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Friends of 12

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A friend of 12 is a positive integer different from 12 with the same abundancy index. By enlarging the supply of methods of Ward (2008), it is shown that (i) if n is an odd friend of 12, then $n = m^2$, where m has at least 5 distinct prime factors, including 3, and (ii) if n is an even friend of 12 other than 234, then $n = 2 \cdot q^e \cdot m^2$, in which q is a prime, e is a positive integer, $29 \le q \equiv e \equiv 1 \mod 4$, and m has at least 3 distinct odd prime factors, one of which is 3, and the other, none equal to q, are greater or equal to 29.

The Abundancy Index

Let $\mathbb{P}$ denote the set of positive integers. For $n \in \mathbb{P}$, let $\sigma(n)$ denote the sum of all positive divisors of n, including n itself. It is well know, and not hard to see, that σ is *weakly multiplicative*: That is, if $m, n \in \mathbb{P}$ and $\gcd(m, n) = 1$, then $\sigma(mn) = \sigma(m)\sigma(n)$. Therefore, if $q_1, \ldots, q_t \in \mathbb{P}$ are distinct primes, and $e_1, \ldots, e_t \in \mathbb{P}$, then $\sigma(\prod_{i=1}^t q_i^{e_i}) = \prod_{i=1}^t \sigma(q_i^{e_i}) = \prod_{i=1}^t (\sum_{j=0}^{e_i} q_i^j) = \prod_{i=1}^t \frac{q_i^{e_i+1}-1}{q_i-1}$. For instance, $\sigma(12) = \sigma(3)\sigma(4) = (1 + 3)(1 + 2 + 4) = 28$.

The abundancy ratio, or abundancy index, of $n \in \mathbb{P}$, is $I(n) = \frac{\sigma(n)}{n}$. From previous remarks about σ we have the following facts about properties of the abundancy index. Ward (2008) also mentioned these properties.

1. I is weakly multiplicative.

2. If $q_1, \ldots, q_t \in \mathbb{P}$ are distinct primes, and $e_1, \ldots, e_t \in \mathbb{P}$, then $I(\prod_{i=1}^t q_i^{e_i}) = \prod_{i=1}^t I(q_i^{e_i}) = \prod_{i=1}^t \frac{\sum_{j=0}^{e_i} q_i^j}{q_i^{e_i}} = \prod_{i=1}^t \frac{q_i^{e_i+1}-1}{q_i^{e_i}(q_i-1)}$.

3. If $q \in \mathbb{P}$ is a prime, then, as $e \in \mathbb{P}$ increases from 1, $I(q^e)$ is strictly increasing, from $\frac{q+1}{q}$, and tends to $\frac{q}{q-1}$ as $e \to \infty$.

4. If $e \in \mathbb{P}$, as q increases among the positive primes, $I(q^e)$ is strictly decreasing.

5. If $m, n \in \mathbb{P}$ and $m \mid n$, then $I(m) \le I(n)$, with equality only if $m = n$.

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Interest in the abundancy index arises from interest in perfect numbers, positive integers with abundancy index 2. Mathematicians have been curious about perfect numbers since antiquity, and the abundancy index offers a context within which to study them indirectly. Perhaps asking questions about the abundancy index will lead to the development of theory applicable to question about the perfect numbers. See Ward (2008) for more references on the abundancy index.

Friends

Positive integers m and n are friends if and only if $m \ne n$ and $I(m) = I(n)$. Thus, different perfect numbers are friends. As in Ward (2008), it is easy to see that 1 has no friend, and that no prime power has a friend. It is not known if any positive integer has infinitely many friends.

Every element of $\{1, \ldots, 9\}$ is a prime power except for 1 and 6. $I(1) = 1$, and 1 has no friend; 6 is the smallest perfect number. (Actually, everything we would want to know about even perfect numbers, except whether or not there are infinitely many of them, is known, thanks to Euler.) Therefore, the first frontier of the study of friends is made up of the integers 10, 12, 14, 15, and 18.

In Ward (2008), Ward took on 10, and proved, among other things, that any friend of 10 must be a square with at least 6 distinct prime factors, including 5, the smallest. It is still unknown whether or not 10 has a friend; however, we feel that Ward has done service by pioneering methods other than "computer search" for hunting for friends of given integers.

The aim here is to apply Ward's methods, with a few new tricks thrown in, to the search for friends of 12. Computer search has already discovered 234 to be a friend of 12. This discovery can also be made rationally, by Ward-like arguments. Using these arguments, with a few twists, we shall obtain a theorem about the friends of 12 similar to Ward's theorem about friends of 10. The following lemma will be useful. The proof is straightforward.

Lemma 1. *If p is an odd prime and $e \in \mathbb{P}$, then $\sigma(p^e)$ is odd if and only if e is even. If $p \equiv 3 \mod 4$ and e is odd, then $4 \mid \sigma(p^e)$. If $p \equiv 1 \mod 4$ then $\sigma(p^e) \equiv e+1 \mod 4$.*

Corollary 1. *If $n \in \mathbb{P}$ and both n and $\sigma(n)$ are odd, then $n = k^2$ for some $k \in \mathbb{P}$. If n is even and $\sigma(n)$ is odd, then $n = 2^f k^2$ for some $f, k \in \mathbb{P}$.*

Friends of 12

Theorem 1. *If $n \notin \{12, 234\}$ is a positive integer such that $I(n) = I(12) = I(234) = \frac{7}{3}$ and n is odd, then $n = 3^{2a}\prod_{i=1}^{k} p_i^{2e_i}$, where $p_1, \ldots, p_k$ are distinct primes greater than 3, $a, e_1, \ldots, e_k \in \mathbb{P}$, and $k \geq 4$. If $n \notin \{12, 234\}$, $I(n) = \frac{7}{3}$ and n is even, then $n = 2 \cdot 3^{2a} \cdot q^e \prod_{i=1}^{k} p_i^{2e_i}$, where $q, p_1, \ldots, p_k$ are distinct primes greater than or equal to 29, $a, e, e_1, \ldots, e_k \in \mathbb{P}$, $a \geq 3$, $k \geq 2$, and $q \equiv e \equiv 1 \mod 4$.*

Proof. Preassuming its existence, let $n \notin \{12, 234\}$ be a positive integer such that $I(n) = \frac{7}{3}$. Since $\frac{\sigma(n)}{n} = I(n) = \frac{7}{3}$, $3\sigma(n) = 7n$ and therefore $3 \mid n$.

Suppose n is odd. Since $3\sigma(n) = 7n$ and $7n$ is odd, $\sigma(n)$ is odd. Since both n and $\sigma(n)$ are odd, n must be a square, by Corollary 2.1. Therefore, $n = 3^{2a}m^2$, where $a, m \in \mathbb{P}$, m is odd and $3 \nmid m$.

If m is 1, then $I(n) = I(3^{2a}) < \frac{3}{2} < \frac{7}{3}$=I(n), a contradiction. Therefore $m > 1$, so $n = 3^{2a}\prod_{i=1}^{k} p_i^{2e_i}$, where $p_1, \ldots, p_k$ are distinct primes greater than 3, $a, e_1, \ldots, e_k \in \mathbb{P}$, and $k \geq 1$.

If $k \leq 2$, then $I(n) \leq I(3^{2a}5^{2e_1}7^{2e_2}) < \frac{3}{2}\frac{5}{4}\frac{7}{6} < \frac{7}{3}$. Therefore $k \geq 3$.

If $k = 3$, then since $I(3^{2a}5^{2e_1}7^{2e_2}17^{2e_3}) < \frac{3}{2}\frac{5}{4}\frac{7}{6}\frac{17}{16} < \frac{7}{3}$ and $I(3^{2a}5^{2e_1}11^{2e_2}13^{2e_3}) < \frac{3}{2}\frac{5}{4}\frac{11}{10}\frac{13}{12} < \frac{7}{3}$, $p_1 = 5$, $p_2 = 7$, and $p_3 \in \{11, 13\}$.

Verify that $I(3^4 5^2 7^2 11^2) = \frac{121}{81}\frac{31}{25}\frac{57}{49}\frac{133}{121} > \frac{7}{3}$. Thus if $n = 3^{2a}5^{2e_1}7^{2e_2}11^{2e_3}$, then $a = 1$. But then $7n = 3\sigma(n) = 3\sigma(3^2 5^{2e_1}7^{2e_2}11^{2e_3}) = 3 \cdot 13 \cdot \sigma(5^{2e_1}7^{2e_2}11^{2e_3})$ would imply that $13 \mid n$, which does not hold. Therefore $p_3 \neq 11$.

Similarly, verify that $I(3^6 5^2 7^2 13^2) = \frac{1093}{729}\frac{31}{25}\frac{57}{49}\frac{183}{169} > \frac{7}{3}$. Thus if $n = 3^{2a}5^{2e_1}7^{2e_2}13^{2e_3}$, $a \in \{1, 2\}$. If $a = 1$ then $I(n) = I(3^2 5^{2e_1}7^{2e_2}13^{2e_3}) = \frac{13}{9}I(5^{2e_1}7^{2e_2}13^{2e_3}) < \frac{13}{9}\frac{5}{4}\frac{7}{6}\frac{13}{12} < \frac{7}{3}$ so $a \neq 1$. If $a = 2$ then $7n = 3\sigma(n) = 3\sigma(3^4 5^{2e_1}7^{2e_2}13^{2e_3}) = 3 \cdot 121 \cdot \sigma(5^{2e_1}7^{2e_2}13^{2e_3})$ would imply that $11 \mid n$, which does not hold. Therefore $a \notin \{1, 2\}$ and we conclude that $k \geq 4$.

Therefore, if n is odd and $I(n) = \frac{7}{3}$, then $n = 3^{2a}\prod_{i=1}^{k} p_i^{2e_i}$, where $p_1, \ldots, p_k$ are distinct primes greater than 3, $a, e_1, \ldots, e_k \in \mathbb{P}$, and $k \geq 4$.

Now, suppose n is even. Since $3 \mid n$, 2^2 does not divide n because if it did then 12 would divide n and we would have $I(n) > I(12) = \frac{7}{3}$. So $n = 2 \cdot 3^b \cdot m$, where $b, m \in \mathbb{P}$, m is odd and $3 \nmid m$.

If $m = 1$, then $I(n) = I(2 \cdot 3^b) = I(2)I(3^b) < \frac{3}{2}\frac{3}{2} < \frac{7}{3}$. Therefore $m > 1$, and $n = 2 \cdot 3^b \prod_{i=1}^{k} q_i^{f_i}$ where $b, f_1, \ldots, f_k \in \mathbb{P}$, $k \geq 1$ and $q_1 < \cdots < q_k$ are distinct primes greater than 3.

Since $7n = 3\sigma(n) = 3\sigma(2 \cdot 3^b \prod_{i=1}^{k} q_i^{f_i}) = 3 \cdot 3 \cdot \sigma(3^b \prod_{i=1}^{k} q_i^{f_i})$, $2 \mid \sigma(3^b \prod_{i=1}^{k} q_i^{f_i})$ but $4 = 2^2 \nmid \sigma(3^b \prod_{i=1}^{k} q_i^{f_i})$.

If all of $b, f_1, \ldots, f_k$ are even then $\sigma(3^b \prod_{i=1}^{k} q_i^{f_i})$ would be odd, and if two or more of $b, f_1, \ldots, f_k$ are odd then we would have $4 \mid \sigma(3^b \prod_{i=1}^{k} q_i^{f_i})$. Therefore exactly one of $b, f_1, \ldots, f_k$ is odd. Further, by Lemma 2.1, if $q \equiv 3 \mod 4$ and is a prime and $e \equiv 1 \mod 2$ then $\sigma(q^e) \equiv 0 \mod 4$ and if $q \equiv 1 \mod 4$ and $e \equiv 3 \mod 4$ then $\sigma(q^e) \equiv 0 \mod 4$. Therefore b must be even, and exactly one of f_i, $1 \leq i \leq k$ is congruent to $1 \mod 4$. Also, for such f_i its corresponding prime divisor q_i is also congruent to $1 \mod 4$. That is, $n = q^e(3m)^2$, where m is a positive odd integer, q is prime that does not divide m, and $q \equiv e \equiv 1 \mod 4$.

If $b = 2$, then $7n = 3\sigma(n) = 3\sigma(2 \cdot 3^2 \prod_{i=1}^{k} q_i^{f_i}) = 3 \cdot 3 \cdot 13 \cdot \sigma(\prod_{i=1}^{k} q_i^{f_i})$ would imply that $13 \mid n$. But then $234 = 2 \cdot 3^2 \cdot 13$ would divide n, and therefore $I(n) > I(234) = \frac{7}{3}$. Therefore, $b \neq 2$. Also, if $b = 4$, then $7n = 3\sigma(n) = 3\sigma(2 \cdot 3^4 \prod_{i=1}^{k} q_i^{f_i}) = 3 \cdot 3 \cdot 121 \cdot \sigma(\prod_{i=1}^{k} q_i^{f_i})$ would imply that $11 \mid n$. But then it follows that $I(n) \geq I(2 \cdot 3^4 \cdot 11) = \frac{3}{2}\frac{121}{81}\frac{12}{11} > \frac{7}{3}$, so we conclude that $b \neq 4$. Therefore $b \geq 6$.

Since $\frac{7}{3} = I(n) \geq I(2 \cdot 3^6 \cdot q_1) = \frac{3}{2}\frac{3^7-1}{2 \cdot 3^6}\frac{q_1+1}{q_1}$, it follows that $\frac{q_1+1}{q_1} \leq \frac{7}{3}\frac{2}{3}\frac{2 \cdot 3^6}{3^7-1}$ and thus $q_1 > 26$. Therefore $q_1 \geq 29$.

If $k = 1$, then $I(n) = I(2 \cdot 3^b \cdot q_1^{f_1}) \leq I(2 \cdot 3^b \cdot 29^{f_1}) = I(2)I(3^b \cdot 29^{f_1}) < \frac{3}{2}\frac{3}{2}\frac{29}{28} < \frac{7}{3}$. Therefore $k \geq 2$.

Suppose $k = 2$. Since $\frac{7}{3} = I(n) = I(2 \cdot 3^b \cdot q_1^{f_1} \cdot q_2^{f_2}) > I(2 \cdot 3^6 \cdot q_1 \cdot q_2) = \frac{3}{2}\frac{3^7-1}{2 \cdot 3^6}\frac{q_1+1}{q_1}\frac{q_2+1}{q_2}$, we have

$$\frac{q_1+1}{q_1}\frac{q_2+1}{q_2} \leq \frac{7}{3}\frac{2}{3}\frac{2 \cdot 3^6}{3^7-1} = \frac{1134}{1093}.$$

Also, since $\frac{7}{3} = I(n) = I(2 \cdot 3^b \cdot q_1^{f_1} \cdot q_2^{f_2}) = I(2)I(3^b \cdot q_1^{f_1} \cdot q_2^{f_2}) < \frac{3}{2}\frac{3}{2}\frac{q_1}{q_1-1}\frac{q_2}{q_2-1}$, we have

$$\frac{q_1}{q_1-1}\frac{q_2}{q_2-1} > \frac{7}{3}\frac{2}{3}\frac{2}{3} = \frac{28}{27}.$$

$$\begin{aligned}
&\frac{q_1+1}{q_1}\frac{q_2+1}{q_2} \leq \frac{1134}{1093} \\
&\qquad \iff (q_1+1)(q_2+1) \leq \frac{1134}{1093}q_1 q_2 \\
&\qquad \iff \frac{41}{1093}q_1 q_2 - q_1 - q_2 - 1 \geq 0 \\
&\qquad \iff (\frac{41}{1093}q_1 - 1)(q_2 - \frac{1093}{41}) - \frac{1134}{41} \geq 0 \\
&\qquad \iff q_2 \geq \frac{1239462}{1681 q_1 - 44813} + \frac{1093}{41},
\end{aligned}$$

and

$$\begin{aligned}
\frac{q_1}{q_1-1}\frac{q_2}{q_2-1} &> \frac{28}{27}\\
&\iff \frac{27}{28}q_1q_2 > (q_1-1)(q_2-1)\\
&\iff \frac{1}{28}q_1q_2 - q_1 - q_2 + 1 < 0\\
&\iff (\frac{1}{28}q_1 - 1)(q_2 - 28) - 27 < 0\\
&\iff q_2 < \frac{27}{\frac{1}{28}q_1 - 1} + 28 = \frac{756}{q_1 - 28} + 28,
\end{aligned}$$

so we have the inequality

$$\frac{1239462}{1681q_1 - 44813} + \frac{1093}{41} \le q_2 < \frac{756}{q_1 - 28} + 28. \quad (1)$$

Since $29 \le q_1 < q_2$ and $q_2 < \frac{756}{q_1-28} + 28$, $29 \le q_1 < \frac{756}{q_1-28} + 27$. To satisfy this inequality, q_1 must be in the interval $[29, 55]$ and therefore $q_1 \in \{29, 31, 37, 41, 43, 47, 53\}$.

If $b = 6$, then $7n = 3\sigma(n) = 3\sigma(2 \cdot 3^6 \cdot q_1^{f_1} \cdot q_2^{f_2}) = 3 \cdot 3 \cdot 1093 \cdot \sigma(q_1^{f_1} \cdot q_2^{f_2})$ so 1093 would divide n. But since $q_2 < \frac{756}{q_1-28} + 28$ and $q_1 \in \{29, 31, 37, 41, 43, 47, 53\}$, $q_2 < 784$ and so $1093 \notin \{q_1, q_2\}$. Therefore $b \neq 6$. If $b = 8$, then since $13 \mid 9841 = \sigma(3^8)$ and $\sigma(3^8) \mid 3\sigma(n) = 7n$, 13 would divide n. However, since $13 < 29 \le q_1 < q_2$, 13 does not divide n and therefore $b \neq 8$. Likewise, if $b = 10$, then since $23 \mid \sigma(3^{10}) = 88573$, 23 would divide n. Since $23 < 29 \le q_1 < q_2$, $23 \nmid n$ and therefore $b \neq 10$. Therefore $b \ge 12$.

Suppose $q_1 = 29$. By the inequality (1), $341 < q_2 < 784$. If $f_1 = 1$, then $7n = 3\sigma(n) = 3\sigma(2 \cdot 3^b \cdot 29^1 \cdot q_2^{f_2}) = 3 \cdot 3 \cdot \sigma(3^b) \cdot 30 \cdot \sigma(q_2^{f_2})$ so 5 would divide n. Therefore $f_1 \neq 1$. Likewise, $f_1 \neq 2$ because $13 \mid 871 = \sigma(29^2)$ but $13 \nmid n$. Therefore, $f_1 \ge 3$. So we have $\frac{7}{3} = I(n) \ge I((2 \cdot 3^{12} \cdot 29^3 \cdot q_2) = \frac{3}{2}\frac{3^{13}-1}{2\cdot 3^{12}}\frac{29^4-1}{28\cdot 29^3}\frac{q_2+1}{q_2}$, and therefore $781 < q_2$. Since there is no prime between 781 and 784, this is a contradiction and we conclude that $q_1 \neq 29$.

Suppose $q_1 = 31$. By (1), $196 < q_2 < 280$; f_1 must be even because $q_1 \equiv 3 \mod 4$, and $f_1 \neq 2$ because if $f_1 = 2$ then $7n = 3\sigma(n) = 3\sigma(2 \cdot 3^b \cdot 31^2 \cdot q_2^{f_2}) = 3 \cdot 3 \cdot \sigma(3^b) \cdot 993 \cdot \sigma(q_2^{f_2})$ would imply that $331 \mid n$, which does not hold because $31 = q_1 < q_2 < 280 < 331$. So $f_1 \ge 4$, and we have $\frac{7}{3} = I(n) \ge I(2 \cdot 3^{12} \cdot 31^4 \cdot q_2) = \frac{3}{2}\frac{3^{13}-1}{2\cdot 3^{12}}\frac{31^5-1}{30\cdot 31^4}\frac{q_2+1}{q_2}$, therefore $q_2 > 278$. This is a contradiction because there is no prime between 278 and 280. Therefore $q_1 \neq 31$.

Suppose $q_1 = 37$. By (1), $97 < q_2 < 112$. $f_1 \neq 1$ because $19 \mid \sigma(37^1)$ but $19 \nmid n$. So $f_1 \ge 2$, and therefore $\frac{7}{3} = I(n) \ge I(2 \cdot 3^{12} \cdot 37^2 \cdot q_2) = \frac{3}{2}\frac{3^{13}-1}{2\cdot 3^{12}}\frac{37^3-1}{36\cdot 37^2}\frac{q_2+1}{q_2}$. This inequality is valid only if $q_2 > 110$, and this is a contradiction because there is no prime between 110 and 112. Therefore $q_1 \neq 37$.

Suppose $q_1 = 41$. By (1), $78 < q_2 < 86$, or $q_2 \in \{79, 83\}$. Since $79 \equiv 83 \equiv 3 \mod 4$ and $q_1 = 41 \equiv 1 \mod 4$, $f_1 \equiv 1 \mod 4$ and f_2 is even. Verify that $I(2 \cdot 3^{12} \cdot 41 \cdot 79^2) > \frac{7}{3}$, therefore $q_2 \neq 79$. Verify also that $I(2 \cdot 3^{12} \cdot 41^5 \cdot 83^2) > \frac{7}{3}$, so if $n = 2 \cdot 3^b \cdot 41^{f_1} \cdot 83^{f_2}$ then $f_1 = 1$. But then $I(n) = I(2 \cdot 3^b \cdot 41 \cdot 83^{f_2}) = I(2)I(41)I(3^b 83^{f_2}) < \frac{3}{2}\frac{42}{41}\frac{3}{2}\frac{83}{82} < \frac{7}{3}$, therefore $q_2 \neq 83$. So $q_2 \notin \{79, 83\}$, and this is a contradiction so we conclude that $q_1 \neq 41$.

If $q_1 = 43$, $71 < q_2 < 78$ by (1) so $q_2 = 73$, and f_1 is even because $q_1 \equiv 3 \mod 4$. It would follow that $I(n) \ge I(2 \cdot 3^{12} \cdot 43^2 \cdot 73) = \frac{3}{2}\frac{3^{13}-1}{2\cdot 3^{12}}\frac{43^3-1}{42\cdot 43^2}\frac{74}{73} > \frac{7}{3}$; therefore $q_1 \neq 43$.

If $q_1 = 47$, then $62 < q_2 < 68$ by (1) so $q_2 = 67$. But since $47 \equiv 67 \equiv 3 \mod 4$, $q_1 \neq 47$, for if $q_1 = 47$ then $n = 2 \cdot 3^b \cdot 47^{f_1} \cdot 67^{f_2}$ and $\sigma(n) = \sigma(2 \cdot 3^b \cdot 47^{f_1} \cdot 67^{f_2})$ would either be odd or divisible by 4.

If $q_1 = 53$, then $54 < q_2 < 59$. Since there is no prime between 54 and 59, $q_1 \neq 53$.

Therefore, $q_1 \notin \{29, 31, 37, 41, 43, 47, 53\}$. This is contradictory to the assumption that $k = 2$, so we conclude that $k \ge 3$.

Therefore, $n = 2 \cdot 3^{2a} \cdot q^e \prod_{i=1}^{k} p_i^{2e_i}$, where $q, p_1, \ldots, p_k$ are distinct primes greater or equal to 29, $a, e, e_1, \ldots, e_k \in \mathbb{P}$, $a \ge 3$, $k \ge 2$, and $q \equiv e \equiv 1 \mod 4$.

□

References

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