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Repetitions in the Number of Vertices of Iterated Line Graphs

Emily Grinstead
Huntingdon College
lgrinstead@huntingdon.edu

John Holbrook
Morehead State University
hjholbrook@moreheadstate.edu

Peter Johnson
Auburn University
johnspd@auburn.edu

Matthew Simpson
North Carolina State University
mcsimps2@ncsu.edu

The line graph $L(G)$ of a simple graph G is defined by $V(L(G)) = E(G)$, with any two vertices in $L(G)$ adjacent if and only if the corresponding edges in G are incident. A subject of interest is the integer sequence $(|V(G)|, |V(L(G))|, |V(L(L(G)))|, \ldots)$. For a finite, simple, connected graph G, we look at the number of times a member of the sequence can occur in consecutive indices. For graphs for which the sequence is convergent, a member can occur consecutively any number of times. However, beyond these classes of graphs, numbers in the sequence can only repeat twice in a row. Repetitions more than two in a row are impossible.

Introduction

Let the vertex and edge sets of a simple graph G be denoted $V(G)$ and $E(G)$, respectively. The line graph $L(G)$ of a simple graph G is the graph whose vertices are the edges of G, with any two vertices in $L(G)$ adjacent if and only if the corresponding edges in G are incident. The line graph sequence of G is $(L^m(G))_{m\in\mathbb{Z}^*} = (G, L(G), L^2(G), \ldots)$, where $\mathbb{Z}^*$ is the set of nonnegative integers, $L^k(G)$ is the k-th iteration of the line graph operator applied to G, and $L^0(G) = G$.

Of interest is the number of vertices in each iterated line graph; consider the sequence $(|V(L^m(G))|)_{m\in\mathbb{Z}^*} = (|V(G)|, |V(L(G))|, |V(L^2(G))|, \ldots)$. van Rooij and Wilf (1965) showed that for a connected graph G on n vertices, the sequence behaves in one of four ways:

(1) If G is the cycle graph C_n, then for each nonnegative integer i, $L^i(G)$ is isomorphic to the original cycle. Thus, $(|V(L^m(G))|) = (n, n, n, \ldots)$.

(2) If G is P_n, the path graph on n vertices, then for each nonnegative integer i, we have

$$L^i(G) = \begin{cases} P_{n-i} & \text{if } i < n \\ P_0 & \text{if } i \geq n \end{cases}$$

Thus, $(|V(L^m(G))|) = (n, n-1, n-2, \ldots, 0, 0, \ldots)$.

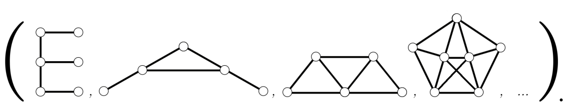
(3) If G is the claw $K_{1,3}$, then for each positive integer i, $L^i(G)$ is K_3. Thus, $(|V(L^m(G))|) = (4, 3, 3, 3, \ldots)$.

(4) In all other cases, the $(|V(L^m(G))|)$ eventually increases without bound. Thus, $\lim_{m\to\infty} |V(L^m(G))| = \infty$.

Remark. *Alternative (2) above acknowledges the existence of $P_0 = L(P_1)$, a graph with no vertices and, therefore, certainly no edges. Why not call this the empty graph? Because, in graph theory, the term "empty graph" means something else: a graph with vertices but no edges. When P_0 is referred to, which is rare, the leading contender for a name for it is: the null graph.*

In some precincts, the null graph is not considered to be a graph – or if it is a graph, its line graph is undefined. In this paper, we take the view that P_0 is a graph, satisfying $L(P_0) = P_0$.

A sequence of iterated line graphs $(L^m(G))$ converges to a graph H if and only if there exists a positive integer N such that $L^k(G) \cong H$ for all $k \geq N$. Thus, $K_{1,3}$ converges to K_3, C_n converges to C_n, and P_n converges to the null graph. The corresponding vertex sequences converge to 3, n, and 0 respectively. For $k \geq 2$, define a k-repetition in the sequence $(|V(L^m(G))|)$ as the occurrence in the sequence of $|V(L^i(G))| = |V(L^{i+1}(G))| = \ldots = |V(L^{i+k-1}(G))|$ for some nonnegative integer i. Clearly, any k-repetition is possible if the sequence $(L^m(G))$ is convergent. The ones of interest, however, are the nontrivial repetitions that occur in divergent sequences. The following example shows that "isolated" 2-repetitions are possible.



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The corresponding sequence of orders is $(6, 5, 5, 7, \ldots)$.

Our purpose here is to prove that 3-repetitions in the sequence $(|V(L^m(G))|)_{m\in\mathbb{Z}^*}$ are impossible if G is finite, connected, and simple, and the sequence $(|V(L^m(G))|)$ is not convergent. This result is surely known, at least folklorically, and proving it will be little more than a pleasant exercise for a graph theorist. In fact, we know of two proofs other than the one we will give here; still, the proof here has some novel aspects that make it worth giving, we think.

The proof here will seem longer than necessary to anyone knowledgeable of graph theory, and it is, because we are writing it for the readers that know little about graphs beyond the fundamental terms. The only fact from graph theory that we will use which novice readers might not be able to verify for themselves is this: if G is a tree – i.e., a connected graph with no cycle subgraphs – then $|E(G)| = |V(G)| - 1$.

The impossibility of 3-repetitions and greater

For positive integers n and d, $d < n$, define the following:

$$e'(n, d) = \min\{|E(L(G))| : \textit{G is a simple, connected graph on n vertices and has a vertex of degree d}\}$$

$$B(n, d) = \{X : \textit{X is a simple, connected graph on n vertices, has a vertex of degree d, and } |E(L(X))| = e'(n, d)\}$$

Lemma 1. *For positive integers n and d, $d < n$, all elements of $B(n, d)$ are trees.*

Proof. Let $X \in B(n, d)$ and $v \in V(X)$ of degree $d < n$. Suppose X contains a cycle. Then that cycle contains an edge e not adjacent to v. Then $X - e$, the graph obtained by removing the edge e from X, is a connected graph on n vertices. Thus, $|E(L(X - e))| < |E(L(X))|$, which is a contradiction to $|E(L(X))| = e'(n, d)$. □

Lemma 2. *For positive integers n and d, $d < n$, $e'(n, d) < e'(n + 1, d)$.*

Proof. Let $X \in B(n + 1, d)$ and $v \in V(X)$ with degree $d < n$. By Lemma 1, X is a tree, so let $u \in V(X)$ be of degree 1. If all vertices of X of degree 1 are adjacent to v, then $X \cong K_{1,n}$, with v being the one vertex of X of degree n. But then $n = d$, contradicting $n > d$. Therefore, there is a vertex $u \in V(X)$ of degree 1 which is not adjacent to v. $X - u$, the graph obtained by removing the vertex u from X, is a connected graph on n vertices containing a vertex, namely v, of degree d. Therefore, $e'(n, d) \leq |E(L(X - u))| < |E(L(X))| = e'(d, n + 1)$. □

Lemma 3. *For a finite, simple, non-null, connected graph G, $|V(G)| = |E(G)|$ if and only if G contains exactly one cycle subgraph. Furthermore, $|E(G)| > |V(G)|$ if and only if G contains at least two cycle subgraphs.*

Proof. We depart from the well known fact that for any (non-null) tree T, $|E(T)| = |V(T)| - 1$. If $n = |V(G)| = |E(G)|$, then G is not a tree, but is connected, by hypothesis, so G must contain a cycle. If G contains another cycle, then G contains an edge e in one cycle but not the other. Then $G - e$ is a connected simple graph (because G is, and e is on a cycle) containing a cycle. Additionally, $E(G - e) = n - 1$. Whether $G - e$ has more than one cycle, or only one, we can go to a tree T from $G - e$ by removing edges to destroy cycles. Then $|E(T)| < n - 1 = |V(T)| - 1$, an impossibility for a tree. Thus, $|E(G)| = |V(G)|$ implies that G contains exactly one cycle subgraph.

Conversely, if G (connected by hypothesis) contains exactly one cycle subgraph, then removing one edge from G gets us to a tree T on $n = |V(G)|$ vertices. Then $|E(G)| - 1 = |E(T)| = n - 1$, implying $|E(G)| = |V(G)|$.

The remaining claim of the lemma is a corollary of the first claim:

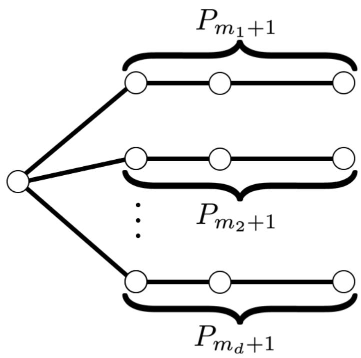
$$\begin{aligned} & G \text{ contains 2 or more different cycles} \\ \iff & G \text{ is neither a tree nor unicyclic} \\ \iff & |E(G)| \geq |V(G)| - 1 = n - 1 \text{ and } |E(G)| \notin \{n, n-1\} \\ \iff & |E(G)| > n = |V(G)| \end{aligned}$$

(The fact that $|E(G)| \geq |V(G)| - 1$ if G is connected is an auxiliary of the fact that $|E(T)| = |V(T)| - 1$ for every non-null tree T.) □

Theorem 1. *For positive integers $n > d$:*

(i) If $d \in 1, 2$, then $B(n, d) = \{P_n\}$.

(ii) If $d > 2$, then all elements of $B(n, d)$ are of the form shown, where $m_1, \ldots, m_d$ are non-negative integers satisfying $m_1 + \ldots + m_d + d + 1 = n$.



Proof. In every case, by Lemma 1, every element $X \in B(n, d)$ is a tree, so $n = |V(X)| = |E(X)| + 1 = |V(L(X))| + 1 \leq |E(L(X))| + 2$; the last inequality holds because $L(X)$ is connected.

Suppose that $d \in 1, 2$. Then P_n has a vertex of degree d, and $|E(L(P_n))| = |E(P_{n-1})| = n - 2 \leq |E(L(X))|$ for any $X \in B(n, d)$, by the remarks above. Therefore, $n-2 = e'(n, d)$ and $P_n \in B(n, d)$.

If X is a tree on n vertices which is not a path, then X contains a vertex of degree strictly greater than 2 in X. Therefore, $L(X)$ contains a C_3, arising from 3 edges incident to the same vertex. By Lemma 3, since $L(X)$ is connected and simple, $|E(L(X))| \geq |V(L(X))| = |E(X)| = n - 1 > n - 2 = e'(n, d)$. Therefore, $X \notin B(n, d)$. Thus, $B(n, d) = \{P_n\}$.

Now suppose that $d \geq 3$. We proceed by induction on $n > d$. If $n = d + 1$, then clearly $K_{1,d}$ is the only element of $B(n, d)$, since it is the only tree on $d + 1$ vertices containing a vertex of degree d. This verifies the conclusion when $n = d + 1$: $m_1 = \ldots = m_d = 0$.

Now suppose that the claim holds for some $n > d$. Let $Y \in B(n, d)$. By the induction hypothesis, Y is of the given form. Let X, a tree on $n + 1$ vertices with a vertex v of degree d, be obtained from Y by adding a leaf to one of the branches (paths) emanating from v. Then $e'(n + 1, d) \geq e'(n, d) + 1 = |E(L(Y))| + 1 = |E(L(X))| \geq e'(n + 1, d)$ (Lemma 2). Therefore, $e'(n + 1) = e'(n, d) + 1$ and every such X, which is of the desired form, is in $B(n + 1, d)$.

On the other hand, suppose that $X \in B(n + 1, d)$. Because X is a tree on $n + 1 > d + 1 \geq 4$ vertices, X contains an edge yx such that $d_X(x) = 1$, and $y \neq v$, where v is a vertex in X of degree $d \geq 3$.

(To see this, consider d walks starting from v and heading out along the d different edges incident to v. Because X is a tree on $n + 1 > d + 1$ vertices, at least one of these walks will have a continuation, after that first edge traversal, to a walk of length greater than or equal to 2. Keep walking on that continuation, making arbitrary choices at forks in the road, but never turning back, until you get to a vertex of X of degree 1.)

Let $Y = X - x$. Then

$$\begin{aligned} e'(n, d) &\leq |E(L(Y))| \\ &= |E(L(X))| - d_Y(y) \\ &= e'(n + 1, d) - d_Y(y) \\ &= e'(n, d) + 1 - d_Y(y) \end{aligned}$$

Therefore, $d_Y(y) = 1$ and $Y \in B(n, d)$. By the induction hypothesis applied to Y, and the way X is obtainable from Y, we see that X has the required form. □

Theorem 2. *Let G be a non-null, finite, simple, connected graph. If $|V(G)| = |V(L(G))| = |V(L^2(G))|$, then G is a cycle graph.*

Proof. Since $n = |V(G)| = |V(L(G))| = |E(G)|$ and G is not the null graph, G is not a tree; in fact, by Lemma 3, G is unicyclic.

Let $d = \triangle(G)$, the maximum degree in G. If $d = 1$, then G, connected, must be a single edge, which is a tree; if $d = 2$, then, because G is not a tree, G must be a cycle. Therefore, we may as well assume that $d \geq 3$. Since the elements of $B(n, d)$ are trees, $G \notin B(n, d)$. Therefore, $e'(n, d) \leq |E(L(G))| - 1 = |V(L^2(G))| - 1 = |V(G)| - 1 = n - 1$.

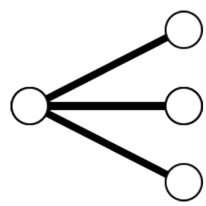
Any $X \in B(n, d)$ is of the form in Theorem 4 for $d \geq 3$. Therefore, for some non-negative integers $m_1, \ldots, m_d$ satisfying $m_1 + \ldots m_d = n - d - 1$,

$$\begin{aligned} e'(n, d) &= |E(L(X))| \\ &= \binom{d}{2} + m_1 + \ldots + m_d \\ &= \frac{d(d-1)}{2} + n - d - 1 \leq n - 1 \\ &\iff d(d-3) \leq 0 \end{aligned}$$

Since $d \geq 3$, we conclude that $d = 3$.

Since G, unicyclic, contains a vertex of degree 3, then $L(G)$ contains at least two cycles, one from the cycle in G and a C_3 from 3 edges incident to the same vertex. By Lemma 3, it follows that $|V(L^2(G))| = |E(L(G))| > |V(L(G))|$, contradicting the supposition that $|V(G)| = |V(L(G))| = |V(L^2(G))|$. □

Corollary 1. *If G is a non-null, finite, simple, connected graph and there is a k-repetition in the sequence $(|V(L^m(G))|)_{m \in \mathbb{Z}^*}$ for some $k \geq 3$, then G is either a path, a cycle, or $K_{1,3}$.*



$K_{1,3}$, The Claw

Proof. If $|V(L^m(G))| = |V(L^{m+1}(G))| = |V(L^{m+2}(G))|$ for some $m \geq 0$, then either $H = L^m(G)$ is the null graph, in which case G is a path, or H is a cycle, by Theorem 5. In the latter case, if G is not a cycle, then we can let $m > 0$ be the smallest integer such that $L^m(G)$ is a cycle. Letting $X = L^{m-1}(G)$, we have that $H = L(X)$, H is a cycle, and X is not a cycle. Therefore, $|V(X)| \neq |E(X)| = |V(L(X))| = |V(H)| = |V(L(H))| = |V(L^2(X))|$, for, otherwise, X would be a cycle, by Theorem 5. So either $|V(X)| > |E(X)|$ or $|V(X)| < |E(X)|$.

If $|V(X)| < |E(X)|$, then, by Lemma 3, X contains 2 or more different cycles. Since each cycle in a graph gives rise to a cycle in its line graph, it follows that $H = L(X)$

has two or more cycles, whence, by Lemma 3, $|V(H)| < |E(H)| = |V(L(H))|$, contrary to assumption. Therefore, $|V(X)| > |E(X)|$. Since X is connected, X must be a tree.

Since X is a tree and $L(X)$ is a cycle, X must have exactly one vertex of degree 3 and no edges other than the 3 edges incident to that vertex; i.e., $X = K_{1,3}$. Since, as is well known and easy to see, $K_{1,3}$ is not the line graph of any graph, $m = 1$ and $G = X = K_{1,3}$. □

References

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