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Exponential Exposition

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The fact that the sequence of functions $(1 + \frac{z}{n})^n$ is locally bounded in $\mathbb{C}$ is enough to prove that the sequence converges to a function that equals its own derivative. This limit can be taken as the definition of the complex exponential, and properties of the real exponential and trigonometric functions can easily be derived. This article is a hybrid of pure mathematical theory and mathematical pedagogy. The theory comes in proving that the limit mentioned above exists using some sophisticated techniques in the theory of functions. The pedagogy comes from the fact that except for the justification of the limit, this can serve as a satisfying introduction to the calculus properties of exponential and trigonometric functions in a first semester calculus course.

In a typical first semester calculus course students will see that $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$. In fact, for every complex number z,

$$e^z = \lim_{n\to\infty}\left(1 + \frac{z}{n}\right)^n;$$

see (Conway, 1973, Lemma 7.19 in Chapter 7). However both of these results follow from previously derived properties of exponential and logarithm functions. A typical proof of the first uses L'Hopital's rule and the second involves some careful estimates using the the Taylor expansion of $\log(1 + z)$.

Independently of a previously defined exponential or logarithm function, this article starts by showing that the above limit exists and then takes it as the *definition* of the complex exponential. Using some complex analysis, which can be justified independently of the elementary transcendental functions, the basic properties of the exponential and trigonometric functions can be derived in a very straightforward manner. In particular by doing the most obvious computation, this sequence converges to a function that equals its own derivative. By first restricting to the real axis and then to the imaginary axis, this becomes a very satisfying approach to exponential and trigonometric functions in a first semester calculus course.

Some Complex Background

In this section we state some of the standard definitions and results from complex variable theory that will be needed, being careful to not use anything that requires exponential or trigonometric functions.

Let G be a domain in the complex plane; i.e. G is open and connected. Recall that $H(G)$ stands for the set of complex valued functions defined on G that are differentiable on G. (Here the "H" refers to the word "holomorphic", which is a standard term used for functions that are differentiable.)

We assume all the basic calculus properties of $H(G)$ such as the product, quotient, and chain rules. Basic properties of contour integration are also required. The most important of these is Cauchy's Theorem that states that the integral of a function in $H(G)$ around a closed contour is zero provided the contour and its interior are contained in G. (Those familiar with the theory will recall that in a lot of arguments it is enough to have this result in the special case of triangles.)

Cauchy's Integral Formula, in the classical development, depends on knowing the value of

$$\int_\sigma \frac{dz}{z},$$

where σ is the unit circle. But the standard way of computing this is to parametrize σ with the complex exponential. In the spirit of this article, we want to avoid that computation until after the complex exponential has been properly developed.

Instead, consider the value of

$$\int_\lambda \frac{dz}{z},$$

where λ represents the "unit square", i.e. the square with vertices $(\pm 1, \pm 1)$.

A straightforward computation shows that this integral equals κi where

$$\kappa = 4\int_{-1}^{1}\frac{dt}{1 + t^2}.$$

Using standard arguments (for example, see (Howie, 2003, Section 7.1)), Cauchy's Integral Formula becomes

$$f(a) = \frac{1}{\kappa i}\int_\gamma \frac{f(z)}{z - a}dz,$$

where $f \in H(G)$, γ is a simple closed contour in G with interior in G, and a is in the interior of γ.

This formula implies Morera's Theorem (Howie, 2003, Theorem 7.8) which characterizes functions in $H(G)$ as those that are continuous on G and have zero contour integrals over simple closed contours which are contained in G together with their interiors. As well, Liouville's Theorem (Howie, 2003, Theorem 7.9) holds, which states that the only bounded functions in $H(\mathbb{C})$ are the constant functions.

Cauchy's Integral Formula also implies that every function $f \in H(G)$ is *analytic* in the sense that given $a \in G$, $f(z)$ is equal to its Taylor series centered at a for z in some disk centered at a; (Howie, 2003, Theorem 7.16). This, in turn, implies the Rigidity Principle which states that if $f \in H(G)$ is not identically zero, then its zeros are isolated; (Stalker, 1998, Theorem 18).

Finally, we need some results concerning sequences in $H(G)$. In (Conway, 1973, Chapter 7) it is shown that if a sequence $\{f_n\}$ converges uniformly on compact subsets of G to a function f, then $f \in H(G)$ and for every positive integer j, $\{f_n^{(j)}\}$ converges uniformly on compact subsets to $f^{(j)}$. This is simply an application of Morera's Theorem and the Cauchy Integral Formula. This is a very powerful result that is not true in real variables. For example, on $(-1, 1)$ there is a sequence of polynomials that converge uniformly to the absolute value function which fails to be differentiable at 0.

A combination of the Arzela-Ascoli Theorem (Conway, 1973, Theorem 1.23 of Chapter 7) and the Cauchy Integral Formula is used to prove Montel's Theorem (Conway, 1973, Theorem 2.9 of Chapter 7), which states that if a family of functions $\mathcal{F} \subset H(G)$ is uniformly bounded on compact subsets of G (i.e. $\mathcal{F}$ is *locally bounded*), then every sequence of functions in $\mathcal{F}$ has a subsequence that converges uniformly on compact subsets of G; i.e. $\mathcal{F}$ is a *normal family*.

Exponential Function

We are now ready to construct our function. First observe that there can be at most one function $f \in H(\mathbb{C})$ so that $f(0) = 1$ and $f' = f$. Let f and g be two such functions. Then it is easy to check that $\frac{d}{dz} f(z)g(-z) = 0$, and thus $f(z)g(-z)$ is a constant which must be 1 by evaluating the product at 0. Applying this when $g(z) = f(z)$ shows that any function f with these properties is non-zero and $f(-z) = \frac{1}{f(z)}$ for all z. Thus if two functions f and g satisfy these conditions then $\frac{f(z)}{g(z)} = 1$ for all z, i.e. $f = g$.

Now, define $f_n(z) := (1 + \frac{z}{n})^n$. If $R > 0$, and $|z| \leq R$ then

$$|f_n(z)| \leq f_n(R) = \sum_{j=0}^{n} \binom{n}{j} \frac{R^j}{n^j} \leq \sum_{j=0}^{n} \frac{R^j}{j!} \leq \sum_{j=0}^{\infty} \frac{R^j}{j!} < \infty.$$

This shows that the sequence $\{f_n\}$ is a locally bounded subset of $H(\mathbb{C})$. By Montel's Theorem, every subsequence of $\{f_n\}$ has a further subsequence that converges uniformly on compact subsets of $\mathbb{C}$.

Let $n_1 < n_2 < \ldots$ and suppose that the subsequence $\{f_{n_k}\}$ converges uniformly on compact subsets of $\mathbb{C}$ to a function f. By the discussion in the previous section, it follows that $f \in H(\mathbb{C})$ and $\{f'_{n_k}\}$ converges uniformly on compact subsets of $\mathbb{C}$ to f'. Fix z. Then for k suitably large,

$$f'_{n_k}(z) = (1 + \frac{z}{n_k})^{n_k - 1} = \frac{(1 + \frac{z}{n_k})^{n_k}}{(1 + \frac{z}{n_k})}.$$

Thus $\{f'_{n_k}(z)\}$ converges to $f(z)$. As such, $f' = f$. It is also clear that $f(0) = 1$.

This, combined with the uniqueness discussed above, shows that there exists a function $f \in H(\mathbb{C})$ so that $f' = f$, $f(0) = 1$, and with the property that every subsequence of $\{f_n\}$ has a further subsequence that converges to f uniformly on compact subsets of $\mathbb{C}$. Now $H(\mathbb{C})$ is a complete metric space so that convergence of a sequence in the metric space is equivalent to uniform convergence on compact subsets of $\mathbb{C}$; see (Conway, 1973, Chapter 7). A standard metric space argument implies that the entire sequence $\{f_n\}$ converges to f uniformly on compact subsets of $\mathbb{C}$. Thus

$$f(z) := \lim_{n\to\infty} (1 + \frac{z}{n})^n \text{ satisfies } f' = f \text{ and } f(0) = 1.$$

It is possible, using tedious estimates, to show directly that $\{f_n\}$ is a Cauchy sequence in $H(\mathbb{C})$. That approach would also work if one were only interested in this convergence for real values of the variable. What makes the complex variable approach so appealing is the ease with which it handles the interchange of sequential limits and differentiation.

In a first semester calculus course one need only mention that it can be shown that $\lim_{n\to\infty}(1 + \frac{x}{n})^n$ exists and that it is justified to interchange the limit and differentiation. A purely real variable proof would involve the tedious estimates mentioned above together with a general result about interchange of limits and differentiation; for example see (Rudin, 1976, Theorem 7.17). This is similar to the fact that in many second semester calculus courses it is usually mentioned, without proof, that power series can be differentiated term-by-term inside the interval of convergence; for example see (Hughes-Hallett et al., 2007, Section 10.3). Actually, the theory that justifies the power series manipulations is exactly the theory that justifies the limit used here.

Properties

We include here a brief discussion of the properties of the function f defined at the end of the previous section. These properties are mostly consequences of the fact that $f' = f$ and $f(0) = 1$. (Note: Most of this material could be used in a first semester calculus course; the part that has to do with purely imaginary numbers would probably be better for a honors section or a semester project.) The treatment here is somewhat similar to those given in (Rudin, 1987) and (Lang, 1983). The main difference is the way in which the function is defined. None of what has been discussed in this article requires power series, which is particularly nice when one wants to introduce the exponential function in a first semester calculus course. Also, we assume here the standard geometric definitions of the trigonometric functions as they relate to the unit circle, and show that it is easy to relate the complex exponential to these functions and thereby prove that the trigonometric functions are differentiable.

Fix w. Then $\frac{d}{dz} f(z + w)f(-z) = 0$ and thus $f(z + w)f(-z)$ is a constant which must be $f(w)$ by evaluating the product at 0, i.e. $f(z + w) = f(z)f(w)$. From this it is easy to check that $f(r) = f(1)^r$ for r a rational number. Define

$$e := f(1) = \lim_{n\to\infty} (1 + \frac{1}{n})^n.$$

Thus f is the unique analytic continuation to $\mathbb{C}$ of e^r, for r rational, by the Rigidity Principle. This is where it makes sense to denote $f(z)$ by e^z.

Restricting to real variables we have $\frac{d}{dx}e^x = e^x > 0$. Since $e > 1$, $\lim_{n\to\infty} e^n = \infty$ and thus $\lim_{n\to\infty} e^{-n} = 0$. It follows that $f : \mathbb{R} \to (0, \infty)$ is a homeomorphism.

The limit definition implies that $e^{\bar{z}} = \overline{e^z}$. Thus if t is real,

$$\overline{e^{it}} = e^{-it},\ |e^{it}|^2 = e^{it}e^{-it} = e^0 = 1, \text{ and } \frac{d}{dt}e^{it} = ie^{it}.$$

Write $e^{it} = c(t) + is(t)$. Thus $c(-t) = c(t)$, $s(-t) = -s(t)$, $c(t)^2 + s(t)^2 = 1$, $c'(t) = -s(t)$, and $s'(t) = c(t)$. Also, $c(0) = 1$ and $s(0) = 0$.

Now, by Liouville's Theorem, it follows that the range of $f(z)$ is dense in $\mathbb{C}$, for otherwise there would be some a so that $\frac{1}{f(z)-a}$ is bounded, and hence constant. In particular there exist real numbers α and β so that $\Re e^{\alpha+i\beta} < 0$. But $e^{\alpha+i\beta} = e^{\alpha}(c(\beta) + is(\beta))$, and thus $c(\beta) < 0$. Since c is an even function we may assume, without loss of generality, that there exists $t_0 > 0$, the smallest positive value at which c vanishes.

Therefore on $[0, t_0]$, c is strictly decreasing and s is strictly increasing. Thus $s(t_0) = 1$. It follows that $e^{it}, 0 \le t \le t_0$, traces the unit circle in the first quadrant from 1 to i. As a result

$$\frac{\pi}{2} = \int_0^{t_0} |e^{i\tau}| d\tau = t_0.$$

More is true. If $0 \le t \le \frac{\pi}{2}$, the arc-length from 1 to e^{it} is given by

$$\int_0^t |e^{i\tau}| d\tau = t,$$

and from the standard geometric definitions we get $c(t) = \cos t$ and $s(t) = \sin t$, where t is measured in radians.

Now $e^{i\frac{\pi}{2}} = i$. This implies that $s(t) = c(t - \frac{\pi}{2})$, $c(\pi - t) = -c(t)$, and $c(2\pi - t) = c(t)$. This interprets graphically to say that the graph of $y = s(t)$ is the shift of the graph of $y = c(t)$ by $\frac{\pi}{2}$, the graph of $y = c(t)$ is anti-symmetric with respect to the line $t = \frac{\pi}{2}$, and the graph of $y = c(t)$ is symmetric with respect to the line $t = \pi$. It is also easy to check that $c(t + 2\pi) = c(t)$.

Since $\cos t$ has each of these properties, $\sin t = \cos(t - \frac{\pi}{2})$, and $c(t) = \cos t$ for $0 \le t \le \frac{\pi}{2}$, then for all t, $c(t) = \cos t$ and $s(t) = \sin t$.

This simultaneously shows the differentiability of $\cos t$ and $\sin t$ and their standard differentiation formulas.

Letting σ be the unit circle and λ the unit square, Cauchy's Theorem implies

$$4i \int_{-1}^{1} \frac{dt}{1 + t^2} = \int_{\lambda} \frac{dz}{z} = \int_{\sigma} \frac{dz}{z} = 2\pi i.$$

and thus

$$\int_{-1}^{1} \frac{dt}{1 + t^2} = \frac{\pi}{2}.$$

The following formula now holds:

$$e^{x+it} = e^x(\cos t + i \sin t).$$

The special case where $x = 0$ can be used to prove many standard trigonometric identities.

Conclusion

We have seen that $f(z) = \lim_{n\to\infty}(1 + \frac{z}{n})^n$ is a function that has the property that $f'(z) = f(z)$ and $f(0) = 1$. For a rational number r, $f(r) = e^r$ where $e = f(1) = \lim_{n\to\infty}(1 + \frac{1}{n})^n$. The function is then denoted by e^z. It is then shown that $e^{x+it} = e^x(\cos t + i \sin t)$.

These facts can be used to prove (even to first semester calculus students) the differentiability of the exponential and trigonometric functions, as well as to provide the basic differentiation formulas.

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