left(\frac{k^2-1}{2k^2}\right)^{q+n}\right\}, \tag{39}$$

for $\frac{1}{3} < k^2 < 1$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\rho) < \frac{1}{2}$, $|\delta| < 1$, $q = 0, 1, 2, \ldots$, $\alpha \neq 1, 2, \ldots$.

For the other case, when $\mu + \frac{1}{2} = \beta - q,\ q = 0, 1, 2, \ldots$, making use (Lebedev, 1965, (9.7.8)) into (13) yields

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\beta-q-\frac{1}{2})}(\rho,\delta;\ k) = \frac{(2k^2)^{q-\beta}}{(1-\rho)^{\lambda}}\sum_{l,m=0}^{\infty}\frac{(\lambda)_l(\gamma)_m}{l!m!}\left(\frac{\rho}{\rho-1}\right)^l(-\delta)^m \times\left\{\frac{1}{(\alpha+l+m)_q}\sum_{n=0}^{q-1}\frac{(q-n-1)!(\beta-q)_n(1-\alpha-l-m-q)_n}{n!}\left(\frac{1-k^2}{2k^2}\right)^n + (-1)^q(1-\beta)_q\sum_{n=0}^{\infty}\frac{(\beta)_n(1-\alpha-l-m)_n}{(q+n)!n!}[\psi(n+1)+\psi(q+n+1) - \psi(\beta+n)-\ \psi(\alpha+l+m-n)+\log\left(\frac{2k^2}{1-k^2}\right)]\left(\frac{k^2-1}{2k^2}\right)^{q+n}\right\}, \tag{40}$$

for $\frac{1}{3} < k^2 < 1$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\rho) < \frac{1}{2}$, $|\delta| < 1$, $q = 0, 1, 2, \ldots$, $\alpha \neq 1, 2, \ldots$, $\beta - q \neq 0, -1, -2, \ldots$.

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