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The Copositive-Plus Matrix Completion Problem

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We derive a useful characterization of 3-by-3 copositive-plus matrices in terms of their entries, and use it to solve the copositive-plus matrix completion problem in the case of specified diagonal by showing that a partial copositive-plus matrix with graph G has a copositive-plus completion if and only if G is a pairwise disjoint union of complete subgraphs.

Introduction

In this article, the superscript "$^\top$" denotes transposition and $\mathbb{R}^n$ the set of all real n-vectors. A vector $x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$ is said to be *nonnegative*, denoted by $x \geqslant 0$, if $x_i \geqslant 0$ for all $i = 1, \ldots, n$.

Let S_n denote the set of all $n \times n$ real symmetric matrices. A matrix $A \in S_n$ is said to be

(1) *(real) positive semidefinite* if $x^\top Ax \geqslant 0$ for all $x \in \mathbb{R}^n$;

(2) *copositive* if $x^\top Ax \geqslant 0$ for all $x \geqslant 0$;

(3) *copositive-plus* if A is copositive and if $x^\top Ax = 0$ with $x \geqslant 0$ implies $Ax = 0$;

(4) *strictly copositive* if $x^\top Ax > 0$ for all nonzero $x \geqslant 0$.

Positive semidefinite matrices are copositive by definition. Moreover, if $A \in S_n$ is positive semidefinite and $x \in \mathbb{R}^n$, then $x^\top Ax = 0$ if and only if $Ax = 0$ (Horn & Johnson, 1985, p.400, Problem 1). A parallel comparison of positive semidefinite matrices and copositive-plus matrices was made in Cottle, Habetler, and Lemke (1970b).

Following Cottle, Habetler, and Lemke (1970a), we denote the class of copositive (copositive-plus, strictly copositive, resp.) matrices by C (C^+, C^*, resp.). Obviously, $C^* \subset C^+ \subset C$. Each of the three classes of copositive matrices has three important properties: *inheritance*, *closure under permutation similarity*, and *closure under positive diagonal congruence*, i.e., if S denotes any class of copositive matrices and $A \in S$, then every principal submatrix of A is in S, $P^\top AP \in S$ for any permutation matrix P, and $DAD \in S$ for any positive diagonal matrix D.

Copositive matrices have applications in control theory, optimization modeling, linear complementarity problems, and many other branches of pure and applied mathematics (cf. Hiriart-Urruty and Seeger (2010)). See recent surveys Hiriart-Urruty and Seeger (2010) and Ikramov and Savel'eva (2000) on copositive matrices and references therein.

A *partial matrix* is one in which some entries are specified, while the remaining entries are unspecified and free to be chosen. A *completion* of a partial matrix is a choice of values for the unspecified entries. A *matrix completion problem* asks which partial matrices have completions with a desired property. A partial C^+ (C, C^*, resp.) matrix is a real symmetric partial matrix such that every fully specified principal submatrix is C^+ (C, C^*, resp.). The C and C^* matrix completion problems were solved in Hogben, Johnson, and Reams (2005) and Hogben (2007), but the C^+ matrix completion problem has remained open.

Our main interest here is in the following C^+ matrix completion problem: Under the assumption that the main diagonal is specified, which patterns for the specified entries of a partial C^+ matrix ensure that each partial matrix with one of these patterns can be completed to a C^+ matrix?

In Section 2, we give a new characterization of 3×3 copositive-plus matrices in terms of their entries. In Section 3, this result is used to solve the C^+ matrix completion problem listed above.

A characterization of copositive-plus matrices of order $n \leqslant 3$

We begin with the following characterization of 2×2 copositive-plus matrices in terms of their entries. We omit the proof since it follows from straightforward computation.

Lemma 1. *The real symmetric matrix* $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$ *is copositive-plus if and only if the following conditions are satisfied:*

(1) $a \geqslant 0$ *and* $d \geqslant 0$*,*

(2a) $b = 0$ *when* $ad = 0$*,*

(2b) $b \geqslant -\sqrt{ad}$ *when* $ad > 0$*.*

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Moreover, it is strictly copositive if and only if $a > 0$, $d > 0$, *and* $b > -\sqrt{ad}$.

Combining (Hadeler, 1983, Theorem 4) and (Simpson & Spector, 1983, Theorem 2.2), we have the following characterization of 3×3 copositive matrices, which will be useful in characterizing 3×3 copositive-plus matrices in terms of their entries.

Lemma 2. *The real symmetric matrix* $A = \begin{bmatrix} a & b & s \\ b & d & c \\ s & c & e \end{bmatrix}$ *is copositive if and only if the following conditions are satisfied:*

(1) $a \geqslant 0$, $d \geqslant 0$, $e \geqslant 0$,

(2) $b \geqslant -\sqrt{ad}$, $c \geqslant -\sqrt{de}$, $s \geqslant -\sqrt{ae}$,

(3) $\sqrt{ade} + b\sqrt{e} + c\sqrt{a} + s\sqrt{d} + \sqrt{2(b + \sqrt{ad})(c + \sqrt{de})(s + \sqrt{ae})} \geqslant 0$.

Moreover, A is strictly copositive if and only if all the above conditions are satisfied with strict inequality. If $\sqrt{ade} + b\sqrt{e} + c\sqrt{a} + s\sqrt{d} \leqslant 0$, *Condition (3) is equivalent to* $\det A \geqslant 0$.

The following characterization of 3×3 copositive-plus matrices with unit diagonal will be crucial in the proof of the main results.

Lemma 3. *The matrix* $A = \begin{bmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{bmatrix}$ *is copositive-plus if and only if*

(1) it is permutation similar to $\begin{bmatrix} 1 & \alpha & -\alpha \\ \alpha & 1 & -1 \\ -\alpha & -1 & 1 \end{bmatrix}$ *with* $-1 \leqslant \alpha \leqslant 1$, *or*

(2) $\alpha, \beta, \gamma > -1$ *and* $1 + \alpha + \beta + \gamma + \sqrt{2(1 + \alpha)(1 + \beta)(1 + \gamma)} \geqslant 0$.

Proof. Since $C^+ \subset C$, by Lemma 2 we may assume that $\alpha, \beta, \gamma \geqslant -1$ and

$$1 + \alpha + \beta + \gamma + \sqrt{2(1 + \alpha)(1 + \beta)(1 + \gamma)} \geqslant 0. \tag{1}$$

Denote $Q(\mathbf{v}) =: \mathbf{v}^\top A\mathbf{v}$ for $\mathbf{v} = (x, y, z)^\top \geqslant 0$. We consider the following two cases:

Case 1: at least one of α, β, γ is -1. Then after applying permutation similarity, we can assume $\gamma = -1$. Then (1) reduces to $\alpha + \beta \geqslant 0$. We claim that $A \in C^+$ if and only if $\alpha + \beta = 0$. Note that

$$\begin{aligned} Q(\mathbf{v}) &= x^2 + y^2 + z^2 + 2\alpha xy + 2\beta xz - 2yz \\ &= x^2 + (y - z)^2 + 2\alpha x(y - z) + 2(\alpha + \beta)xz \\ &= (\alpha x + y - z)^2 + (1 - \alpha^2)x^2 + 2(\alpha + \beta)xz. \end{aligned}$$

If $\alpha + \beta = 0$, then $Q(\mathbf{v}) = 0$ if and only if

$$\alpha x + y - z = 0 \quad \text{and} \quad (1 - \alpha^2)x = 0$$

i.e., if and only if

$$\begin{cases} x + \alpha(y - z) = 0 & \text{if } \alpha = \pm 1 \\ x = 0 \text{ and } y = z & \text{if } \alpha \neq \pm 1 \end{cases}$$

i.e., if and only if $A\mathbf{v} = 0$. Hence, $A \in C^+$.

On the other hand, if $\alpha + \beta \neq 0$, then $Q(\mathbf{v}) = 0$ and $A\mathbf{v} = (\alpha + \beta, 0, 0)^\top$ for $\mathbf{v} = (0, 1, 1)^\top$, hence $A \notin C^+$. Note that $\alpha + \beta = 0$ and $\alpha, \beta \geqslant -1$ imply $-1 \leqslant \alpha, \beta \leqslant 1$.

Case 2: $\alpha, \beta, \gamma > -1$. We claim that $A \in C^+$ if and only if (1) holds. Since $A \in C^*$ if and only if strict inequality in (1) holds, it remains to show that $A \in C^+$ if

$$1 + \alpha + \beta + \gamma + \sqrt{2(1 + \alpha)(1 + \beta)(1 + \gamma)} = 0, \tag{2}$$

which implies that $1 + \alpha + \beta + \gamma \leqslant 0$. Since we assume $\alpha, \beta, \gamma > -1$, it follows that

$$-1 < \alpha, \beta, \gamma < 1. \tag{3}$$

Moreover, we have $\det A = 0$ by Lemma 2. Now that

$$1 + 2\alpha\beta\gamma - \alpha^2 - \beta^2 - \gamma^2 = \det A = 0, \tag{4}$$

we then have

$$\alpha\gamma - \beta = \sqrt{(1 - \alpha^2)(1 - \gamma^2)} \geqslant 0; \tag{5}$$

$$\alpha\beta - \gamma = \sqrt{(1 - \alpha^2)(1 - \beta^2)} \geqslant 0; \tag{6}$$

$$\beta\gamma - \alpha = \sqrt{(1 - \beta^2)(1 - \gamma^2)} \geqslant 0. \tag{7}$$

By solving the linear equation $A\mathbf{v} = 0$, we know that every solution is a scalar multiple of $(\alpha\gamma - \beta, \alpha\beta - \gamma, 1 - \alpha^2)^\top$, i.e., a scalar multiple of $(\sqrt{1 - \gamma^2}, \sqrt{1 - \beta^2}, \sqrt{1 - \alpha^2})^\top$.

Now we write

$$\begin{aligned} Q(\mathbf{v}) &= x^2 + y^2 + z^2 + 2\alpha xy + 2\beta xz + 2\gamma yz \\ &= p\left(\frac{x}{\sqrt{1 - \gamma^2}} - \frac{y}{\sqrt{1 - \beta^2}}\right)^2 + q\left(\frac{x}{\sqrt{1 - \gamma^2}} - \frac{z}{\sqrt{1 - \alpha^2}}\right)^2 \\ &\quad + r\left(\frac{y}{\sqrt{1 - \beta^2}} - \frac{z}{\sqrt{1 - \alpha^2}}\right)^2 \\ &=: p(\hat{x} - \hat{y})^2 + q(\hat{x} - \hat{z})^2 + r(\hat{y} - \hat{z})^2, \end{aligned}$$

where by (4), (5), (6), and (7)

$$p = -\alpha\sqrt{(1 - \beta^2)(1 - \gamma^2)} = \alpha(\alpha - \beta\gamma) = \frac{1 + \alpha^2 - \beta^2 - \gamma^2}{2}; \tag{8}$$

$$q = -\beta\sqrt{(1 - \alpha^2)(1 - \gamma^2)} = \beta(\beta - \alpha\gamma) = \frac{1 + \beta^2 - \alpha^2 - \gamma^2}{2}; \tag{9}$$

$$r = -\gamma\sqrt{(1 - \alpha^2)(1 - \beta^2)} = \gamma(\gamma - \alpha\beta) = \frac{1 + \gamma^2 - \alpha^2 - \beta^2}{2}. \tag{10}$$

From (2) and (3), we see that at most one of α,β,γ is nonnegative. If $-1 < \alpha,\beta,\gamma \leqslant 0$, then $p,q,r \geqslant 0$ and hence $Q(\mathbf{v}) = 0$ if and only if $\hat{x} = \hat{y} = \hat{z}$, i.e., $A\mathbf{v} = 0$, hence $A \in C^+$. If exactly one of α,β,γ is positive, say $-1 < \beta,\gamma < 0 < \alpha < 1$ and so $-1 < p < 0 < q,r < 1$, we claim that $Q(\mathbf{v}) = 0$ if and only if $\hat{x} = \hat{y} = \hat{z}$, i.e., $A\mathbf{v} = 0$.

We first note that $p+q = 1-\gamma^2 > 0$, and $p+r = 1-\beta^2 > 0$ by (8), (9) and (10). Then it is easy to see that $Q(\mathbf{v}) > 0$ if exactly two of $\hat{x},\hat{y},\hat{z}$ are equal. It remains to show that $Q(\mathbf{v}) > 0$ if $\hat{x},\hat{y},\hat{z}$ are distinct.

Since $p < 0$, $p + q > 0$, and $p + r > 0$, we assume that $|\hat{x}-\hat{y}| = |\hat{x}-\hat{z}| + |\hat{y}-\hat{z}|$, i.e., $\hat{z}$ is between $\hat{x}$ and $\hat{y}$ (The other two cases are trivial. For example, if $|\hat{x}-\hat{z}| = |\hat{x}-\hat{y}| + |\hat{y}-\hat{z}|$, then $Q(\mathbf{v}) > p(\hat{x}-\hat{y})^2 + q(\hat{x}-\hat{z})^2 > (p+q)(\hat{x}-\hat{y})^2 > 0$). Denote $u =: |\hat{x}-\hat{z}| > 0$ and $v =: |\hat{y}-\hat{z}| > 0$. Then

$$\begin{aligned}
Q(\mathbf{v}) &= p(u+v)^2 + qu^2 + rv^2 \\
&= (p+q)u^2 + (p+r)v^2 + 2puv \\
&= (1-\gamma^2)u^2 + (1-\beta^2)v^2 + 2\alpha(\alpha-\beta\gamma)uv \\
&= (\sqrt{1-\gamma^2}u - \sqrt{1-\beta^2}v)^2 \\
&\quad + 2(\sqrt{(1-\beta^2)(1-\gamma^2)} - \alpha(\beta\gamma-\alpha))uv \\
&= (\sqrt{1-\gamma^2}u - \sqrt{1-\beta^2}v)^2 + 2(1-\alpha)(\beta\gamma-\alpha)uv \text{ (by (7))} \\
&> 0
\end{aligned}$$

This completes the proof. □

The following result characterizes 3×3 copositive-plus matrices in terms of their entries.

Theorem 4. *A real symmetric matrix A of order 3 is copositive-plus if and only if A is permutation similar to one of the following forms:*

(1) $\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *with $a \geqslant 0$,*

(2) $\begin{bmatrix} a & b & 0 \\ b & d & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *with $a,d > 0$ and $b \geqslant -\sqrt{ad}$,*

(3) $\begin{bmatrix} a & b & s \\ b & d & c \\ s & c & e \end{bmatrix}$ *with $a,d,e > 0$ and $c = -\sqrt{de}$, $-\sqrt{ad} \leqslant b \leqslant \sqrt{ad}$, $-\sqrt{ae} \leqslant s \leqslant \sqrt{ae}$, and $b\sqrt{e} + s\sqrt{d} = 0$,*

(4) $\begin{bmatrix} a & b & s \\ b & d & c \\ s & c & e \end{bmatrix}$ *with $a,d,e > 0$ and $c > -\sqrt{de}$, $b > -\sqrt{ad}$, $s > -\sqrt{ae}$, and $\sqrt{ade} + b\sqrt{e} + c\sqrt{a} + s\sqrt{d} + \sqrt{2(b+\sqrt{ad})(c+\sqrt{de})(s+\sqrt{ae})} \geqslant 0$.*

Proof. Let $A = \begin{bmatrix} a & b & s \\ b & d & c \\ s & c & e \end{bmatrix}$ be real symmetric with nonnegative diagonal entries. According to Lemma 1 and the inheritance property of C^+ matrices, the rows and columns containing a zero diagonal entry are zero. Therefore, if $ade = 0$, then $A \in C^+$ if and only if A is in Form (1) or Form (2), up to permutation similarity. On the other hand, if $a,d,e > 0$, then $A \in C^+$ if and only if $\begin{bmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{bmatrix} \in C^+$, since

$$A = \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{d} & 0 \\ 0 & 0 & \sqrt{e} \end{bmatrix} \begin{bmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{bmatrix} \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{d} & 0 \\ 0 & 0 & \sqrt{e} \end{bmatrix},$$

when $\alpha = \frac{b}{\sqrt{ad}}, \beta = \frac{s}{\sqrt{ae}}, \gamma = \frac{c}{\sqrt{de}}$. The proof is then completed by applying Lemma 3. □

Remark 5. A matrix having Form (3) in Theorem 4 corresponds to a matrix having Form (1) in Lemma 3, while one having Form (4) with the following equality

$$\sqrt{ade}+b\sqrt{e}+c\sqrt{a}+s\sqrt{d}+\sqrt{2(b+\sqrt{ad})(c+\sqrt{de})(s+\sqrt{ae})} = 0 \tag{11}$$

corresponds to one having Form (2) in Lemma 3 with (2).

As seen from the proof of Lemma 3, a matrix having Form (3) in Theorem 4 or Form (4) satisfying (11) is positive semidefinite (its determinant is 0). This yields the following characterization of C^+ matrices of order no more than 3.

Corollary 6. *(Väliaho, 1986, Theorem 5.4) A matrix $A \in S_n$ with $n \leqslant 3$ is copositive-plus if and only if it is positive semidefinite or, after deleting the possible zero rows and zero columns, strictly copositive.*

Remark 7. In determining whether a 3×3 partial C^+ matrix has a C^+ completion or whether a conventional 3×3 matrix is C^+, Lemma 3 or Theorem 4 is more useful than Corollary 6.

Example 8. Consider the partial C^+ matrix $B = \begin{bmatrix} 1 & 2 & ? \\ 2 & 1 & -1 \\ ? & -1 & 1 \end{bmatrix}$. Appealing to Lemma 3, we see that B has no C^+ completion.

The copositive-plus matrix completion problem

To solve the C^+ matrix completion problem with specified main diagonal, we will need some concepts from graph theory. See Johnson (1990) for a similar study of the positive definite matrix completion problem.

We follow Golumbic (1980) for terminology and results needed from graph theory. An (undirected) *graph* is a pair $G = (V, E)$ in which V, the *vertex set*, is finite and E, the *edge set*, is a collection of two-element subsets of V. A vertex u is *adjacent* to a vertex v if $\{u, v\} \in E$. A *complete graph* is one with the property that every pair of distinct vertices is adjacent. A *path* $[v_1, \ldots, v_k]$, in which $v_1, \ldots, v_k$ are distinct, is a sequence of vertices such that $\{v_j, v_{j+1}\} \in E$ for $j = 1, \ldots, k-1$; in this case, the *length* of the path $[v_1, \ldots, v_k]$ is $k - 1$. If $W \subset V$, the *subgraph induced by W* is the graph $G[W] = (W, E[W])$ in which $E[W] = \{\{x, y\} \in E : x, y \in W\}$. An *induced path* in G is a path that is an induced subgraph of G. It is trivial to see that the connected graphs with no induced path of length two are the complete graphs.

Given a partial symmetric matrix A with specified main diagonal, we can associate a graph $G = (V, E)$ in which $V = \{1, \ldots, n\}$ and $E = \{\{i, j\} : a_{ij}$ is specified and $i \neq j\}$. In the remainder of the paper, G is a graph on n vertices with $n \geqslant 3$.

We will need the following lemma to solve the C^+ matrix completion problem with specified main diagonal.

Lemma 9. *If the graph G contains an induced path of length* 2*, then there is a partial copositive-plus matrix A with graph G that has no copositive-plus completion.*

Proof. Let G contain an induced path of length 2, say $[u, v, w]$. Note that $[u, v, w]$ is the graph of the partial C^+ matrix B in Example 8. Let A be any partial C^+ matrix with graph G and a principal submatrix B. Since B is not completable to a C^+ matrix, neither is A by inheritance. □

A graph G contains no induced path of length 2 if and only if G is the union of pairwise disjoint complete subgraphs. This fact leads to the following characterization of patterns that ensure C^+ completability.

Theorem 10. *Every partial copositive-plus matrix A with graph G has a copositive-plus completion if and only if G is the union of pairwise disjoint complete subgraphs.*

Proof. Lemma 9 establishes necessity. For sufficiency, just assign 0 to each unspecified off-diagonal entry; the resulting matrix is a direct sum of C^+ matrices and, hence, it is C^+ also. □

Finally, we consider the general copositive-plus matrix completion problem (with some unspecified diagonal entries). The following lemma is useful.

Lemma 11. *(Hogben, 2007, Corollary 3) Every partial strictly copositive matrix can be completed to a strictly copositive matrix.*

We have the following two results of sufficient conditions and necessary conditions, respectively, about the completability of a partial C^+ matrix.

Theorem 12. *If A is partial C^+ matrix that has either no specified diagonal entry or exactly one positive specified diagonal entry, then A has a C^+ completion.*

Proof. Suppose A is a partial C^+ matrix that has either no specified diagonal entry or exactly one positive specified diagonal entry. Then A is a partial C^* matrix and so, by Lemma 11, it has a C^* (and hence C^+) completion. □

Theorem 13. *If A is partial C^+ matrix that has a C^+ completion, then*

(1) the row (column) containing a 0 *diagonal entry must be zero row (column);*

(2) every 3×3 *principal submatrix of A that is of the form* $\begin{bmatrix} ? & \alpha & \beta \\ \alpha & 1 & -1 \\ \beta & -1 & 1 \end{bmatrix}$*, after permutation and positive diagonal scaling, must have* $\alpha + \beta = 0$.

Proof. Since A has a C^+ completion, every principal submatrix of A is C^+ completable. The two necessary conditions then follow from Lemma 1 and Lemma 3, respectively. □

It seems that it is not easy to find sufficient and necessary conditions for a partial C^+ matrix to be C^+ completable. We leave this as an open problem.

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