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Generating Non-negative Matrices with Specified Row and Column Sums

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Introduction

The topic announced in the title is a spinoff of a problem recently solved, more or less, in (Foster and Johnson, 2012): Given an alphabet $S = \{s_1, \ldots, s_m\}$, an integer $k > 1$, and a k-dimensional array $[f] = [f(i_1, \ldots, i_k); 1 \le i_1, \ldots, i_k \le m]$ of non-negative numbers, under what conditions on the array does there exist a "statistically stable source" producing text over S (a hypothetically endless stream of letters in S) such that whenever $1 \le i_1, \ldots, i_k \le m$, $f(i_1, \ldots, i_k)$ is the relative frequency of $s_{i_1} \ldots s_{i_k}$ among blocks of k consecutive letters in the source text; in other words, $f(i_1, \ldots, i_k)$ is the probability that a block of k consecutive letters chosen at random from the source text will be $s_{i_1} \ldots s_{i_k}$.

From elementary probability comes a necessary condition on $[f]$ for the existence of such a source, sometimes called the *consistency condition*: for any $i_1, \ldots, i_{k-1} \in \{1, \ldots, m\}$,

$$\sum_{i=1}^{m} f(i, i_1, \ldots, i_{k-1}) = \sum_{j=1}^{m} f(i_1, \ldots, i_{k-1}, j)$$

The common sum is the relative frequency of $s_{i_1} \ldots s_{i_{k-1}}$ among blocks of $k-1$ consecutive integers in the source text.

In the case $k = 2$, therefore, the array $[f(i, j)]$ is an $m \times m$ matrix, which, if it is to be the matrix of relative frequencies of the "digrams" $s_i s_j$ in the text written by some statistically stable text-writing machine, enjoys the property that, for each $j \in 1, \ldots, m$, the sum down the j^{th} column of $[f(i, j)]$ is equal to the sum across the j^{th} row; that common sum will be the relative frequency of the single letter s_j in the text, if such a text exists. There are other requirements on $[f]$ for the existence of a statistically stable source over S: the sum of the entries of $[f]$ must be 1, and the matrix cannot be permutation-equivalent to a diagonal block matrix, but evidently these are not difficult to achieve.

The main problem in generating matrices of relative digram frequencies is in getting the corresponding row and column sums equal; as an additional complication, it would be useful to get those sums equal to pre-specified values, the relative frequencies of the single letters.

During the 2011 Auburn University math REU one of the co-authors of (Hankerson et al., 2003), noting that it was sometimes desirable to have a good supply of non-symmetric digram-relative-frequency matrices for pedagogical purposes, tossed to the participants the vague problem of finding ways of generating such matrices. This paper is a response to that challenge.

Definitions and results

We will say a square matrix is *consistent* if all of its entries are non-negative and, for each i, the sum of the i^{th} row is equal the sum of the i^{th} column. We will say two $n \times m$ matrices, $A = [a_{ij}]$ and $B = [b_{ij}]$, *correspond* if $\sum_{j=1}^{m} a_{ij} = \sum_{j=1}^{m} b_{ij}$ for $1 \le i \le n$ and $\sum_{i=1}^{n} a_{ij} = \sum_{i=1}^{n} b_{ij}$ for $1 \le j \le m$.

A *4-point transformation matrix* is a matrix with row and column sums all equal to zero and with exactly 4 non-zero entries, at positions (i_1, j_1), (i_2, j_2), (i_2, j_1), (i_1, j_2), for some $i_1 \ne i_2$ and $j_1 \ne j_2$. That is, a 4-point transformation matrix is a matrix of zeroes, except for a 2×2 sub-matrix

$$\begin{bmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix}$$

An example of a 3×3 4-point transformation matrix is given below.

$$\begin{bmatrix} \lambda & 0 & -\lambda \\ -\lambda & 0 & \lambda \\ 0 & 0 & 0 \end{bmatrix}$$

Note that adding a 4-point transformation matrix to another matrix A results in a matrix with the same row and column sums as A.

Theorem. *If $A \ne B$ are $n \times m$ non-negative corresponding matrices, then there exists a finite sequence $T_1, \ldots, T_k$ of 4-point transformation matrices such that $B = A + \sum_{i=1}^{k} T_i$ and $A + \sum_{i=1}^{j} T_j$ has non-negative entries for each $j \in \{1, \ldots, k\}$.*

Corollary. *Suppose that $d_1, \ldots, d_m > 0$. Every non-negative consistent matrix with row (and therefore column) sums $d_1, \ldots, d_m$ can be obtained from the diagonal matrix*

$$\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_m \end{bmatrix}$$

by successively adding $m \times m$ 4-point transformation matrices, chosen so that the result at each stage is non-negative.

So, for instance, if a professor teaching a course on, say, data compression wanted for problems and examples a supply of matrices of relative digram frequencies for source text over a 3-element alphabet in which the single letters were to have relative frequencies .5, .3, .2, the professor could simply start with the matrix

$$\begin{bmatrix} .5 & 0 & 0 \\ 0 & .3 & 0 \\ 0 & 0 & .2 \end{bmatrix}$$

and add 4-point transformation matrices chosen to give non-negative results, discarding block diagonal matrices such as

$$\begin{bmatrix} .4 & .1 & 0 \\ .1 & .2 & 0 \\ 0 & 0 & .2 \end{bmatrix}$$

as invalid, and symmetric matrices as uninteresting. For instance, it is possible to go from

$$\begin{bmatrix} .5 & 0 & 0 \\ 0 & .3 & 0 \\ 0 & 0 & .2 \end{bmatrix}$$

to the relatively interesting matrix

$$\begin{bmatrix} .3 & .1 & .1 \\ .15 & .1 & .05 \\ .05 & .1 & .05 \end{bmatrix}$$

in just three steps by adding 4-point transformation matrices. Incidentally, it is shown in (Foster and Johnson, 2012) that there is a machine that will produce statistically stable text over a 3-letter alphabet exhibiting the relative digram frequencies in that resultant matrix.

The corollary obviously follows from the theorem. The theorem will be proven in the next section. Although it has no obvious practical application, it is worth noting that an efficient algorithm can be extracted from the proof for producing $T_1, \ldots, T_k$, given A and B as in the statement of the theorem.

Proofs and intermediate results

Lemma. *Suppose that* $c_1, \ldots, c_k, d_1, \ldots, d_t \geq 0$, *and* $\sum_{i=0}^{k} c_i, \sum_{j=0}^{t} d_j \geq e > 0$. *Then there exist* $\lambda_1, \ldots, \lambda_r \geq 0$ *and partitions* $C_1, \ldots, C_k$ *and* $D_1, \ldots, D_t$ *of* $\{1, \ldots, r\}$ *such that:*

1. *for each* $i \in \{1, \ldots, k\}$, $\sum_{u \in C_i} \lambda_u \leq c_i$
2. *for each* $j \in \{1, \ldots, t\}$, $\sum_{v \in D_j} \lambda_v \leq d_j$
3. $\sum_{i=1}^{r} \lambda_i = e$

Proof. We will go by induction on $k + t$. If $k = t = 1$ then $c_1, d_1 \geq e$: take $r = 1, \lambda_1 = e$, and $C_1 = D_1 = \{1\}$.

Now suppose that $k + t > 2$. If $min(c_1, d_1) \geq e$, take $r = 1, \lambda_1 = e, C_1 = D_1 = \{1\}$, and $C_i = D_j = \emptyset$ for all $i, j > 1$. If, without loss of generality, $c_1 = min(c_1, d_1) < e$ take $\lambda_1 = c_1$ and set $C_1 = \{1\}$. Applying the induction hypothesis to the sequences $c_2, \ldots, c_k$ and $d_1 - c_1, d_2, \ldots, d_t$, with e replaced by $e - c_1$, we obtain $\lambda_2, \ldots, \lambda_r \geq 0$ and partitions $C_2, \ldots, C_k$ and $D'_1, \ldots, D'_t$ of $\{2, \ldots, r\}$ such that (i) for each $i \in \{2, \ldots, k\}, \sum_{u \in C_i} \lambda_u \leq c_i$; (ii) for each $j \in \{2, \ldots, t\}, \sum_{v \in D'_j} \lambda_v \leq d_j$ and $\sum_{v \in D'_1} \lambda_v \leq d_1 - c_1$; and (iii) $\sum_{i=2}^{r} \lambda_i = e - c_1$.

Take $D_1 = D'_1 \cup \{1\}$, and $D_j = D'_j$ for $j \geq 2$. It is straightforward to see that $\lambda_1, \lambda_2, \ldots, \lambda_r$ and the partitions $C_1, \ldots, C_k$ and $D_1, \ldots, D_t$ of $\{1, \ldots, r\}$ satisfy the requirements (i), (ii), and (iii). □

Proof of Theorem. Let A and B be as hypothesized in the theorem. Note that if either $m = 1$ or $n = 1$, and A and B are corresponding, then $A = B$. So $m, n \geq 2$. We will go first by induction on n, and then by induction on m.

If $m = n = 2$ then it is easy to see that $T = A - B$ is a 4-point transformation matrix. Now suppose that $n > m = 2$. If any row of A were equal to the corresponding row of B, we could replace A and B by A' and B', $(n-1) \times 2$ corresponding matrices, $A \neq B$, and invoke the induction hypothesis to obtain $(n-1) \times 2$ 4-point transformation matrices $T'_1, \ldots, T'_k$ satisfying the conclusion of the theorem with respect to A' and B'. Then forming $T_1, \ldots, T_k$ by inserting a zero row into each of $T'_1, \ldots, T'_k$, at the row number equal to that of the row removed before the inductive step, will give a sequence satisfying the conclusion of the theorem with respect to A and B. So we may suppose that $[a_{i1}, a_{i2}] \neq [b_{i1}, b_{i2}]$ for every $i = 1, \ldots, n$.

If we can find 4-point transformation matrices $T_1, \ldots, T_s$ so that $A + \sum_{i=1}^{j} T_i$ is non-negative, $1 \leq j \leq s$, and the last row of $A + \sum_{i=1}^{s} T_i$ is $[b_{n1}, b_{n2}]$, then either $A + \sum_{i=1}^{s} T_i = B$ and we would be done with the induction on n, or, by the argument above, there would exist 4-point transformation matrices $T_{s+1}, \ldots, T_k$ such that $T_1, \ldots, T_k$ satisfy the conclusion of the theorem with respect to A and B. We will show that such $T_1, \ldots, T_s$ exist.

Without loss of generality, $a_{n1} < b_{n1}$. Keep in mind that $b_{n1} - a_{n1} = a_{n2} - b_{n2}$. Because $\sum_{i=1}^{n}(b_{i1} - a_{i1}) = 0$, we have that $b_{n1} - a_{n1} = \sum_{i=1}^{n-1}(a_{i1} - b_{i1})$. Let $P = \{i \in \{1, \ldots, n-1\} \mid a_{i1} - b_{i1} > 0\}$. For purposes of simplifying the discussion let $P = \{1, \ldots, s\}$. Then $0 < b_{n1} - a_{n1} \leq \sum_{i=1}^{s}(a_{i1} - b_{i1})$. Therefore, there exist $\lambda_1, \ldots, \lambda_s, 0 < \lambda_i \leq a_{i1} - b_{i1}$, such that $\sum_{i=1}^{s} \lambda_i = b_{n1} - a_{n1}$. Let T_i be the 4-point transformation matrix with λ_i in positions $(n, 1)$ and $(i, 2)$, and $-\lambda_i$ in positions $(i, 1)$ and $(n, 2)$. Observing that $a_{i1} - \lambda_i \geq b_{i1} \geq 0$ and that for each $j \in \{1, \ldots, s\}$, $a_{n2} - \sum_{i=1}^{j} \lambda_i \geq a_{n2} - \sum_{i=1}^{s} \lambda_i = a_{n2} - (b_{n1} - a_{n1}) = a_{n2} - (a_{n2} - b_{n2}) = b_{n2} \geq 0$, we see that the list $T_1, \ldots, T_s$ has the desired properties. Thus, the theorem holds if $m = 2$, for all n.

Now we suppose that $m > 2$, and that the conclusion holds for m replaced by $m - 1$, for all n. If the last columns of A and B are the same, then we may delete them, invoke the induction hypothesis, and we are done. Therefore, if we can go from A, through the non-negative matrices, by adding 4-point transformation matrices, to a matrix whose last column agrees with the last column of B at at least one more place than does the last column of A, then we are done, because we can then repeat the procedure until the last column of the resultant matrix is the same as the last column of B.

Since $\sum_{i=1}^{n} a_{im} = \sum_{i=1}^{n} b_{im}$, if the last columns of A and B are not identical, then $a_{im} < b_{im}$ for some i. To simplify the discussion, assume $a_{nm} < b_{nm}$. Let $P = \{j \in \{1, \dots, m-1\} \mid a_{nj} - b_{nj} > 0\}$ and let $Q = \{i \in \{1, \dots, n-1\} \mid a_{im} - b_{im} > 0\}$. To simplify discussion, we may assume that $P = \{1, \dots, k\}$ and $Q = \{1, \dots, t\}$.

By the argument by which it was shown that $0 < b_{n1} - a_{n1} \leq \sum_{i \in P}(a_{i1} - b_{i1})$ in the $m = 2$ case, we have that $0 < b_{nm} - a_{nm} \leq \sum_{j \in P}(a_{nj} - b_{nj}), \sum_{i \in Q}(a_{im} - b_{im})$. Thus, the hypothesis of the Lemma holds, with $e = b_{nm} - a_{nm}, c_j = a_{nj} - b_{nj}, j \in P, d_i = a_{im} - b_{im}, i \in Q$. By the Lemma, there exist $\lambda_1, \dots, \lambda_r \geq 0$ and partitions $C_1, \dots, C_k$ and $D_1, \dots, D_t$ of $\{1, \dots, r\}$ such that $\sum_{i=1}^{r} \lambda_i = b_{nm} - a_{nm}$, $\sum_{u \in C_j} \lambda_u \leq a_{nj} - b_{nj}$, for $j = 1, \dots, k$, and $\sum_{v \in D_i} \lambda_v \leq a_{im} - b_{im}$, for $i = 1, \dots, t$.

For each $s \in \{1, \dots, r\}$ such that $\lambda_s > 0$ let T_s be the 4-point transformation matrix with λ_s in positions (n, m) and (i, j), and $-\lambda_s$ in positions (n, j) and (i, m), where i and j are such that $s \in C_j \cap D_i$. It is straightforward to verify that A+(any sum of the T_s) is non-negative, and $A' = A + \sum_{\{s \mid \lambda_s > 0\}} T_s$ has b_{nm} in the (n, m) position. Furthermore, the only positions besides the (n, m) position where A' differs from A in the last column are the positions (i, m) such that $a_{im} - b_{im} > 0$. That is, A' differs from A in the last column only at places where B differs from A, so A' agrees with B at least at one more position than does A, the position (n, m). □

This is one of those proofs that is easy to "see", but difficult to write, and probably even more difficult to read. However, if the reader has "seen" the proof through the tedium, it is hoped that the reader will agree that the theorem is just the case $k = 2$ of a more general theorem about k-dimensional matrices, or arrays, of non-negative real numbers. In the general case, the role of the 4-point transformation matrices is played by 2^k-point transformation arrays, which are arrays in which every line sum is zero, and the entries are all zero except in a $2 \times 2 \times \ldots \times 2$ subarray, in which every entry is either λ or $-\lambda$ for some $\lambda \neq 0$.

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