Alabama Journal of Mathematics
Spring/Fall 2011

Möbius Group, Bergman Metric on the Unit Disk and Reproducing Formulas

Zhijian Wu
The University of Alabama, Tuscaloosa, AL

Invited Lewis-Parker Lecture at the annual meeting of AACTM; Huntingdon College; Montgomery, AL; February 19, 2011

Introduction

I would like to thank the AACTM for inviting me to give the 2011 Lewis-Parker Lecture. This paper, which is based on the lecture, revisits some basic ideas and results about the Möbius group and Bergman metric on the unit disk, and discusses the theory of atomic decomposition and some applications. We refer the reader to the references listed at the end of this paper for more information.

Möbius group

Denote the complex plane by $\mathbb{C}$ and the extended complex plane by $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$. A Möbius transformation of $\hat{\mathbb{C}}$ is a rational function of the form

$$\varphi(z) = \frac{az+b}{cz+d}$$

with coefficients a, b, c and d in $\mathbb{C}$ satisfying $ad - bc \neq 0$.

It is standard that if $\varphi(z) = \frac{az+b}{cz+d}$ is a Möbius transformation, then $\varphi^{-1}(z) = \frac{dz-b}{-cz+a}$ is also a Möbius transformation, and

1. φ is one-to-one and onto;
2. φ is the composition of translation, dilation, and the inversion;
3. φ maps circles to circles.

Under the composition, the set of all Möbius transformations forms a group, called the Möbius group.

As an example of Möbius transformations map circles to circles, we check the image of the circle $|z - z_0| = r$ ($r > 0$), under the inversion $w = \varphi(z) = \frac{1}{z}$. Indeed, it is clear that the image is the circle $|w| = \frac{1}{r}$ if $z_0 = 0$. If $z_0 \neq 0$, the image under φ can be described by all $w \in \hat{\mathbb{C}}$ satisfying

$$\left|\frac{1}{w} - z_0\right| = r \iff |1 - z_0 w| = r|w| \quad \text{or equivalently}$$

$$\begin{cases} 1 - 2\mathrm{Re}(z_0 w) = 0, & \text{if } |z_0| = r; \\ \left|w - \frac{\overline{z_0}}{|z_0|^2 - r^2}\right| = \frac{r}{\left||z_0|^2 - r^2\right|}, & \text{if } |z_0| \neq r. \end{cases}$$

The first equation above represents a line (circle with radius ∞), and the second equation is a circle with center $\frac{\overline{z_0}}{|z_0|^2 - r^2}$ and radius $\frac{r}{\left||z_0|^2 - r^2\right|}$.

Denote by $\mathbb{D}$ the unit disk of the complex plane $\mathbb{C}$ and by $\partial\mathbb{D}$ the unit circle, which is the boundary of $\mathbb{D}$. The Möbius group on the unit disk, denoted by $\mathrm{Aut}(\mathbb{D})$, is the set of all conformal self-map on $\mathbb{D}$. There are infinitely many such conformal mappings. For example, for any fixed $a \in \mathbb{D}$, it is not hard to verify that the Möbius transformation

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z},$$

which interchanges 0 and a, maps $\mathbb{D}$ to itself conformally, and maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$. It is also easy to check that $\varphi_a^{-1} = \varphi_a$.

To characterize $\mathrm{Aut}(\mathbb{D})$, we need the following lemma (see, for example, (Garnett, 2007)).

Schwarz's Lemma. *If an analytic function $f : \mathbb{D} \to \overline{\mathbb{D}}$ satisfies $f(0) = 0$, then*

$$|f(z)| \leq |z|, \ \forall z \in \mathbb{D} \setminus \{0\} \quad \textit{and} \quad |f'(0)| \leq 1.$$

Equality holds at some point z or $|f'(0)| = 1$ if and only if $f(z) = cz$ for all $z \in \mathbb{D}$ with c a unit module constant.

Proof sketch: Define $g : \mathbb{D} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z)}{z}, & \text{if } z \in \mathbb{D} \setminus \{0\}; \\ f'(0), & \text{if } z = 0. \end{cases}$$

Then g is analytic in $\mathbb{D}$. Using the Maximum Modulus Principle (see, for example, (Conway, 1973)), which says that if f is an analytic function then the modulus $|f|$ cannot exhibit a true local maximum properly within the domain of f, we can conclude that

$$|g(z)| \leq 1, \qquad \text{for all } z \in \mathbb{D},$$

and equality holds at some point inside $\mathbb{D}$ implies $|g(z)| \equiv 1$ in $\mathbb{D}$.

Schwarz's Lemma can be used to prove the following characterization of the Möbius group $\mathrm{Aut}(\mathbb{D})$:

$$\mathrm{Aut}(\mathbb{D}) = \{c\varphi_a : a \in \mathbb{D} \text{ and } c \in \mathbb{C} \text{ with } |c| = 1\}.$$

In fact, we know from the above that $c\varphi_a \in \mathrm{Aut}(\mathbb{D})$ if $a \in \mathbb{D}$ and $c \in \mathbb{C}$ with $|c| = 1$. Suppose $f \in \mathrm{Aut}(\mathbb{D})$.

Then there is a point $a \in \mathbb{D}$ such that $f(a) = 0$. Consider $F = f \circ \varphi_a^{-1}$ and $G = \varphi_a \circ f^{-1}$. Clearly $F, G \in \mathrm{Aut}(\mathbb{D})$ and $F(0) = G(0) = 0$. Using Schwarz's Lemma, we have

$$|F'(0)| \le 1 \quad \textit{and} \quad |G'(0)| \le 1 .$$

On the other hand, $(G \circ F)(z) = z$ implies $G'(0)F'(0) = 1$. Therefore we must have

$$|F'(0)| = 1 \text{ and } |G'(0)| = 1 .$$

Hence, by Schwarz's Lemma, $(f \circ \varphi_a^{-1})(z) = F(z) = cz$, or equivalently $f = c\varphi_a$.

Möbius group $\mathrm{Aut}(\mathbb{D})$ has some interesting properties. For example, it easy easy to check that $\varphi_a'(z) = -\frac{1-|a|^2}{1-\bar{a}z}$, and

$$\begin{aligned} 1 - |\varphi_a(z)|^2 &= 1 - \left|\frac{a-z}{1-\bar{a}z}\right|^2 \\ &= \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2} \\ &= |\varphi_a'(z)|(1-|z|^2). \end{aligned}$$

We have, therefore

$$\frac{|\varphi'(z)|}{1-|\varphi(z)|^2} = \frac{1}{1-|z|^2}, \qquad \forall \varphi \in \mathrm{Aut}(\mathbb{D}). \tag{0.1}$$

Bergman metric

Equation (0.1) suggests that the measure

$$ds = \frac{|dz|}{1-|z|^2}$$

is invariant under the action of $\mathrm{Aut}(\mathbb{D})$. We refer this measure as Möbius invariant.

The hyperbolic length of a rectifiable arc $\gamma : [0,1] \to \mathbb{D}$ is defined by

$$\ell(\gamma) = \int_0^1 \frac{2|\gamma'(t)|dt}{1-|\gamma(t)|^2}.$$

The Bergman distance of two points z and w in $\mathbb{D}$ is defined by

$$d(z,w) = \inf\{\ell(\gamma) : \gamma([0,1]) \subset \mathbb{D},\ \gamma(0) = z \text{ and } \gamma(1) = w\}.$$

Clearly, the Bergman metric is Möbius invariant. Therefore

$$\begin{aligned} d(z,w) &= d(0, \varphi_z(w)) \\ &= \inf \int_0^1 \frac{2|\gamma'(t)|dt}{1-|\gamma(t)|^2} \\ &= 2\int_0^{|\varphi_z(w)|} \frac{dr}{1-r^2} \\ &= \log \frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}. \end{aligned}$$

The Bergman metric is so called because it is derived from the Bergman kernel (see, for example, (Krantz, 2001)). In general, let Ω be a domain in $\mathbb{C}^n$ and $K_\Omega(z,w)$ be the Bergman kernel on Ω. Define a Hermitian metric by

$$g_{ij}(z) = \frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log K_\Omega(z,z), \quad z \in \Omega .$$

The length of a rectifiable arc $\gamma : [0,1] \to \Omega$ is defined as

$$\ell(\gamma) = \int_0^1 \sqrt{\sum_{i,j} g_{ij}(\gamma(t)) \frac{\partial \gamma_i(t)}{\partial t} \frac{\partial \overline{\gamma_j(t)}}{\partial t}} dt.$$

We will see later that the Bergman kernel for the unit disk $\mathbb{D}$ is

$$K_{\mathbb{D}}(z,w) = \frac{1}{(1-\overline{w}z)^2}.$$

Therefore

$$g(z) = \frac{\partial^2}{\partial z \partial \bar{z}} \log \frac{1}{1-|z|^2} = \frac{1}{(1-|z|^2)^2}.$$

Bergman metric is also referred as Poincaré metric in view of $\mathbb{C}^n = \mathbb{R}^{2n}$. For example, the unit ball of $\mathbb{R}^n$, the associated metric tensor of the Poincaré metric is given by

$$ds^2 = 4\frac{\sum_j dx_j^2}{(1-\sum_j x_j^2)^2}.$$

For fixed $a \in \mathbb{D}$ and $t > 0$, the Bergman disk of center a and radius t is defined by

$$D(a,t) = \{\, w \in \mathbb{D} \,:\, d(w,a) < t\} .$$

Since $d(w,a) < t$ is equivalent to

$$|\varphi_a(w)| < \frac{e^t - 1}{e^t + 1}$$

and φ_a is a Möbius transformation, we conclude that $D(a,t)$ is also a Euclidean disk. The Euclidean center and radius of $D(a,t)$ can be easily calculated from the equation $|\varphi_a(w)| = \frac{e^t-1}{e^t+1}$. They are, respectively,

$$\frac{1-r^2}{1-|a|^2r^2} a \quad \text{and} \quad r\frac{1-|a|^2}{1-|a|^2r^2}$$

where $r = \frac{e^t-1}{e^t+1}$.

For a set E in $\mathbb{D}$, denote the normalized area of E by

$$|E| = \int_E \frac{dxdy}{\pi}.$$

For $t > 0$, let $r = \frac{e^t-1}{e^t+1}$. The following result is standard (see, for example, (Zhu, 2007)).

- $|D(a,t)| = r^2 \dfrac{(1-|a|^2)^2}{(1-|a|^2r^2)^2}$;
- $\inf\{|1-\bar{a}z| : z \in D(a,t)\} = \dfrac{1-|a|^2}{(1-r|a|)^2}$;

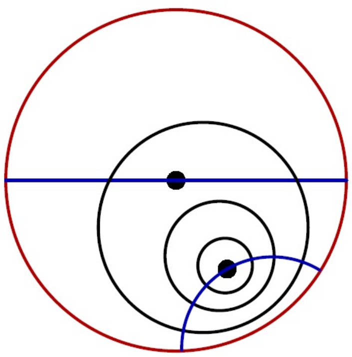
- $\sup\{|1-\bar{a}z| : z \in D(a,t)\} = \dfrac{1-|a|^2}{(1+r|a|)^2}$;
- There exists a constant $C(t)$ such that

$$\left|\frac{1-\bar{a}z}{1-\bar{a}w} - 1\right| \le C(t)d(z,w)$$

for all $a, z, w \in \mathbb{D}$ and $d(z,w) \le t$.

We note that under the Bergman metric, the geodesics in the unit disk $\mathbb{D}$ are circular arcs in $\mathbb{D}$ orthogonal to $\partial\mathbb{D}$. If these arcs are called lines, then the parallel postulate is false. Therefore the Bergman metric on $\mathbb{D}$ provides a good model for Lobachevsky geometry.

The following figure illustrates the Bergman disks $D(a,t)$ with $t = 1/2, 1$ and 2. It illustrates also geodesics passing through a and 0 respectively.



δ-lattice in $\mathbb{D}$

For given $\delta > 0$, a sequence $\{z_j\}$ in $\mathbb{D}$ is called a δ-lattice in the Bergman metric if $d(z_i, z_j) \ge \delta/5$ for $i \ne j$ and $\{D(z_k, \delta)\}$ covers $\mathbb{D}$, i.e.,

$$\mathbb{D} = \bigcup_{j=1}^{\infty} D(z_j, \delta)\,.$$

It is not hard to see that there exists an integer $M_\delta > 0$ such that every point in $\mathbb{D}$ is covered by $\{D(z_k, \delta)\}$ at most M times.

The following theorem, which is proved in (Coifman and Rochberg, 1980) (see also (Zhu, 2007)), says that a δ-lattice always exists.

Theorem 1. *For any $\delta > 0$, there exists a δ-lattice in $\mathbb{D}$.*

Mathematical induction can be employed to prove above theorem. We provide here a constructive approach.

Let $z_{0,0} = 0$. For any integer $k > 0$, consider the Bergman circle $\gamma_k = \{z \in \mathbb{D} : d(z,0) = k\delta/5\}$. It is easy to compute

$$\ell(\gamma_k) = \int_0^1 \frac{2|\gamma_k'(t)|dt}{1-|\gamma_k(t)|^2} = \frac{4\pi R_k}{1-R_k^2}.$$

Here $R_k = \frac{e^{k\delta/5}-1}{e^{k\delta/5}+1}$.

Since the Bergman length of the circle γ_k is finite, there are finite many points $\{z_{k,j} : j = 0, 1, \cdots, J_k\}$ evenly spread out on γ_k, such that the Bergman distance between any two points next to each other is a constant which is in $[\delta/5, 2\delta/5)$.

It is easy to see that for any integer $k \ge 0$, the smallest Bergman distance between points in $\{z_{k,j} : j = 0, 1, \cdots, J_k\}$ and points in $\{z_{k+1,j} : j = 0, 1, \cdots, J_{k+1}\}$ is within $[\delta/5, 3\delta/5)$. Therefore it can be seen that the set of the points $\{z_{k,j} : j = 0, 1, \cdots, J_k; k = 0, 1, 2, \cdots\}$ is a δ-lattice.

Given a δ-lattice $\{z_j\}$. There is an associated decomposition of unity as in the following theorem.

Theorem 2. *For any δ-lattice $\{z_j\}$ in $\mathbb{D}$, there are $C^\infty(\mathbb{D})$ functions ψ_j with $0 \le \psi_j(z) \le \psi(z_j) = 1$ and $\mathrm{supp}\psi_j \subseteq D(z_j, \delta)$, such that*

$$\sum_{j=1}^{\infty} \psi_j(z) \equiv 1\,, \qquad \textit{for all } z \in \mathbb{D}\,.$$

In fact, pick any $C^\infty(\mathbb{D})$ function ξ such that $0 \le \xi(z) \le \xi(0) = 1$ and $\mathrm{supp}\xi = D(0,\delta)$. Let $\xi_j = \xi \circ \varphi_{z_j}$. Clearly $\mathrm{supp}\xi_j = D(z_j, \delta)$, $\xi_j \in C^\infty(\mathbb{D})$ and

$$\sum_{j=1}^{\infty} \xi_j(z) \le M_\delta, \qquad \text{for all } z \in \mathbb{D}\,.$$

It is easy to see now that

$$\psi_j = \frac{\xi_j}{\sum_k \xi_k}, \qquad j = 1, 2, \cdots$$

is what needed.

Bergman spaces

For $0 < p < \infty$, Bergman space A_p is the set of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_p^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty\,.$$

Here $dA(z) = \frac{1}{\pi}dxdy$. It is standard that $\mathcal{P}$, the set of all polynomials on $\mathbb{D}$, is dense in A_p.

It is easy to see that for $1 \le p < \infty$, A_p with norm $\|\cdot\|_p$, is a Banach space. Since for any $z \in \mathbb{D}$, the point evaluation is a bounded linear functional of A_p, there exists a function K_z such that

$$f(z) = \int_{\mathbb{D}} f(w)\overline{K_z(w)}dA(w), \quad \forall f \in A_p\,.$$

The above formula is referred as the reproducing formula for Bergman space A_p.

It can be shown that $K_{\mathbb{D}}(w,z) = K_z(w)$, called the Bergman kernel on $\mathbb{D}$, has the expression

$$K_{\mathbb{D}}(w,z) = \frac{1}{(1-\bar{z}w)^2}.$$

Indeed, with $K_{\mathbb{D}}(w,z)$ we can verify the Bergman reproducing formula for all $f \in \mathcal{P}$ in the following. Write $f(z) = \sum_{j\geq 0} f_j z^j$. Write

$$K_{\mathbb{D}}(w,z) = \frac{1}{(1-\bar{z}w)^2} = \sum_{k\geq 0}(k+1)\bar{z}^k w^k .$$

It is easy to check that

$$\begin{aligned} \int_{\mathbb{D}} f(w)\overline{K_{\mathbb{D}}(w,z)}dA(w) &= \sum_{j,k\geq 0}(k+1)f_j z^k \int_{\mathbb{D}} w^j \overline{w}^k dA(w) \\ &= \sum_{j\geq 0}(j+1)f_j z^j \int_{\mathbb{D}} |w|^{2j} dA(w) \\ &= \sum_{j\geq 0} f_j z^j \\ &= f(z) . \end{aligned}$$

Here are some useful properties (see, for example, (Zhu, 2007)).

- If $\varphi \in \mathrm{Aut}(\mathbb{D})$, then

$$K_{\mathbb{D}}(z,w) = \overline{\varphi'(z)} K_{\mathbb{D}}(\varphi(z),\varphi(w))\varphi'(w);$$

- For $1 < p < \infty$, the integral operator

$$\int_{\mathbb{D}} f(w)|K_{\mathbb{D}}(z,w)|dA(w) = \int_{\mathbb{D}} \frac{f(w)}{|1-\overline{w}z|^2} dA(w)$$

is bounded on $L^p(\mathbb{D}, dA)$.

Atomic decomposition

In this section, C and c denote positive constants that may change from one step to the next. We say that two positive functions a and b are equivalent, denoted by $a \asymp b$, if there are two positive constants c and C such that $ca \leq b \leq Ca$.

The following lemma plays an important role in the establishment of the atomic decomposition.

Schur's Lemma. *Suppose $1 < p < \infty$ and $p' = \frac{p}{p-1}$. Suppose $Q(z,w)$ is a positive function on $\mathbb{D}\times\mathbb{D}$. If there is a positive function g on $\mathbb{D}$, such that*

$$\int_{\mathbb{D}} Q(z,w) g^{p'}(w) dA(w) \leq C g^{p'}(z)$$

and

$$\int_{\mathbb{D}} Q(w,z) g^{p}(w) dA(w) \leq C g^{p}(z)$$

hold for all $z \in \mathbb{D}$, then the linear map given by

$$f \mapsto \int_{\mathbb{D}} Q(z,w) f(w) dA(w)$$

is a bounded map on $L^p(\mathbb{D}, dA)$.

Suppose $\alpha > -1$. With the Taylor series

$$\frac{1}{(1-x)^{\alpha+2}} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)} x^n ,$$

it is easy to check that for $f \in \mathcal{P}$, the following reproducing formula holds:

$$f(z) = (\alpha+1)\int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^{\alpha+2}}(1-|w|^2)^{\alpha} dA(w) .$$

In fact, the kernel $\frac{\alpha+1}{(1-\overline{w}z)^{\alpha+2}}$ is the Bergman kernel for the weighted Bergman space A_p^{α}, which is the completion of $\mathcal{P}$ in $L^p(\mathbb{D}, (1-|z|^2)^{\alpha} dA(z))$. Moreover, by Schur's lemma, one can prove that the map

$$f \mapsto (\alpha+1)\int_{\mathbb{D}} \frac{f(w)}{|1-\overline{w}z|^{\alpha+2}}(1-|w|^2)^{\alpha} dA(w)$$

is bounded on $L^p(\mathbb{D}, (1-|z|^2)^{\alpha} dA(z))$ for $1 < p < \infty$.

The following theorem is proved in (Coifman and Rochberg, 1980) under several variables setting. One can refer to (Rochberg, 1985) also.

Atomic Decomposition of A_p^{α}. *Suppose $0 < p < \infty$, $\alpha > -1$ and $b > \max\left(1, \frac{1}{p}\right) + \frac{\alpha+1}{p}$. There exists an $\delta_0 > 0$, such that for $0 < \delta < \delta_0$ and any δ-lattice $\{z_j\}$ in $\mathbb{D}$, we have*

(a) If $f \in A_p^{\alpha}$, then

$$f(z) = \sum_j \lambda_j \frac{(1-|z_j|^2)^{b-\frac{2+\alpha}{p}}}{(1-\bar{z}_j z)^b} \tag{0.2}$$

and

$$\left\|\{\lambda_j\}\right\|_{\ell^p} \leq C\|f\|_{A_p^{\alpha}}.$$

(b) For any $\{\lambda_j\} \in \ell^p$, the function f, defined by (0.2), is in A_p^{α} and

$$\|f\|_{A_p^{\alpha}} \leq C \left\|\{\lambda_j\}\right\|_{\ell^p} .$$

Proof sketch: The part (b) of the theorem is a consequence of Schur's lemma. For part (a), we consider the case of $p > 1$ only.

Starting with the reproducing formula and using the decomposition of unity, we have

$$\begin{aligned} f(z) &= (b-1)\int_{\mathbb{D}} \frac{f(w)(1-|w|^2)^{b-2}}{(1-\overline{w}z)^b} dA(w) \\ &= (b-1)\sum_j \int_{D(z_j,\delta)} \frac{f(w)(1-|w|^2)^{b-2}}{(1-\overline{w}z)^b}\psi_j(w) dA(w) . \end{aligned}$$

For the integration over the disk $D(z_j,\delta)$ above, if we replace the function $\frac{f(w)(1-|w|^2)^{b-2}}{(1-\overline{w}z)^b}$ by $\frac{f(z_j)(1-|z_j|^2)^{b-2}}{(1-\overline{z}_j z)^b}$, we obtain the following approximation for f:

$$A(f)(z) = \sum_j c_j \frac{f(z_j)(1-|z_j|^2)^{b-2}}{(1-\overline{z_j}z)^b}$$

where

$$c_j = (b-1)\int_{D(z_j,\delta)} \psi_j(w)dA(w)$$
$$\le c|D(z_j,\delta)| \asymp (1-|z_j|^2)^2, \qquad j = 1, 2, \cdots .$$

Denote $\Lambda(f) = \{\lambda_j(f)\}$, with $\lambda_j(f) = c_j(1 - |z_j|^2)^{\frac{2+\alpha}{p}-2} f(z_j)$. We can see that the linear operator Λ is bounded from A^α_p to ℓ^p. Indeed, since the mean value property for analytic functions implies

$$|f(z_j)|^p \le \frac{c}{|D(z_j,\delta)|}\int_{D(z_j,\delta)} |f(w)|^p dA(w),$$

we have

$$\begin{aligned} \|\Lambda(f)\|^p_{\ell^p} &\le c\sum_J (1-|z_j|^2)^{2+\alpha}|f(z_j)|^p \\ &\le c\sum_j \int_{D(z_j,\delta)} |f(w)|^p(1-|w|^2)^\alpha dA(w) \\ &\le cM_\delta \|f\|_{A^\alpha_p}. \end{aligned}$$

We can write

$$A(f)(z) = \sum_j \lambda_j(f)\frac{(1-|z_j|^2)^{b-\frac{2+\alpha}{p}}}{(1-\overline{z_j}z)^b}.$$

Let's look at the error of the approximation of f by $A(f)$.

$$\begin{aligned} &|f(z) - A(f)(z)| \\ &\le c\sum_j \int_{D(z_j,\delta)} \left|\frac{f(w)(1-|w|^2)^{b-2}}{(1-\overline{w}z)^b} - \frac{f(z_j)(1-|z_j|^2)^{b-2}}{(1-\overline{z_j}z)^b}\right| \psi_j(w)dA(w) \\ &\le c\sum_j \frac{1}{|1-\overline{z_j}z|^b}\int_{D(z_j,\delta)} |f(w)-f(z_j)|(1-|w|^2)^{b-2}dA(w) \\ &\quad + c\sum_j \int_{D(z_j,\delta)} |f(w)|\left|\frac{(1-|w|^2)^{b-2}}{(1-\overline{w}z)^b} - \frac{(1-|z_j|^2)^{b-2}}{(1-\overline{z_j}z)^b}\right| dA(w) \end{aligned}$$

By properties of Bergman kernel, properties of Bergman spaces and Schur's Lemma, we can prove, from the above estimate, that

$$\|f - A(f)\|_{A^\alpha_p} \le C\delta\|f\|_{A^\alpha_p}, \qquad \forall f \in A^\alpha_p.$$

It is clear that we can choose $\delta > 0$ small, say $\delta < \frac{1}{2C}$, so that the linear operator A is invertible and that

$$A^{-1} = \sum_{n=0}^\infty (I_d - A)^n$$

is bounded (on A^α_p).

For any $f \in A^\alpha_p$, we have $A^{-1}(f) \in A^\alpha_p$ and therefore $\Lambda(A^{-1}(f)) \in \ell^p$. We can write

$$f(z) = A\left(A^{-1}(f)\right)(z) = \sum_j \lambda_j\left(A^{-1}(f)\right)\frac{(1-|z_j|^2)^{b-\frac{2+\alpha}{p}}}{(1-\overline{z_j}z)^b}.$$

This is the part (a) of the atomic decomposition theorem.

Atomic decomposition can be established for many analytic function spaces, such as

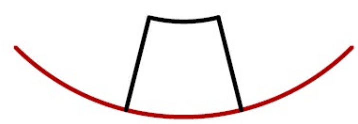
- Besov spaces B_p ($p > 1$):
$$\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2}dA(z) < \infty$$
- Bloch space B:
$$\sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty$$
- BMOA or Q_s ($0 < s \le 1$) spaces (BMOA= Q_1):
$$\|f\|^2_{Q_s} = \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f'(z)|^2\left(\log\frac{1}{|\varphi_a(z)|}\right)^s dA(z) < \infty$$

Atomic decomposition for BMOA was established by Rochberg and Semmes (1986). As a generalization of BMOA space, Q_s space was first introduced in (Aulaskari and Lappan, 1994; Aulaskari et al., 1995) (see also (Xiao, 2001)). It was proved in (Wu and Xie, 2003) that Q_s space is a potential space of Morry space.

To end this paper we introduce the following atomic decomposition theorem for Q_s spaces which was proved in (Wu and Xie, 2002).

The Carleson square in $\mathbb{D}$, based on an arc I, is defined by

$$S(I) = \{z \in \mathbb{D} : |z| > 1 - |I| \text{ and } z/|z| \in I\}.$$



Carleson square in $\mathbb{D}$ based on an arc I

Atomic Decomposition of Q_s. *Suppose $0 < s \le 1$ and $b > \frac{1+s}{2}$. There exists an $\delta_0 > 0$, such that for $0 < \delta < \delta_0$ and any δ-lattice $\{z_j\}$ in $\mathbb{D}$, we have*

(a) If $f \in Q_s$, then

$$f(z) = \sum_j \lambda_j \frac{(1-|z_j|^2)^{b-\frac{s}{2}}}{(1-\bar{z}_j z)^b} \tag{0.3}$$

and

$$\sup_{I\subset\partial\mathbb{D}} \frac{\sum_{j:z_j\in S(I)} |\lambda_j|^2}{|I|^s} \le C\|f\|_{Q_s}.$$

(b) For any $\{\lambda_j\}$ with $\sup_{I\subset\partial\mathbb{D}} \frac{\sum_{j:z_j\in S(I)}|\lambda_j|^2}{|I|^s} < \infty$, the function f, defined by (0.3), is in Q_s and

$$\|f\|_{Q_s} \le C\sup_{I\subset\partial\mathbb{D}} \frac{\sum_{j:z_j\in S(I)} |\lambda_j|^2}{|I|^s}.$$

Above theorem can be proved by establishing the estimate

$$\|f - A(f)\|_{Q_s} \le C\delta\|f\|_{Q_s}, \qquad \forall f \in Q_s.$$

Schur's type estimate and Carleson measures are needed, and we refer the reader to (Wu and Xie, 2002) for details.

References

Aulaskari, R. and Lappan, P. (1994). Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal. In *Complex analysis and its applications (Hong Kong, 1993)*, volume 305 of *Pitman Res. Notes Math. Ser.*, pages 136–146. Longman Sci. Tech., Harlow.

Aulaskari, R., Xiao, J., and Zhao, R. H. (1995). On subspaces and subsets of BMOA and UBC. *Analysis*, 15(2):101–121.

Coifman, R. R. and Rochberg, R. (1980). Representation theorems for holomorphic and harmonic functions in L^p. In *Representation theorems for Hardy spaces*, volume 77 of *Astérisque*, pages 11–66. Soc. Math. France, Paris.

Conway, J. B. (1973). *Functions of one complex variable*. Springer-Verlag, New York. Graduate Texts in Mathematics, 11.

Garnett, J. B. (2007). *Bounded analytic functions*, volume 236 of *Graduate Texts in Mathematics*. Springer, New York, first edition.

Krantz, S. G. (2001). *Function theory of several complex variables*. AMS Chelsea Publishing, Providence, RI. Reprint of the 1992 edition.

Rochberg, R. (1985). Decomposition theorems for Bergman spaces and their applications. In *Operators and function theory (Lancaster, 1984)*, volume 153 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 225–277. Reidel, Dordrecht.

Rochberg, R. and Semmes, S. (1986). A decomposition theorem for BMO and applications. *J. Funct. Anal.*, 67(2):228–263.

Wu, Z. and Xie, C. (2002). Decomposition theorems for Q_p spaces. *Ark. Mat.*, 40(2):383–401.

Wu, Z. and Xie, C. (2003). Q spaces and Morrey spaces. *J. Funct. Anal.*, 201(1):282–297.

Xiao, J. (2001). *Holomorphic Q classes*, volume 1767 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin.

Zhu, K. (2007). *Operator theory in function spaces*, volume 138 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition.