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Convergence of Certain Classes of Numerical Series and Series of Functions

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Introduction

Series of the form $\sum \frac{1}{n}$ and $\sum(-1)^{n-1}\frac{1}{n}$ are known as harmonic and alternating harmonic series, respectively. An interesting property of these series is that the former diverges while the latter converges. In general, then given a sequence $\{a_n\}$ of $1s$ and $-1s$, it is interesting to investigate the convergence of the corresponding series $\sum a_n/n$. It turns out that in some cases the corresponding series converges while in others diverges. Feist and Naimi in their paper 'Almost Alternating Harmonic Series' (See (Feist and Naimi, 2004)) investigated the properties of such series, where they came up with necessary and sufficient conditions on a sequence $\{a_n\}$ of real numbers, for which the corresponding series converges.

In this paper, we will first show that similar necessary and sufficient conditions on an arbitrary sequence $\{a_n\}$ of real numbers result in a convergent series of the form $\sum a_n/b_n$, where $\{b_n\}$ is any, non-void, convergent sequence of real numbers. Then, we will continue our discussion for sequences and series of real functions on some subset $E \in \mathbb{R}$. Namely, we will demonstrate certain conditions under which similar statements as those presented for sequences of real numbers hold in this case as well.

Some Technical Background

Definition 1. Let $\{a_n\}$ be a sequence of $1s$ and $-1s$. Let P_n denote the number of $1s$ and Q_n the number of $-1s$ among the fist n terms. We say that $\{a_n\}$ is almost alternating if $P_n/Q_n \to 1$ as $n \to \infty$.

Definition 2. Let $\{a_n\}$ be a sequence of $1s$ and $-1s$. Define the n^{th} partial average of $\{a_n\}$ to be

$$A_n = \frac{1}{n}\sum_{i=1}^{n} a_i.$$

We say that $\{a_n\}$ is almost alternating if $A_n \to 0$ as $n \to \infty$.

Note: The n^{th} partial average A_n can be defined in exactly the same way for any sequence of real numbers as well.

Remark: We invite the readers to show that Definition 1 and Definition 2 are equivalent!

Definition 3. Let $\{a_n\}$ be an almost alternating sequence. Then, we say that $\sum a_n/n$ is an almost alternating harmonic series.

Theorem 4. *Given two sequences $\{a_n\}$ and $\{b_n\}$, put $S_k = \sum_{n=1}^{k} a_n$ if $n \geq 0$ and $S_{-1} = 0$. Then if $0 \leq p \leq q$, we have*

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} S_n(b_n - b_{n+1}) + S_q b_q - S_{p-1} b_p. \quad (1)$$

In particular, if $p = 1$ and $q = k$ the above formula becomes

$$\sum_{n=1}^{k} a_n b_n = S_k b_{k+1} + \sum_{n=1}^{k} S_n(b_n - b_{n+1}). \quad (2)$$

This is known as *summation by parts* and is extensively used in investigating the nature of series of the form $\sum a_n b_n$.

Preliminaries

Below we state some of the results by Naimi and Feist, relevant to our topic.

Necessary and Sufficient Conditions for Convergence

Theorem 5. *Let $\{a_n\}$ be a sequence of real numbers. If $\sum_{n=1}^{\infty} a_n/n$ converges, then $\lim_{n\to\infty} A_n = 0$.*

Theorem 6. *Let $\{a_n\}$ be a sequence of real numbers. Then $\sum_{n=1}^{\infty} a_n/n$ converges if and only if $\lim_{n\to\infty} A_n = 0$ and $\sum_{n=1}^{\infty} A_n/n$ converges.*

Lemma 7. *Let $\{x_n\}$ be a sequence of real numbers such that $n|x_{n+1} - x_n|$ is bounded above. If $\sum_{n=1}^{\infty} x_n/n$ converges, then $\lim_{n\to\infty} x_n = 0$.*

Now, I state the main theorem given by Naimi and Feist.

Theorem 8. *Let $\{a_n\}$ be a sequence of $1s$ and $-1s$. Then $\sum_{n=1}^{\infty} a_n/n$ converges if and only if $\sum_{n=1}^{\infty} A_n/n$ converges.*

Main Results

The first result that we state is a variation of one of the results given by Naimi and Feist. In their paper they give necessary and sufficient conditions on a sequence of $1s$ and $-1s$ for which the corresponding series $\sum a_n/n$ converges. Here, we will provide a necessary and sufficient condition on a larger/different class of sequences $\{a_n\}$ for which the corresponding series $\sum a_n/n$ converges. We prove the following lemma first.

Lemma 9. *Let $\{a_n\}$ be a sequence of real numbers such that the partial sums, S_n, of $\sum a_n$ form a bounded sequence. then $n|A_{n+1} - A_n|$ is bounded above.*

Proof. Let $S_n = \sum_{i=1}^{n} a_i$. Since the partial sums S_n form a bounded sequence, there exists some number $M > 0$ such that

$$|S_n| \leq M, \text{ for all } n \in \mathbb{N}.$$

Then

$$|S_{n+1} - 2S_n| \leq 3M. \tag{3}$$

Now consider

$$\begin{aligned} |A_{n+1} - A_n| &= \left| \frac{A_n + a_{n+1}}{n+1} - A_n \right| \\ &= \left| \frac{a_{n+1} - nA_n}{n+1} \right| \\ &= \frac{1}{n+1} |a_{n+1} - S_n| \\ &\leq \frac{1}{n} |S_{n+1} - 2S_n| \\ &\leq \frac{3M}{n}, \end{aligned}$$

where we have used (3) in the last inequality. The statement of the lemma now follows.

□

Theorem 10. *Let $\{a_n\}$ be a sequence of real numbers such that the partial sums S_n of $\sum a_n$ form a bounded sequence. Then $\sum a_n/n$ converges if and only if $\sum A_n/n$ converges.*

Proof. The proof follows readily from Lemma 9, Lemma 7 and Theorem 6. □

A Necessary Condition for Convergence

Theorem 11. *Let $\{a_n\}$ be a sequence of real numbers and $\{b_n\}$ be a convergent sequence of real numbers such that $b_n \neq 0$ for all n. If $\sum a_n/b_n$ converges then $\lim_{n\to\infty} A_n = 0$.*

Proof. Let

$$S_k = \sum_{n=1}^{k} \frac{a_n}{b_n}, \text{ and } A_k = \frac{1}{k} \sum_{n=1}^{k} a_n.$$

Applying Theorem 4 as given in equation (2), we get

$$\begin{aligned} \sum_{n=1}^{k} a_n &= \sum_{n=1}^{k} \frac{a_n}{b_n} b_n \\ &= S_k b_{k+1} + \sum_{n=1}^{k} S_n (b_n - b_{n+1}). \end{aligned}$$

Then

$$A_k = \frac{b_{k+1}}{k} S_k + \frac{1}{k} \sum_{n=1}^{k} S_n (b_n - b_{n+1}).$$

Letting $u_n = b_n - b_{n+1}$ in the above equation, we get

$$A_k = \frac{b_{k+1}}{k} S_k + \frac{1}{k} \sum_{n=1}^{k} S_n u_n. \tag{4}$$

Now, by hypothesis $\sum a_n/b_n$ converges, say to S. Then, $S_k \to S$ as $k \to \infty$. Also, since $\{b_k\}$ converges to some number b, $\{u_k\}$ will converge to 0 as $k \to \infty$. Thus

$$\frac{b_{k+1}}{k} S_k \to 0, \text{ as } k \to \infty,$$

and

$$\frac{1}{k} \sum_{n=1}^{k} S_n u_n \to 0, \text{ as } k \to \infty. \tag{5}$$

For a proof of (5) see (Rankin, 1963, p. 50). Therefore $A_k \to 0$ as $k \to \infty$. This concludes the proof!

□

We notice that the contrapositive of the above theorem is the most useful when solving specific problems, and for this reason we will state it in the form of a corollary. This provides us with another divergence test for these types of series.

Corollary 12. Let $\{a_n\}$ be a sequence of real numbers and $\{b_n\}$ be a convergent sequence of real numbers, such that $b_n \neq 0$ for all n. If $\lim_{n\to\infty} A_n \neq 0$, then the series $\sum a_n/b_n$ diverges.

Sequences and Series of Functions

In this section we will show that similar statements as those given in some of the previous theorems hold for sequences $\{f_n\}$ of real functions as well.

A Necessary Condition for Convergence. Let us first consider an example that will serve as a motivation for the theorem to come.

Example 13. Let $\{f_n\}$ be a sequence of real functions given by

$$f_n(x) = nx(1 - x^2)^n \qquad (0 \leq x \leq 1, n = 1, 2, 3, \ldots). \tag{6}$$

Let

$$A_n(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x),$$

be the n^{th} partial average function of the sequence $\{f_n\}$. Consider the series

$$\sum_{n=1}^{\infty} \frac{f_n(x)}{n} = x \sum_{n=1}^{\infty} (1 - x^2)^n = \frac{1 - x^2}{x} = f(x).$$

So the series on the LHS converges point-wise to the function $f(x) = \frac{1-x^2}{x}$ on $(0,1)$. Next, we wil show that $\lim_{n\to\infty} A_n(x) \equiv 0$. Applying summation by parts we get

$$\begin{aligned} A_n(x) &= \frac{x}{n}\sum_{i=1}^{n} i(1-x^2)^i \\ &= \frac{x}{n}\left[\sum_{i=1}^{n}(1-x^2)^i(n+1) - \sum_{i=1}^{n}\left(\sum_{k=1}^{i}(1-x^2)^k\right)\right] \\ &= \frac{1-x^2}{x}\left[\frac{n+1}{n}(1-(1-x^2)^n) - 1 + \left(\frac{1}{n} - \frac{(1-x^2)^n)}{n}\right)\right]. \end{aligned}$$

From the above expression, we see that $A_n(x) \to 0$ as $n \to \infty$ for all $x \in (0,1)$.

Lemma 14. *Let $\{f_n(x)\}$ be a sequence of uniformly bounded real functions converging uniformly to a function f on $E \subset \mathbb{R}$. Let $\{g_n(x)\}$ be another sequence of real functions whose n^{th} term is given by:*

$$g_n(x) = \frac{1}{n}\sum_{i=1}^{n} f_i(x).$$

Then $\{g_n(x)\}$ converges uniformly to f on E.

Proof. Since $\{f_n\}$ converges uniformly to f, given an $\epsilon > 0$, there exists some N such that for all $x \in E$ and $n \geq N$ we have

$$|f_n(x) - f(x)| < \epsilon. \tag{7}$$

On the other hand, since $\{f_n\}$ are uniformly bounded on E, there exists a number M such that

$$|f_n(x)| < M \qquad (x \in E, n = 1,2,3,...). \tag{8}$$

Now consider

$$\begin{aligned} |g_n(x) - f(x)| &= \left|\frac{1}{n}\sum_{i=1}^{n} f_i(x) - f(x)\right| \\ &= \frac{1}{n}\left|\sum_{i=1}^{n} f_i(x) - nf(x)\right| \\ &= \frac{1}{n}\left|\sum_{i=1}^{N}[f_i(x) - f(x)] + \sum_{j=N+1}^{n}[f_j(x) - f(x)]\right| \\ &\leq \frac{MN}{n} + \frac{\epsilon}{2}. \end{aligned}$$

The last inequality follows from (7) and (8). In the above expression, if we let $X(\epsilon) = max\left\{N, \frac{2MN}{\epsilon}\right\}$, we obtain

$$|g_n(x) - f(x)| < \epsilon, \text{ for all } n > X(\epsilon),$$

as desired. □

Theorem 15. *Let $\{f_n(x)\}$ be a sequence of uniformly bounded real functions on $E \subset \mathbb{R}$. If $\sum_{n=1}^{\infty} f_n(x)/n$ converges uniformly to a function f on E, then $A_n(x)$ converges uniformly to 0 on E, where*

$$A_n(x) = \frac{1}{n}\sum_{i=1}^{n} f_i(x).$$

Proof. Let $S_k(x) = \sum_{n=1}^{k} f_n(x)/n$. Applying summation by parts we get:

$$\begin{aligned} \sum_{n=1}^{k} f_n(x) = \sum_{n=1}^{k} \frac{f_n(x)}{n}n \\ &= S_k(x)(k+1) + \sum_{n=1}^{k} S_n(x)(-1) \\ &= (k+1)S_k(x) - \sum_{n=1}^{k} S_n(x). \end{aligned}$$

So, for the average function we have

$$A_k(x) = \frac{k+1}{k}S_k(x) - \frac{1}{k}\sum_{n=1}^{k} S_n(x). \tag{9}$$

By the hypothesis of the theorem $\sum_{n=1}^{\infty} f_n(x)/n$ converges uniformly to some function $f(x)$, so $S_k(x)$ converges uniformly to $f(x)$. Thus, since $\{S_k(x)\}$ is a sequence of bounded functions we have

$$\frac{k+1}{k}S_k(x) \to f(x), \text{ uniformly on } E \text{ as } k \to \infty.$$

Also, by Lemma 14 we have

$$\frac{1}{k}\sum_{n=1}^{k} S_n(x) \to f(x), \text{ uniformly on } E \text{ as } k \to \infty.$$

Therefore, by (9) , as $k \to \infty$, $A_k(x) \to 0$, uniformly on E. □

Again, notice that the contrapositive of this theorem is the one that would be the most useful in solving problems involving uniform convergence of series of this form.

A Necessary and Sufficient Condition for Convergence. Before we state and prove the main theorem of this section, and the last in this paper, we will give two convergence tests applicable to series of functions.

Proposition 16. (Comparison Test for Series of Functions, CTSF) Suppose $\sum_{n=1}^{\infty} f_n(x)$ and $\sum_{n=1}^{\infty} g_n(x)$ are two infinite series of functions on E, with $g_n(x) \geq 0$, for all $x \in E$ and $n = 1,2,3,\ldots$.

(a) If $|f_n(x)| \leq g_n(x)$ for all $x \in E$ and $n \geq N_o$, where N_o is some fixed integer, and if $\sum g_n(x)$ converges uniformly on E, then $\sum f_n(x)$ converges uniformly on E as well;

(b) If $f_n(x) \geq g_n(x)$, for all $x \in E$ and $n \geq N_o$, and if $\sum g_n(x)$ diverges on E, then $\sum f_n(x)$ diverges as well.

Proof. Since $\sum g_n(x)$ converges uniformly on E, given $\epsilon > 0$, there exists some $N \geq N_o$ such that $n \geq m \geq N$ implies

$$\sum_{k=n}^{m} g_k(x) \leq \epsilon,$$

by the Cauchy criterion for convergence. Then

$$\left|\sum_{k=n}^{m} f_n(x)\right| \leq \sum_{k=n}^{m} |f_k(x)| \leq \sum_{k=n}^{m} g_k(x) \leq \epsilon.$$

Therefore, by the Cauchy criterion for convergence, $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on E. This concludes the proof of part (a). Part (b) is proved in a similar fashion. □

Proposition 17. (Limit Comparison Test for Sequences of Functions, LCTSF). Suppose $\sum_{n=1}^{\infty} f_n(x)$ and $\sum_{n=1}^{\infty} g_n(x)$ are two infinite series of functions on a set E, with $g_n(x) > 0$, and $f_n(x) > 0$, for all $x \in E$ and $n = 1, 2, 3, \ldots$. Let $F_n(x) = f_n(x)/g_n(x)$, for all $x \in E$, and suppose that $F_n(x)$ converges point-wise to a function $h(x)$ on E.

If $h(x) > 0$ and is bounded on E, then $\sum f_n(x)$ converges uniformly on E if and only if $\sum g_n(x)$ converges uniformly on E.

Proof. Given $\epsilon = \frac{1}{2} sup\{h(x)\} = \frac{1}{2}M > 0$, for every $x \in E$ there exists some N such that $n \geq N$ implies

$$\left|\frac{f_n(x)}{g_n(x)} - h(x)\right| \leq \frac{M}{2}.$$

Then

$$\frac{M}{2} g_n(x) \leq f_n(x) \leq \frac{3M}{2} g_n(x). \tag{10}$$

Then, the statement of the proposition follows by Proposition 16 and the expression in (10). □

Remark: Notice that if E is compact in the above proposition, then the requirement that $h(x)$ be bounded on E is redundant.

Theorem 18. *Let $\{f_n(x)\}$ be a sequence of uniformly bounded positive real functions on $E \subset \mathbb{R}$. Then, $\sum_{n=1}^{\infty} f_n(x)/n$ converges uniformly to a function $f(x)$ on E if and only if* $\lim_{n\to\infty} A_n(x) = 0$ *and $\sum_{n=1}^{\infty} A_n(x)/n$ converges uniformly to some function $g(x)$ on E.*

Proof. Let $S_k(x) = \sum_{n=1}^{k} f_n(x)$, then $S_k(x) = kA_k(x)$. Applying summation by parts we get

$$\sum_{n=1}^{k} \frac{f_n(x)}{n} = \frac{k}{k+1} A_k(x) + \sum_{n=1}^{k} \frac{A_n(x)}{n+1}. \tag{11}$$

First, notice that since $\{A_k(x)\}$ is a sequence of bounded functions, $\frac{k}{k+1}A_k(x)$ converges uniformly to 0 on E if and only if $A_k(x)$ converges uniformly to 0 on E, as $k \to \infty$. Also, by Proposition 17, $\sum_{n=1}^{k} \frac{A_n(x)}{n+1}$ converges uniformly to some function $f(x)$ on E if and only if $\sum_{n=1}^{k} \frac{A_n(x)}{n}$ converges uniformly to $f(x)$ on E. The conclusion of the theorem now follows readily from Theorem 15 and the relation in (11). □

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