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# Lewis Parker Lecture 2014
# Recovery from Research Doldrums and Resulting Discoveries: A personal mathematical narrative.

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I became chair of the Mathematic Department at Auburn University at a time when I had ceased to do research. Much as I wanted to get back into research, becoming chair of a large department made doing so even more difficult. I will discuss a sequence of fortuitous events along with my continued fascination with indecomposable continua that allowed me to jump start my research once again. This led to my research in the areas of non-metric indecomposable continua, Whitney maps and generalized inverse limits. These concepts will be defined and I will outline my successful research in these areas.

It sometimes happens that a once productive mathematician will stop doing research. As a former chair of a mathematics department I'm aware of such incidents and understand some of these circumstances. The hum-drum nature of everyday life can sap our attention and slow down one's creativity. Some researchers get tired and loose interest in doing research; some have worked on hard problems without any discernible progress and become frustrated with the idea of doing research; some loose the energy of their youthful exuberance as retirement seems not so far down the road; personal issues and family problems can often impact on all of our work; sometimes the other duties and obligations to our institution just does not leave us much time to do research and dedicating all our weekends to research may not be healthy; sometimes one's interest veers away from mathematical research; sometimes a person just feels that he or she is just not capable of it anymore.

One's personal or work environment can in many ways affect one's conduct in a job in general, sometimes to the detriment of our productivity. How does a head or chair in a mathematics department address this problem when he/she sees a faculty member becoming less productive? I feel that the best strategy is to work together with a faculty member to find ways of encouraging them in their research and in the meantime to find alternate ways for them to be productive in the department. A conscientious faculty member will be very willing to work with an encouraging chair. Because of a combination of some of the reasons I listed above, in the 1990's and early 2000's I did not publish any research. But, along with the support of my chair at the time, I did two things. I took on additional administrative duties and I got involved with mathematics education grants. Thus I was able to continue being a productive member of the department – and Auburn's mission includes teaching and outreach such as our grant work. Such was my position in 2002: I was acting as the Associate Chair of the department at the time – my main duties included working out the teaching schedules for the two mathematics departments then at Auburn which included making assignments to 50 – 70 odd graduate students; and I became involved in a multi-million dollar Mathematics Education Grant, first as a writer and eventually as a senior personnel member of the grant. So I was certainly "pulling my weight" for the department. I was still interested in mathematics and enjoyed figuring out challenging problems, but I was not doing "new" research. Then in 2002, for personal reasons, the chair of the Mathematics Department, resigned. Because at that time I was "Associate" Chair, the Dean asked me to serve as acting chair of the department until a permanent chair could be selected by due process. Without going into the messy details, I eventually became the permanent chair. Now this placed a considerable amount of additional responsibility and work on my shoulders and so made the likelihood of returning to research even less likely.

It takes at least two years to learn the job as chair and I had the luck to have access to former chairs for advice – still the work was considerable. And, in addition, I was determined to continue the teaching that I loved so much; in fact that was one of the important reasons that I became a faculty member at Auburn. Then two unrelated events happened; at this time I do not recall which came first.

One: I received an invitation to submit an article for a special volume of a journal that was to be dedicated to David Bellamy a well-known and outstanding mathematician (many thanks to the editor Wayne Lewis for the invitation);

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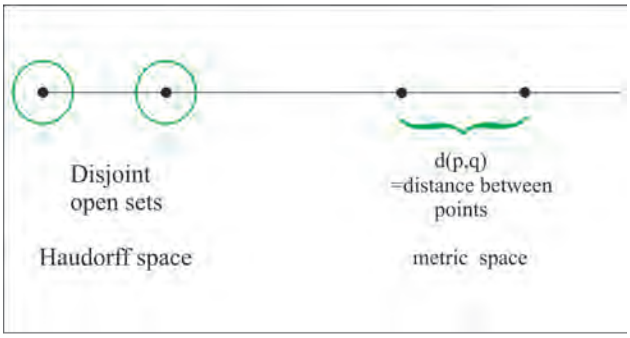
back when I was doing research I had done some work in areas related to his work. Coincidentally I was mulling around an idea about a property of indecomposable subcontinua in what might be called the "long plane," so named because it is the product of two long lines. So let me define these objects and tell you my idea and what became of it. . . . This idea led to the first paper (Smith, 2007) I had written in over a decade! It was reviewed and accepted by the special Bellamy edition of the journal.

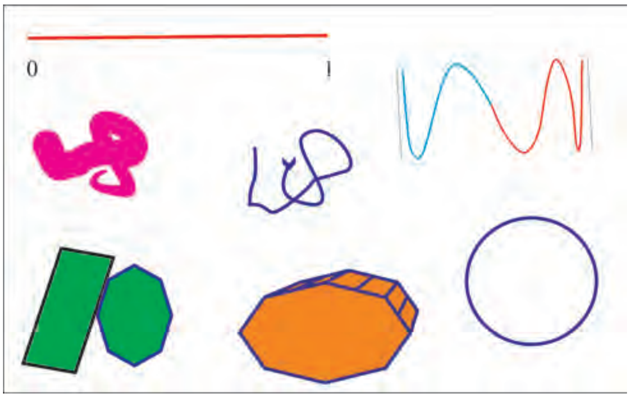
First some definitions: A Hausdorff space is one where for each pair of points there is a pair of disjoint open sets containing them. And a metric space is one that has a distance function that satisfies the triangle inequality.

*Figure 1*. Hausdorff and metric spaces




A continuum is a compact connected Hausdorff space. A continuum that is the union of two of its proper subcontinua is said to be decomposable. Following in Figure (2) is a sample of familiar decomposable continua.

*Figure 2*. Decomposable Continua




A continuum that is not decomposable is called indecomposable. Singleton points are obviously indecomposable; but showing that there is such a thing as a non-degenerate indecomposable continuum takes a bit of work. Not only do I want to convince you that there are such things as indecomposable continua, I want to argue that there are lots of them. This argument is an outline of work due to Bing (1951). To do that I want to look at the space of subcontinua of a space and so will introduce some notation: If $X$ is a topological space then $2^X$ denotes the space of compact subsets of $X$ and $C(X)$ denotes the subspace of $2^X$ consisting of the subcontinua of $X$. If $X$ is a metric space then $C(X)$ is also a metric space. If $d$ is the distance function or metric for $X$ then we construct a metric $D$ for $2^X$ as follows: first for the element $H$ of $2^X$ define for a positive number $\epsilon$:

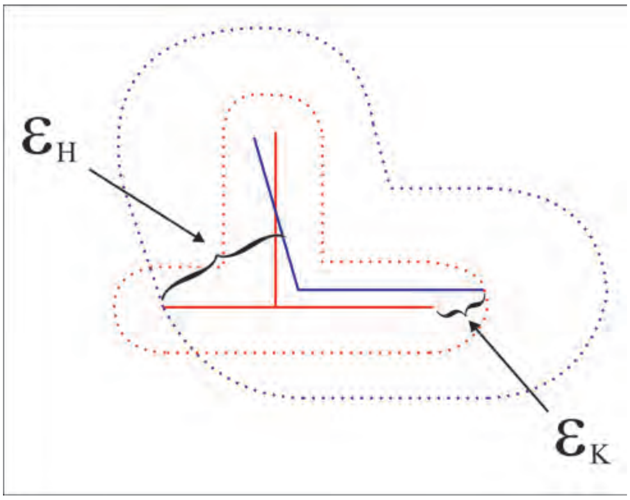
$$D_\epsilon(H) = \{x \in X | \mathrm{d}(x, H) < \epsilon\}$$

This is essentially an $\epsilon$-ball around $H$. Then we define the "distance" from $H \in C(X)$ to $K \in C(X)$ by,

$$D(H, K) = \inf \{\epsilon | K \subset D_\epsilon(H) \text{ and } H \subset D_\epsilon(K)\}.$$

This is called the Hausdorff metric. The distance from $H$ to $K$ can be interpreted as the infimum of the epsilons so that each one of the continua is contained in the epsilon ball of the other. In the following picture the larger of $\epsilon_H$ and $\epsilon_K$ is the Hausdorff distance from $H$ to $K$.

*Figure 3*. The Hausdorff metric




For the Euclidean plane or Euclidean $n$-space ($n > 0$), the space of subcontinua of $E^n$ is a locally compact metric space with the Hausdorff metric and so it is a complete metric space (by which we mean that Cauchy sequences converge.) I will need the important Baire Category theorem. I state two versions of the theorem.

- Version 1: A complete metric space is not the union of a countable set of no-where dense sets.
- Version 2: In a complete metric space, the common part of a countable collection of dense open sets is non-empty and dense in the space.

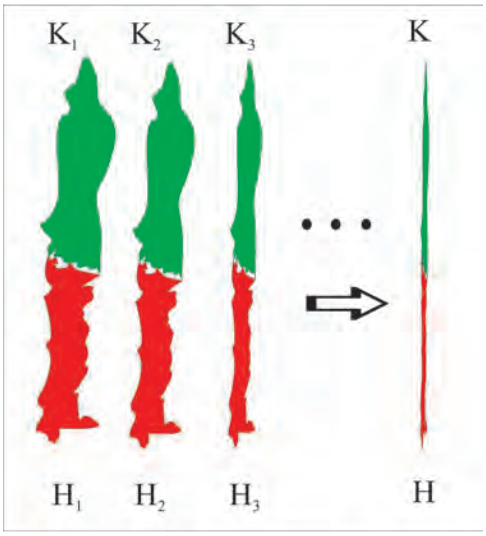
A nowhere dense set $M$ is a set with the property that for every open set $U$ intersecting $M$ there is an open subset of $U$ that misses $M$. The countable common part of a collection of open sets is called a $G_\delta$ set. One of my favorite theorems (due to Bing (1951)) uses the Baire Category theorem to prove the existence of indecomposable continua. It is sort of like proving that there is an irrational (or transcendental) number by showing that the complement only forms a countable (and hence the countable union of nowhere dense sets) subset of the reals – which is a complete metric space - so that there is something left over that has to exist.

The Baire category theorem says that, in a complete metric space, the complement of the countable union of nowhere dense sets is dense in the space. And is, in a sense, a "big" set. For ease of understanding, let us consider the Euclidean plane for our construction. The argument works just as well in higher dimensional Euclidean spaces. Consider now the set $S_n$ of all continua that are the union of two proper subcontinua that are at least distance $\frac{1}{n}$ apart in the Hausdorff metric. One can make the reasonable guess that this set is closed. I want to claim that it is closed and nowhere dense. So, in line with the argument, define a countable collection of sets $S_n$ as follows: for each positive integer $n$ let

$$S_n = \left\{ M | M = H \cup K; D(H, K) \geq \frac{1}{n} \right\}$$

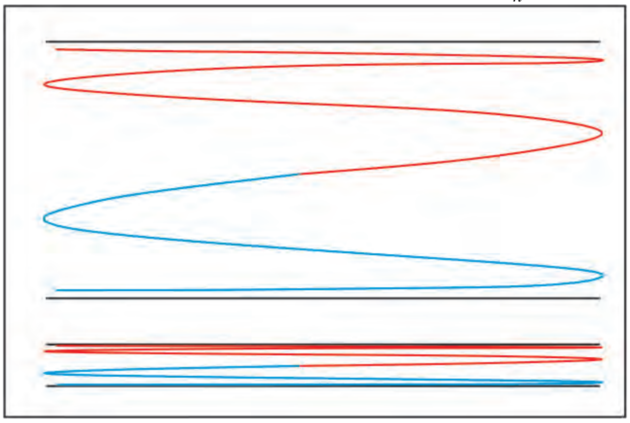
where $H$ and $K$ are subcontinua of $M$. These $S_n$'s will be nowhere dense sets in the hyperspace $C(X)$ of subcontinua of the plane (or $E^n, n \geq 2$, or $n$-manifold, or Hilbert space, etc. ) I will argue that these sets are closed and nowhere dense by picture. First, to see that the sets are closed we need to know that if pairs of points $H_i, K_i$ stay apart by distance $\frac{1}{n}$ then the continua that they limit to also stays apart by at least that much (this is why we need the equality in the ≥ symbol.)

*Figure 4*. Closure property of $S_n$




In Figure 5 below the top continuum lies in $S_n$ where $\frac{1}{n}$ is at most the distance between the middle of the continuum and the top or bottom edge. The lower continuum, which is just a "flattening" of the upper continuum, lies in $S_n$ for a much smaller $\frac{1}{n}$. So this is a "large" continuum that lies in $S_n$ for a relatively small $\frac{1}{n}$. Note that any decomposition of it into two proper subcontinua would have the top and bottom bars in different proper subcontinua. In Figure 5 this is indicated as red and blue. Notice that I can make this distance arbitrarily small by shrinking it along the $y$-axis even further.

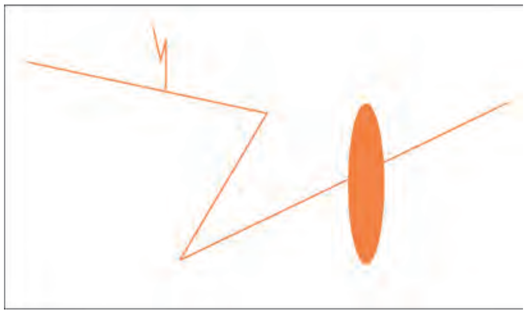
*Figure 5*. A "large" continuum in $S_n$ for a small $\frac{1}{n}$



Following is a three step "picture proof" that each $S_n$ is nowhere dense.

1. Consider $S_N$ for some $N$ and pick an arbitrary continuum in it and an open set $U$ containing that continuum.

*Figure 6*. Proof Step 1



2. Find an arc "close" close to it, close in the sense of lying in $U$ (Figure 7).

3. Fatten the arc to "thin" double $\sin\left(\frac{1}{x}\right)$ curve so that this new continuum still lies in $U$. Make it thin enough so that it is not in $S_N$ for this integer $N$ (Figure 8). Since the set $S_N$ is closed and this double $\sin\left(\frac{1}{x}\right)$ curve is not in $S_N$, then there is an open set containing this double $\sin\left(\frac{1}{x}\right)$ curve that is a subset of $U$.

It therefore follows from the Baire category theorem, that the union of the nowhere dense sets $S_1, S_2, S_3, \ldots$ cannot be all of the subcontinua of the plane. But if $M = H \cup K$ is

*Figure 7*. Proof Step 2

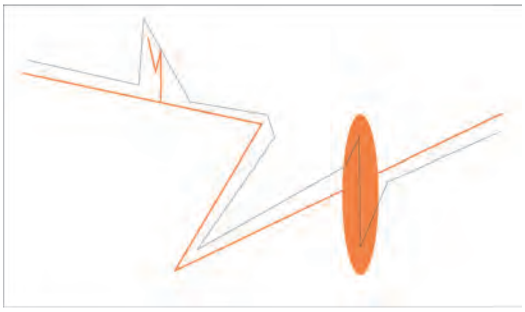
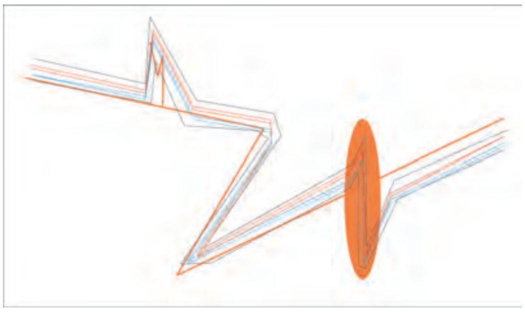


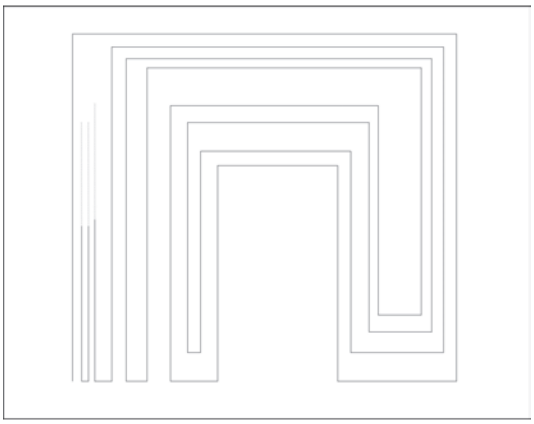
*Figure 8*. Proof Step 3



a decomposable continuum then there is an integer $n$ so that the distance from $H$ to $K$ is greater than $\frac{1}{n}$: $D(H, K) > \frac{1}{n}$ and so the continuum belongs to one of the sets $S_n$. So the "left over stuff" consists of indecomposable continua.

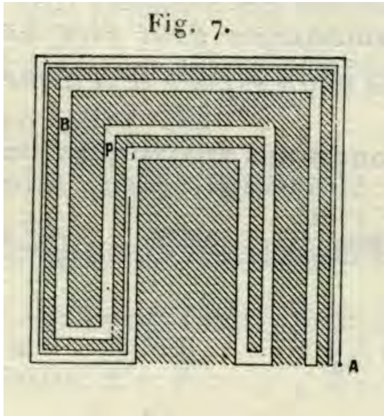
Indecomposable continua were actually discovered some decades before Bing published his proof using the Baire category theorem. Figure 9 is a graphic of one of the earliest published example due to Janiszewski (1912). Figure 10 is an image from his Ph.D. thesis of one of the steps in his construction.

*Figure 9*. Janiszewski Example 1



Let us modify our definition of the sets $S_n$ of continua. We modify the definition by requiring that each element of $S_n$ to contain rather than be a continuum which can be decomposed into two subcontinua at least $\frac{1}{n}$ apart: so now let $M \in S_n$ if and only if $M$ contains a subcontinuum $L = H \cup K$ so that $(H, K) \geq \frac{1}{n}$ . The argument that each $S_n$ is closed is essentially the same. It takes a bit more work to prove each $S_n$ is nowhere dense in $C(X)$, where $X$ is a Euclidean space of dimension greater than 1 (or Hilbert space or manifolds of dimension greater than 1) but Bing did this in (R. Bing, 1951). A continuum that does not contain a non-degenerate decomposable continuum is said to be hereditarily indecomposable. So Bing's result gives us the following theorem.

*Figure 10*. Janiszewski Example 2




**Theorem 1.** *If $X$ is an $N$-dimensional Euclidean space or manifold, $N \geq 2$ or Hilbert space then the subset of the hyperspace $C(X)$ of $X$ consisting of the hereditarily indecomposable subcontinua of $X$ is a dense $G_\delta$ subset of $C(X)$.*

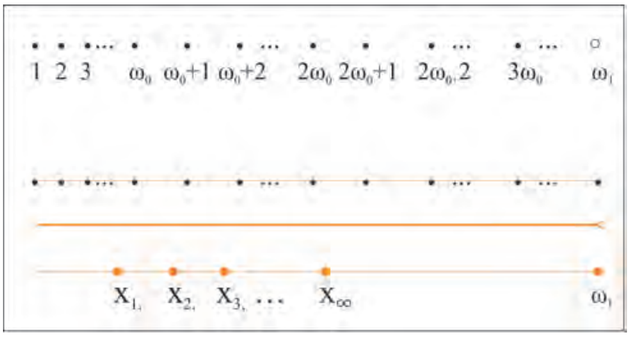
Bing actually proved a stronger theorem. He showed (R. Bing, 1951):

**Theorem 2.** *If $X$ is an $N$-dimensional Euclidean space or manifold, $N \geq 2$ or Hilbert space then the subset of the hyperspace $C(X)$ of $X$ consisting of the hereditarily indecomposable chainable subcontinua of $X$ is a dense $G_\delta$ subset of $C(X)$.*

Furthermore he showed (R. Bing, 1951) that each two non-degenerate hereditarily indecomposable chainable continua are homeomorphic. This object was named the pseudo-arc, because it has the additional property, like the arc, of being homeomorphic to each of its non-degenerate subcontinua (Moise, 1948). The first paper to describe this continuum was by Knaster (1922). Many mathematicians in addition to Bing and Moise have studied this object and I mention the work of Lewis and Mouron as major contributions to the area.

Before I had stopped doing research, I had published several papers on the pseudo-arc. My current interest was in considering non-metric versions of these results and that leads to my discovery. Start with the first uncountable ordinal $\omega_1$ and connect each point to the next one above it in the well ordering with a metric interval. Compactify it by adding the last point $\omega_1$ and you have a non-metric arc $L$. That is: $L$ is a Hausdorff continuum with exactly two non-cut points. (That is every point except for the two "end" points separates the continuum.) One property of the long line is that for any $x < \omega_1$ the interval $[0, x]$ is just a metric arc and so is topologically equivalent to the interval $[0, 1]$.

*Figure 11*. Constructing the long line




Another important property of the long line is that for any infinite countable sequence bounded above by $\omega_1$ there is always an upper bound of it that is below $\omega_1$. So consider a product of two compactified long lines to produce a compactification of a "long plane" or "long rectangle" $R = L \times L$. I wanted to consider the possibility of a version of Bing's theorems for $R$ and to look at the relationship between metric and non-metric indecomposable subcontinua of this long rectangle. Since this space contained usual metric rectangles, it contained lots of metric indecomposable continua. Because every subcontinuum of the long line that misses the $\omega_1$ point is metric it follows that the only non-metric indecomposable subcontinua of $R$ must intersect either the top or left edge, $L \times \{\omega_1\}$ or $\{\omega_1\} \times L$ respectively.

*Figure 12*. A standard metric indecomposable continuum in the long rectangle

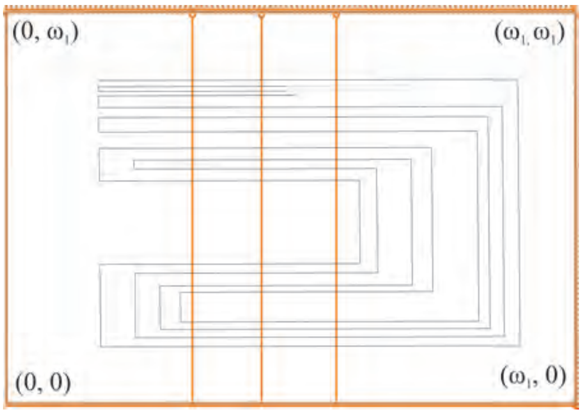
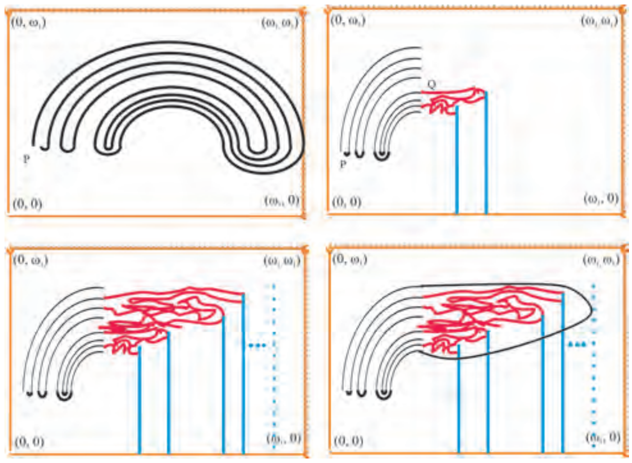



Figure 12 shows an indecomposable continuum lying in my long rectangle. Notice that it is metric because it does not touch the top and bottom edge. What I discovered was that any continuum that touches the top and bottom edge has to be decomposable. So the property of being an indecomposable subcontinuum of this space implies that it is metric. This is a very unusual property. Following is an outline of the proof in pictures.

*Figure 13*



Reading the pictures from left to right as usual in Figure 13: The first picture displays an indecomposable continuum $M$ that touches the right edge $\{\omega_1\} \times L$ of the space; we assume that such a continuum exists. (I have pictured a curvilinear version of the Janiszewski example called the "buckethandle.") For picture 2, we start with an open subset $O$ of the continuum, whose closure is not all the continuum, and analyze how the continuum stretches toward the $\omega_1$ end. Call that point of $M$ that is leftmost in the picture $P$. By properties of indecomposable continua, $O \cup M$ will have uncountably many components; also there will be a point $Q$ so that $M$ is irreducible from $P$ to $Q$. In picture 2 then assume that we are stretching from the point $Q$ inside $M$. The point $Q$ will be an "internal" point in the same sense that the Cantor set has "internal" points that are not endpoints of complementary open sets.

As we connect the small internal component of the part of $M \cap O$ containing $Q$ from the open set $O$ further and further to the right, we connect up more components of $M \cap O$ as picture 3 shows. But as we do this we limit closer and closer to that end point $P$ near $(0, 0)$ by stretching farther and farther from the "internal" point $Q$ of $M$. In a countable number of steps we have stretched from the internal point $Q$ all the way to the end point $P$. But this is done in a countable number of steps; and from the property of the long line, that stretching is bounded to the right by $\omega_1$. Since the same argument works for the top $\omega_1$ edge, we have limited onto the whole continuum inside of a metric box. So no indecomposable subcontinuum of this long rectangle can touch those two edges and so there are no non-metric indecomposable subcontinua of this long rectangle.

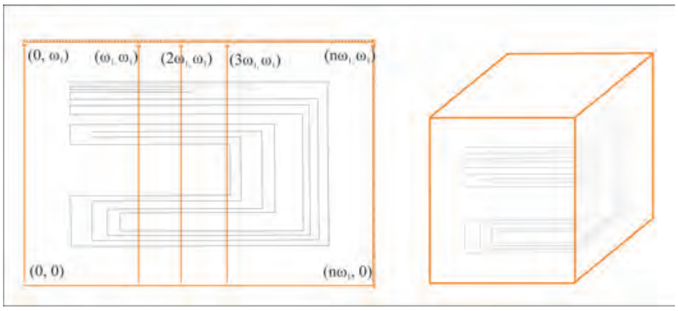
**Theorem 3** ((Smith, 2007))**.** *If $L$ is the compactified long line and $M$ is an indecomposable subcontinuum of $L \times L$ then $M$ is metric.*

I distinctly remember figuring this out and realizing that I had a nice result that I could expand on for that invited paper for that special edition of the journal dedicated to David Bellamy. I was down the road from Auburn in downtown Columbus in a little book shop that had easy chairs and served coffee. I was waiting for my daughter while she took her oboe lessons and I was in the habit of always buying a NY Times there and reading it with a cup of coffee while I waited. That particular day the news must not have been very interesting (or too depressing). So I pulled out pen and paper and did some doodling (I always had a clipboard, pen and paper with me) – I drew some pictures similar to those in Figure 13 and figured out this basic argument. The rest of the paper is built on this insight.

What about the product of three copies of the long line glued together end to end? It turns out that you can kind of go "through" the $\omega_1$ points as is illustrated in Figure 14. The technique generalizes to the following.

**Theorem 4** (Smith (2007))**.** *Every product of three non-metric arcs contains a non-metric indecomposable continuum.*

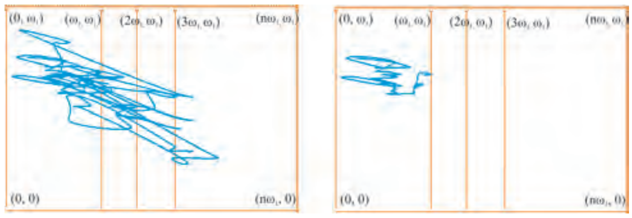
*Figure 14*. A non-metric indecomposable subcontinuum in the product of three non-metric arcs.




So we considered "long planes" made of products of long lines strung end to end. One can go "through" one of the $\omega_1$ points with Janiszewski-like continua. This is done by considering a longer arc that is a sequence of $\omega_1$ long arcs glued end to end. All that has been done to obtain the non-metric indecomposable continuum in this new product space, is that a collection of metric arcs in the Janiszewski example has been replaced by copies of the compactified long lines. This is what makes the indecomposable continuum non-metric. So this product of non-metric arcs contains a non-metric indecomposable continuum. I had to use the rectilinear version of the Janiszewski continuum as opposed to a "bucket-handle" continuum like the one pictured in the first step of Figure 13, because it is not possible to draw a circle in a non-metric plane (or higher dimensional analogue) since there is no distance function for these spaces. The first picture in Figure 13 is very misleading (in fact it is not possible for more than one reason.)

But what about hereditarily indecomposable subcontinua of these long rectangles? A short picture proof shows that this is not possible.

*Figure 15*



Suppose that there were hereditarily indecomposable going across those $\omega_1$ points as in the left frame of Figure 15. Then we would have a subcontinuum of it, as in the right frame of Figure 15, that would lie entirely inside one copy of that first "long plane" which, I already showed, is impossible. So in this situation there are no non-metric hereditarily indecomposable continua in the long rectangle; these hereditarily indecomposable continua have to all be metric because they can not go "over" one of the $\omega_1$ points. The main theorem of that paper was:

**Theorem 5** (Smith (2007))**.** *Let $A$ be a non-metric Hausdorff arc such that the set $Q$ of $\omega_1$- type of points is no-where dense and such that for each point $p \in A - Q$ there is a metric open set containing $p$ (as is the case with the long line) then every hereditarily indecomposable subcontinuum of $A^n$ is metric for $n > 1$ with $n$ finite or countable.*

This connection with a mathematical friend led to a wonderful reassuring experience in doing research. The other event that happened while I was chair around this time was the appearance of a graduate student in my office. She had returned to continue her work on the Ph.D. after leaving the department for a period of time. She said that she was interested in continuum theory and asked about working with me. I told her that I had not done much research in quite a while and that since I was department chair I would have limited time and it would sometimes be difficult to meet; so she would have to be willing to do a lot of the work on her own. I had been recommended to her by one of her mentors (and a good friend of mine). She indicated that she would like to learn some continuum theory anyway. That much I could do.

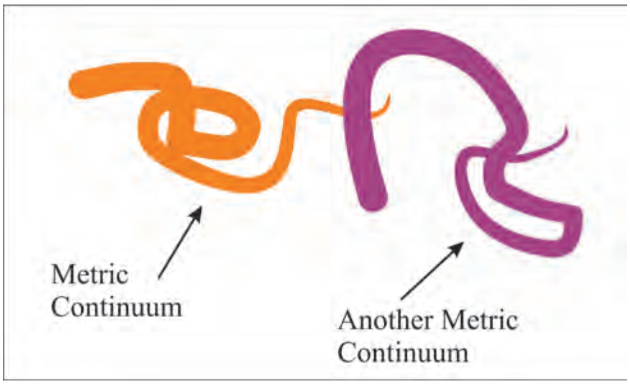
There was concept that I was interested in learning more about and so we spent some time learning about Hyperspaces (which I mentioned earlier) and a function called a Whitney map (see S. Nadler (1978) for background material on hyperspaces.)

**Definition 1.** *If $X$ is a Hausdorff space, then $2^X$ denotes the hyperspace of compact subset of the space $X$ with the Vietoris topology; $C(X)$ denotes the subspace of $2^X$ consisting of the subcontinua of $X$. Classically, if $X$ is a metric space then a mapping $\mu : 2^X \to [0, \infty)$ is called a Whitney map if it is a continuous function so that:*

1. $\mu(\{p\}) = 0$ *for all* $p \in X$,
2. *if* $H, K \in 2^X$ *and* $H \subset K$, $H \neq K$, *then* $\mu(H) < \mu(K)$.

It follows from a theorem of V. E. Sneider (1945) that if $X$ is compact and there is a Whitney map on $2^X$ then $X$ has to be a metric space. However, I wondered, and this was the question that I asked the graduate student, if it is possible to have a Whitney map on $C(X)$ for some non-metric continuum $X$? With my encouragement, she found a continuum that was non-metric every proper subcontinuum of which was an arc; and using that fact she built a Whitney map on it. (Using, of all things, the arctan function.) Back in the 70's I had constructed a continuum every proper subcontinuum of which was metric (Smith, 1976). It is pretty easy to see that the union of two intersecting metric continua is a metric continuum. So a non-metric continuum every proper subcontinuum of which is metric must be indecomposable! – since two intersecting proper subcontinua of it could not add up to the whole thing. (See Figure 16.)

*Figure 16*




Her continuum looked something like the one pictured below in Figure 17. If you take the common part of a sequence of thinner and thinner tori (doughnuts) each one of which does a double loop in the previous one, then you have the metric solenoid.

My student's construction was a lot like the solenoid. As I indicated earlier, in the non-metric setting it is easiest to do things rectilinearly. Which is why the doughnut looks like it was made of Lego blocks. Look at the indicated cut across the continuum.

The left image shows successive cuts that the tori make in the cross sectional cut shown in Figure 18 and the right image is the cross section of common parts of all the tori.

*Figure 17*

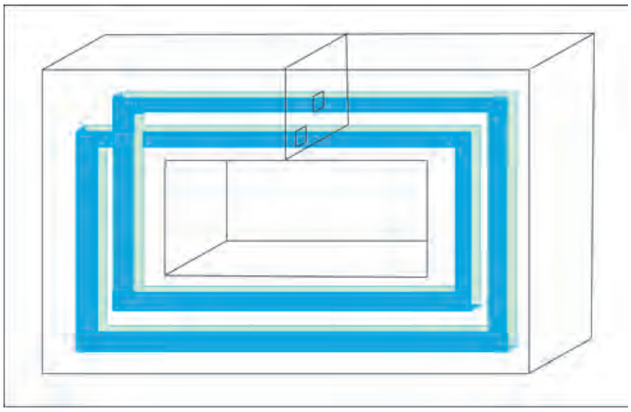
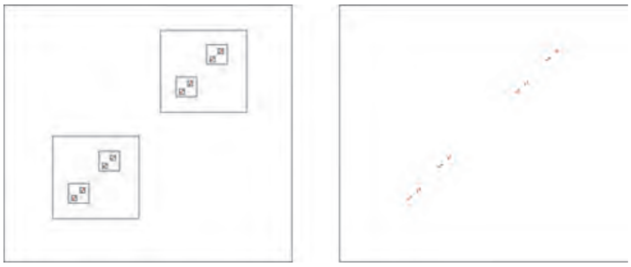


*Figure 18*



For the metric solenoid, this is the well-known Cantor set. For my student's construction, it is a non-metric analogue of the Cantor set called the Double Arrow space. This is a well-studied separable compact totally disconnected non-metric space.

Sadly it turned out that Charatonik (2000) had shown that a non-metric indecomposable example of Guteck and Hagopian (1982) supported a Whitney map of the type that we wanted. So I asked her, "Well can you get one that has the property that every (non-degenerate) subcontinuum of it is non-metric?" This she did, after many hours in my office catching me when she could; her example evolved into her Dissertation. She essentially used a process of "blowing up" uncountably many well distributed points of her first example into a copy of her original continuum; repeating this process infinitely many times and in such a way that every non-degenerate subcontinuum has to pass through one of these blown-up points (Smith & Stone, 2014; Stone, 2008).

**Theorem 6** (Stone (2008))**.** *There exists a non-metric continuum that supports a Whitney map every non-degenerate subcontinuum of which is non-metric.*

So in spite of my research doldrums and now busy schedule as chair I was able to direct a Ph.D. student. It occurred to me that if I could successfully direct a graduate student to do original creative research then (obviously it seemed to me) I should be able to "direct" myself to do the same. So I went back to that paper about hereditarily indecomposable subcontinua of the product of non-metric arcs with those $\omega_1$-type of points. The natural question is what about the existence of hereditarily indecomposable continua in other non-metric arcs. So I considered the lexicographic arc, this is a well-studied non-metric arc. It is actually the square disc with the points ordered according to the dictionary or "lexicographic" ordering.

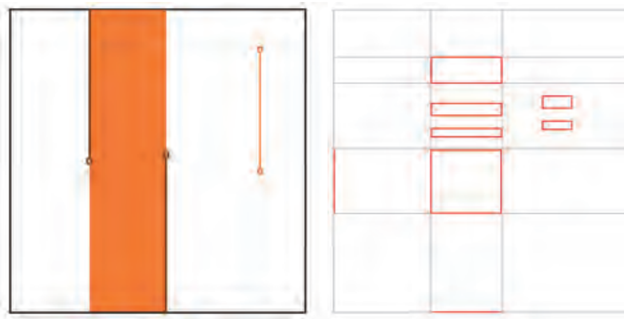
**Definition 2.** *The lexicographic arc: Let* $L = [0,1] \times [0,1]$ *and define a "dictionary" order* $<_L$ *on* $L$ *as follows:* $(a,b) <_L (c,d)$ *if and only if:*

$$a < c \text{ or}$$

$$a = c \text{ and } b < d.$$

*The space* $L$ *with the topology induced by the order* $<_L$ *is a non-metric arc.*

*Figure 19.* Lexicographic arc and the square of two lexicographic arcs.



The left picture of Figure 19 is the lexicographically ordered square disc with a couple of basic open sets shown. Note that it is easy to see that it is not metric because it contains uncountably many disjoint open sets: each set of the form $\{x\} \times (0,1)$ is open. It has the property that every open set contains a copy of the reals. Two typical open subsets of $L$ are shown in the left diagram; note that one of them looks like an ordinary open interval of the reals and the other contains uncountably many disjoint such intervals. The order topology makes it into a Hausdorff arc.

Although I have it pictured as a square, it really is not anything like a square. In the right picture two of these arcs have been stretched out and made into a non-metric square by taking the product of two of them. You can envision $L$ by thinking of $[0,1]$ with each point blown up into a copy of $[0,1]$. Similarly you can envision $L \times L$ as $[0,1] \times [0,1]$ with each point blown up into a copy of $[0,1] \times [0,1]$. My question was what could be said about the non-metric indecomposable and hereditarily indecomposable subcontinua of $L \times L$. As with the long rectangle examined above, this space contains lots of metric indecomposable and hereditarily indecomposable continua. I discovered the following (Greiwe, Stone, & Smith, 2014).

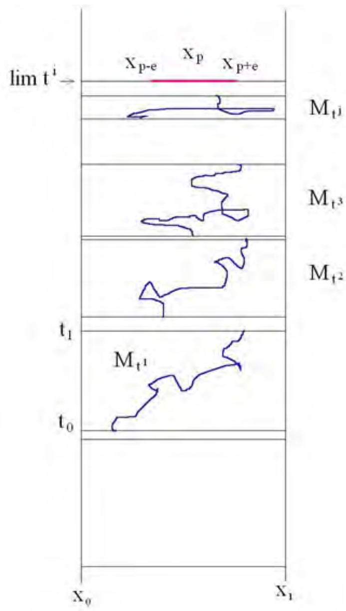
**Theorem 7** (Greiwe et al. (2014))**.** *Every non-metric indecomposable subcontinuum of* $L \times L$ *contains an arc.*

**Corollary 1** (Greiwe et al. (2014))**.** *If* $L$ *is the lexicographic arc then every hereditarily indecomposable subcontinuum of* $L \times L$ *is metric.*

This result eventually led to a joint paper with R. Greiwe and J. Stone (2014). So now I had a "research topic" to direct myself on. I want to outline the argument that $L \times L$ does not contain a non-metric hereditarily indecomposable continuum.

Suppose that $M$ is a subcontinuum of $L \times L$ that is non-metric and lies entirely inside a copy of $[0,1] \times L$ as in the image in Figure 20. Then, since it is non-metric, it has to intersect uncountably many copies of $[0,1] \times [0,1]$. Then there will be a sequence of subcontinua $M_{t^1}, M_{t^2}, M_{t^3}, \ldots$ of $M$ that limit to an arc. Now suppose that $M$ intersects more than one column as in Figure 21. Then there is a non-metric sub-continuum of it (the dark blue) that intersects a copy of $[0,1] \times L$ and so, by the previous result, contains an arc.

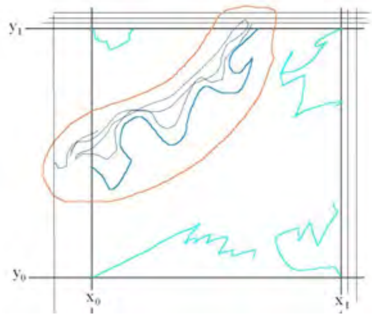
*Figure 20*




It turns out that the argument works for the lexicographically ordered cube and so on to higher dimensions.

So I started looking at products of other non-metric Hausdorff arcs. The plan was to either find non-metric hereditarily indecomposable subcontinua in them or show that none were possible. Next I looked at the Souslin line. It is different from the reals in a particularly interesting way. A Souslin line is a connected linearly ordered space (different from the reals) which does not contain uncountably many disjoint open sets. The existence of Souslin lines is independent of the axioms

*Figure 21*




of set theory and so may or may not exist depending on the axioms of set theory used. A Souslin arc is just a Souslin line that has been compactified by adding a first and last point. The following theorem led to another publication.

**Theorem 8** (Smith (2009))**.** *Suppose that $S$ is the Souslin arc. Then every hereditarily indecomposable subcontinuum of $S \times S$ is a metric continuum.*

I also showed that the only way that $S \times S$ can contain a non-degenerate hereditarily indecomposable continuum is if the Souslin line under consideration contains a copy of the reals and the hereditarily indecomposable continuum lies in the product of two of these copies of the reals.

**Theorem 9** (Smith (2009))**.** *Suppose that $S$ is a Souslin arc, that $Y$ is any Hausdorff arc and that $M$ is a hereditarily indecomposable subcontinuum of $S \times Y$ so that the projection on the second coordinate is all of $Y$. Then $Y$ is a metric arc.*

With my successful production of a Ph.D. student, Jennifer Stone (Actually co-directed between myself and her former mentor Dr. Jo Heath), another student came to me and indicated an interest in working with me. She had written a Master's thesis on the lexicographic arc and so I had the perfect problem for her: Is every hereditarily indecomposable subcontinuum of the product of three Lexicographic arcs metric? It took a while and involved developing new techniques – but she did it. So I produced another Ph. D. student.

**Theorem 10** (Greiwe (2009))**.** *Let $L$ denote the lexicographic arc. Then every hereditarily indecomposable subcontinuum of $L \times L \times L$ is metric.*

Once you start thinking about mathematics problems, it becomes almost a subconscious activity. You will drift off in the middle of something and start puzzling over a problem. Since I am talking about things that inspire, and jump starts, research I want to relate an incident that led to another discovery and subsequent paper. This occurred when the next student Regina Greiwe had her final Ph.D. defense.

As you well know, the committee members test the student's knowledge on the material related to her (or his) dissertation. Dr. Gary Gruenhage was on her committee and he had a question. Since she had been working with non-metric hereditarily indecomposable continua, he asked her about those type of continua. The examples that she had discussed were either metric or not perfectly normal (an important set theoretic property: a perfectly normal space is one where every closed set is a $G_\delta$ set). So he asked, "Do you have an example of a non-metric perfectly normal hereditarily indecomposable continuum." She didn't know the answer to the question! (And was very worried, at a Ph.D. defense, students expect to be able to answer ALL the questions.)

I admitted, at the defense, to the committee that I did not know the answer either. Neither, apparently, did Dr. Gruenhage. She passed her defense. But that problem stuck with me – I should have known the answer; in my past research – now some two decades old - I had produced many non-metric hereditarily indecomposable continua; none were perfectly normal – I checked mentally the ones I remembered at the defense exam. Then, and I want to say it was that night – my memory is fuzzy. But in any case one night soon thereafter, I woke up at 3:00 am in the morning and I knew the answer! It was an example that was in my previous student Jennifer Stone's dissertation. It looked perfectly normal – but we had to prove it!

So the three of us met weekly at a local public house, perhaps in a small way emulating the famous group of mathematicians from the Lwow School of Mathematics in Poland who met at the Schottish Café where they discussed mathematics. Together we showed that the example was indeed perfectly normal, put together some other pieces (e.g. the hereditary indecomposablity) and sent off a paper. I assumed that if Gary Gruenhage did not know the answer, probably nobody did.

**Theorem 11** (Greiwe, Stone, and Smith (2012))**.** *There exists a separable perfectly normal, and hence first countable, non-metric hereditarily indecomposable continuum.*

The paper was eventually accepted for publication. We worked well together and together we re-examined the argument that I had about hereditary indecomposable subcontinua of the product of two lexicographic arcs and produced another joint paper.

**Theorem 12** (Greiwe et al. (2014))**.** *Let $L$ denote the lexicographic arc. Then $L \times L \times L$ contains a non-metric indecomposable continuum that does not contain an arc.*

This showed that one of my results on the product of two lexicographic arcs does not extend to the product of three of them. It is still not known if every hereditarily indecomposable subcontinuum of the product of three Souslin arcs is metric.

One last story. During my non-research period I stopped going to meetings – I figured that since I did not have anything to present I should not attend. I now understand that this was a bad idea! In any case, I started to attend the annual Spring Topology Conference again and present my new results.

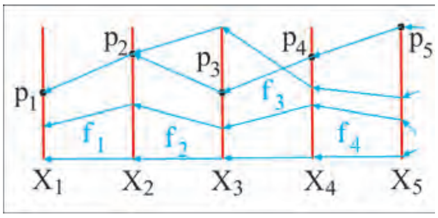
Seeing other people's work gave me more problems to work on and more ideas to try out on my favorite problems. At one of these meetings I touched base again after a few years, with my former professor Bill Mahavier. He told me that since he did not have any graduate students working with him he started working on some problems that he had saved up as potential directions for some future doctoral student. One of these ideas was the concept of generalized inverse limits. He collaborated with Tom Ingram and they converted his ideas into a couple of papers (Ingram & Mahavier, 2006; Mahavier, 2004) and eventually a book (Ingram & Mahavier, 2012). The book appeared just a few years before Mahavier passed away.

As a result of our discussions at that meeting, he gave me copies of these first couple of papers on the subject with the encouragement to work in the area; he was still my major professor. I felt like an excited graduate student again after some decades. Also, as chair, I had some obligation to attend some mathematics meetings and in particular I went to the annual Joint Mathematics Meetings which are traditionally held in January in various large cities that can handle five to seven thousand mathematicians and their groupies. I mainly attended to interview candidates for tenure track positions at Auburn at the employment center part of the meeting.

I distinctly remember being in New Orleans at one of these meetings. And as you know there is nothing to do in New Orleans and so I found myself alone one day eating this excellent, sloppy looking, dish called filé gumbo; and in spite of the culinary pleasure, I started looking at one of these papers. (Okay, I will be honest: I spent much of my free time at this little café that had wonderful shrimp ratatouille, a statue of Louis Armstrong out front and lots of music – and of course, beer. I forget why I was bored at the filé gumbo restaurant; it could have been no music and no beer and I was just hungry. I always doodled mathematics at these meetings – whether or not there was music.) At the time I had acquired another graduate student and he (yes "he" and not "she") had become interested in inverse limits. So I defined the generalized inverse limit concept of Mahavier and Ingram with upper semi-continuous set valued maps and gave him lots of questions and free rein. Like me, he was interested in indecomposable continua. So now I will discuss some background on inverse limits.

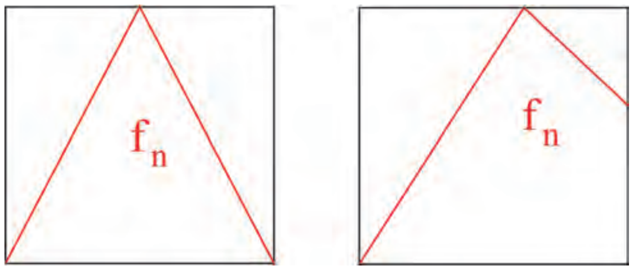
**Definition 3.** *Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of spaces and* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of functions with* $f_n : X_{n+1} \to X_n$. *Then the inverse limit* $\lim_{\leftarrow} \{X_n, f_n\}_{n=1}^{\infty}$ *is the subset of the product* $\prod_{n=1}^{\infty} X_n$ *consisting of all the points* $\{p_n\}_{n=1}^{\infty}$ *so that* $f_n(p_{n+1}) = p_n$.

*Figure 22*. The classical inverse limit.




The beauty of inverse limits is that some apparently simple inverse limits can produce complicated spaces. This gives topologists a tool that can be used to study complex spaces. Consider the function of the left box of Figure 23; the inverse limit so that each coordinate space is the unit interval $[0, 1]$ and so that each bonding map is the function $f_n$ is the Janiszewski indecomposable continuum (also called the bucket-handle). The one where each bonding map is the function on the right produces the topologist's $\sin\left(\frac{1}{x}\right)$ continuum.

*Figure 23*. Inverse limit bonding maps.




Mahavier and Ingram generalized the classical inverse limit as follows.

**Definition 4.** *Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of spaces and* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of set-valued upper semi-continuous functions with* $f_n : X_{n+1} \to 2^{X_n}$. *Then the generalized inverse limit* $\lim_{\leftarrow} \{X_n, f_n\}_{n=1}^{\infty}$ *is the subset of the product* $\prod_{n=1}^{\infty} X_n$ *consisting of all the points* $\{p_n\}_{n=1}^{\infty}$ *so that* $p_n \in f_n(p_{n+1})$.

**Definition 5.** *The function* $f$ *is upper semi-continuous if and only if for each open set* $U$ *containing* $f(x)$ *there exists an open set* $V$ *containing* $x$ *so that if* $t \in V$ *then* $f(t) \subset U$.

The set-valued bonding map on the left of Figure 24 is upper semi-continuous and the one on the right is not. Below in Figure 25 is a diagram for an inverse limit with set-valued bonding maps. Because of the set-valued nature of the function, one point may be mapped onto more than one point.

Generalized inverse limits became the topic of his dissertation. I was pleased that a topic that Mahavier had saved for one of his students became a dissertation topic for one of his mathematical grandchildren. Among other things he proved

*Figure 24*. Set valued bonding maps.

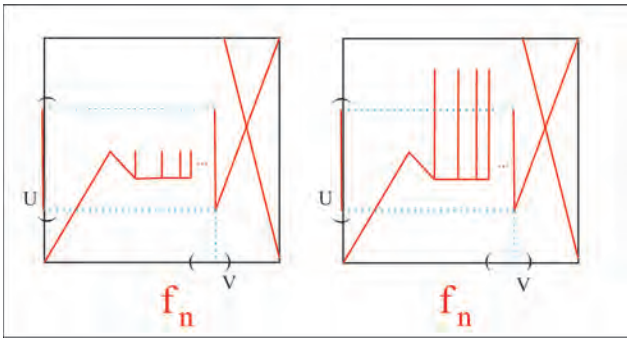
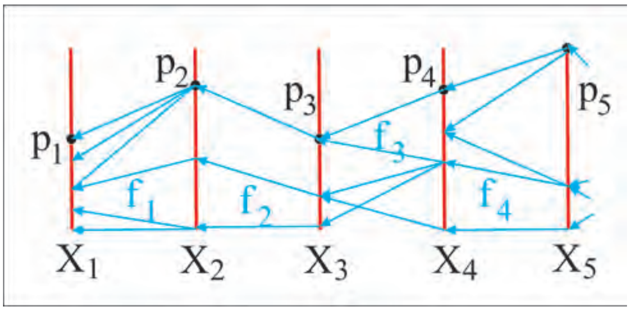
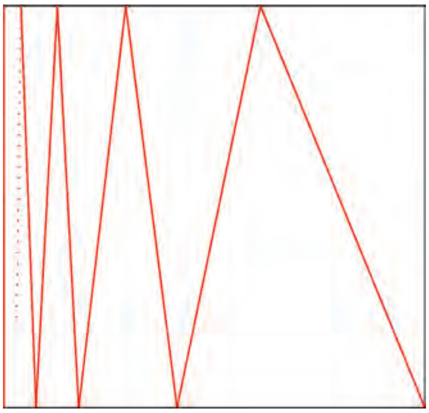



*Figure 25*. Generalized inverse limit with set-valued bonding maps.




that the generalized inverse limit on the unit interval using the $\sin\left(\frac{1}{x}\right)$-like graph of Figure 25 as the bonding map (which only has one point with a non-degenerate image) produces an indecomposable continuum (Varagona, 2011, 2012).

*Figure 26*. Generalized inverse limit yielding an indecomposable continuum.

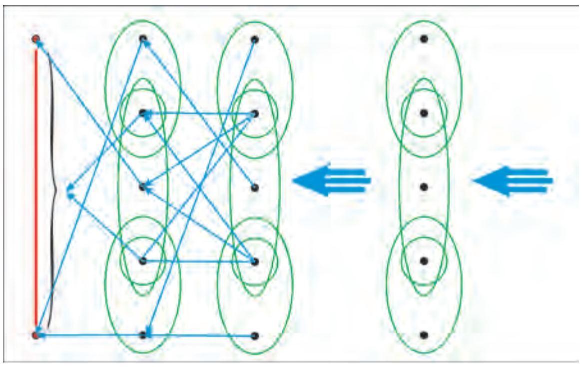


Around this time, at one of the topology meetings, Van Nall asked the following question, "Can every metric continuum be expressed as the generalized inverse limit of the metric interval $[0, 1]$ with the same bonding map?" And I was able to answer it in the negative: I pointed out that since, by a theorem of Van Nall himself 2012, every 2 dimensional generalized inverse limit of the metric arc with the same bonding map contains an arc, every higher dimensional hereditarily indecomposable subcontinuum (such as the examples of these produced by Bing 1951) is a counterexample since, by their property of indecomposability, they do not contain arcs. To me the solution was easy once you know the background and a little about hereditarily indecomposable continua; all I did was observe how the pieces can be put together. Van Nall's question, rephrased in the case where the bonding maps are not required to all be the same, has not been answered as of the time of this writing.

I keep in touch with my former student Scott, and I met him at the 2013 Spring topology conference. I had not done any work in generalized inverse limits myself (except for the observation that answered Van Nall's question.) But I was interested in the area. We were sitting in one of the conference rooms between the talks and he was telling me about his latest result in the area.

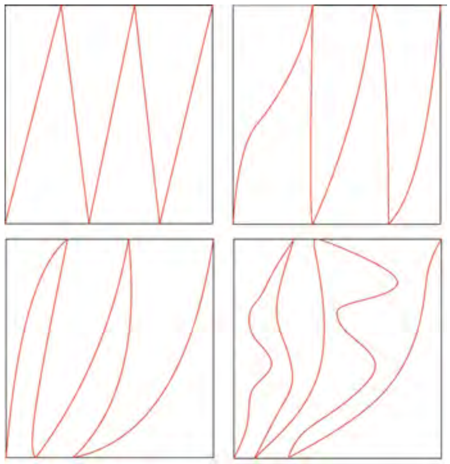
He had shown that a certain generalized inverse limit is indecomposable. I suggested the possibility of an even more generalized inverse limit where the functions are set-valued but not necessarily upper semi-continuous and where, except for the first coordinate, the coordinate spaces are not even Hausdorff - and in fact were finite connected $T1$ spaces. He thought that it had a chance to work, so I ended up proving a theorem that argued that we could express a wide range of metric continua with generalized inverse limit with set-valued maps where, except for the first coordinate, each coordinate space was a finite non- Hausdorff space (Smith & Varagona, 2014), the author was first made aware of this type of construction through work of S. Baldwin 2007 that was presented in the Continuum Theory Seminar at Auburn.)

*Figure 27*. Inverse limit with finite coordinate spaces.



The inverse limit expressed by the diagram of Figure 27 has $[0, 1]$ as its first coordinate space and each subsequent space is a five point space with a basis of the topology expressed by the green sets. This inverse limit is equivalent to the Janiszewski indecomposable continuum already discussed several times above. The theorem proved that all the spaces with the bonding maps picture below in Figure 28 are homeomorphic.

*Figure 28.* Bonding maps that produce equivalent indecomposable continua.



The beauty of the theorem is that a finite set of points is often much easier to work with than an infinite uncountable set.

So what can I recommend to someone who is interested in increasing or resurrecting his or her research program that had fallen off? I suggest:

- Keep in touch with other members of the research community.
- Collaborate with graduate students and/or colleagues, organize seminars.
- Go to meetings.
- Play with the old problems that interest you.
- Work on more than one research line.
- Stay in touch with mentors.
- ~~Become chair of your department.~~

You might skip that last one; but I have to confess that as chair of a mathematics department I wanted to serve as a good example to the department faculty of what a mathematician does: and that includes research as well as teaching (and involvement in grant work.) Most importantly, whether it is new or old, whether it has been done before or not: -- work on mathematics problems and enjoy playing around with it– it is fun.

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