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The matrix Lie algebra on a one-step ladder is zero product determined

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The class of *matrix algebras on a ladder* $\mathcal{L}$ generalizes the class of block upper triangular matrix algebras. It was previously shown that the matrix algebra on a ladder $\mathcal{L}$ is zero product determined under matrix multiplication. In this article, we show that the matrix algebra on a one-step ladder is zero product determined under the Lie bracket.

Keywords: matrix algebra; matrix Lie algebra; zero product determined algebra; zero Lie product determined algebra

Introduction

In Brice and Huang (2015), the authors defined a class of matrix algebras, the *ladder matrix algebras*, that generalizes the class of block upper triangular matrix algebras. They introduce the notion of an upper triangular k-step ladder as a method of parameterizing and indexing these algebras. Certain one-step ladder matrix algebras arise as ideals of derivation algebras of parabolic subalgebras of reductive Lie algebras, which provided the motivation for their study (Brice (2014)).

While these terms are made precise in the sequel, the concepts are perhaps best illustrated with an example. Let $\mathcal{L} = \{(3, 2), (6, 5)\}$. $\mathcal{L}$ is then a 2-step upper triangular ladder on 6. The ladder matrix algebra on $\mathcal{L}$ is the subalgebra

$$M_{\mathcal{L}} = \left\{ \begin{pmatrix} 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix} \right\}$$

of $M^{n \times n}$.

An algebra $(A, *)$ is *zero product determined* if each bilinear map φ on $A \times A$ that preserves zero products necessarily factors as a linear map f on A^2 composed with the algebra multiplication $*$ so that $\varphi(x, y) = f(x * y)$. The notion is motivated by the linear preserver problem in operator theory and has recently become a topic of considerable research (Brešar, Grašič, and Ortega (2009)).

It was previously shown that the ladder matrix algebras are zero product determined when $*$ is matrix multiplication (Brice and Huang (2015)). The purpose of this paper is to show that a one-step ladder matrix algebra is zero product determined when $*$ is the Lie bracket $[x, y] = xy - yx$.

Previous work on zero product determined algebras has also considered the case where $*$ is the Jordan product $x \circ y = xy + yx$ (Brešar et al. (2009)). Extending the present one-step result on ladder matrix algebras to the Jordan product case, and to the k-step case for both the Lie bracket and the Jordan product, remains a topic of interest to the author.

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Preliminaries

Let F be a field. Let n be a positive integer. Let $M_F^{n \times n}$ denote the space of n-by-n matrices with entries in F. Let $e_{i,j}$ denote the matrix whose entry in the ith row jth column is 1_F, and whose other entries are 0_F. We will suppress further mention of the field F when convenient, but the reader as advised that all references to linearearity and tensor that follow refer specifically to F-linearity and tensors over F.

Definition 1. A k-step *ladder* on n is a set of pairs of positive integers

$$\mathcal{L} = \{(i_1, j_1), ..., (i_k, j_k)\}$$

with

$$1 \le i_1 < i_2 < ... < i_k \le n$$

and

$$1 \le j_1 < j_2 < ... < j_k \le n.$$

Each pair (i_t, j_t) is called a *step* of $\mathcal{L}$.

Definition 2. The *ladder matrices* on $\mathcal{L}$ is the subspace

$$M_{\mathcal{L}} = \text{Span} \bigcup_{t=1}^{k} \left\{ e_{i,j} \middle| 1 \le i \le i_t \text{ and } j_t \le j \le n \right\}.$$

Definition 3. A ladder $\mathcal{L}$ is called *upper triangular* if $i_t < j_{t+1}$ for $t = 1, 2, ..., k - 1$.

Theorem 4 (Brice and Huang (2015))**.** *Let $\mathcal{L}$ be a ladder on n. $M_{\mathcal{L}}$ is closed under matrix multiplication (and subsequently under the Lie bracket) if and only if $\mathcal{L}$ is upper triangular.*

We remind the reader that if $x, y \in M_F^{n\times n}$, then the Lie bracket of x and y, denoted $[x, y]$, is the matrix $xy - yx$. A subspace of $M_{\mathcal{L}}^{n\times n}$ closed under $[\cdot,\cdot]$ is termed a *Lie algebra*.

In light of Theorem 4, whenever $\mathcal{L}$ is upper triangular we will call $M_{\mathcal{L}}$ the *matrix algebra on* $\mathcal{L}$ in case we are considering $M_{\mathcal{L}}$ as an algebra under matrix multiplication or the *matrix Lie algebra on* $\mathcal{L}$ in case we are considering $M_{\mathcal{L}}$ as an algebra under the Lie bracket.

The following proposition establishes that the class of block upper triangular matrix algebras is a subclass of the class of ladder matrix algebras.

Proposition 5. *Let* $\mathfrak{q} \subseteq M_F^{n\times n}$ *be a block upper triangular matrix algebra (res. Lie algebra). There is an upper triangular ladder* $\mathcal{L}$ *such that* $\mathfrak{q} = M_{\mathcal{L}}$.

Proof. Block upper triangular matrix algebras (res. Lie algebras) correspond with partitions of n (Knapp (2002)). Let $\pi = (n_1, n_2, ..., n_k)$ be the partition of n corresponding to $\mathfrak{q}$. Let

$$\mathcal{L} = \left\{ \left(\sum_{i=1}^{t} n_i, 1 + \sum_{i=1}^{t-1} n_i \right) \middle| 1 \le t \le k \right\}$$

where $\sum_{i=1}^{0} n_i$ should be understood to be 0. $\mathcal{L}$ is upper triangular by construction, and furthermore is constructed so that $\mathfrak{q} = M_{\mathcal{L}}$. ☐

Stated perhaps more clearly, the block upper triangular matrix algebras are precisely the ladder matrix algebras where $j_{t+1} = i_t + 1$ for $t = 1, 2, ..., k-1$.

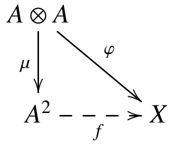
Definition 6. An *algebra* over F is a pair (A, μ) where A is a vector space over F and $\mu : A \otimes A \to A$ is an F-linear map. The image of μ is denoted by A^2.

This definition of algebra does not assume that the multiplication map μ is associative. This definition is chosen because it is indifferent as to whether we are considering $M_{\mathcal{L}}$ as an associative algebra under $\mu : x \otimes y \mapsto xy$ as a Lie algebra under $\mu : x \otimes y \mapsto [x, y]$ or as a Jordan algebra under $\mu : x \otimes y \mapsto x \circ y$.

Definition 7. An algebra is called *zero product determined* if for each F-linear map $\varphi : A \otimes A \to X$ (where X is an arbitrary vector space over F) the condition

$$\forall x, y \in A, \varphi(x \otimes y) = 0 \text{ whenever } \mu(x \otimes y) = 0 \qquad (1)$$

ensures that φ factors through μ.



A linear map satisfying condition 1 is said to *preserve zero products*. By φ *factors through* μ it is meant that there is a linear map $f : A^2 \to X$ such that $\varphi = f \circ \mu$, as illustrated above. If φ factors through μ, then condition 1 holds trivially. We note that in case (A, μ) is zero product determined and $\varphi : A \otimes A \to X$ preserves zero products, then the map f such that $\varphi = f \circ \mu$ is uniquely determined.

The notion of a zero-product determined algebra was introduced by Matej Brešar, Mateja Grašič, and Juana Sánchez Ortega to further the study of near-homomorphisms on Banach algebras (Brešar et al. (2009)). We present below the results of interest to us in this paper.

Theorem 8 (Brešar et al. (2009))**.** $M_F^{n\times n}$ *considered as an algebra under either matrix multiplication or the Lie bracket is zero product determined.*

Theorem 9 (Grašič (2010))**.** *The classical Lie algebras are zero product determined.*

Theorem 10 (Wang, Yu, and Chen (2011))**.** *The simple Lie algebras over* $\mathbb{C}$ *and their parabolic subalgebras are zero product determined.*

Theorem 11 (Brice and Huang (2015))**.** *An abelien Lie algebra is zero product determined.*

Theorem 12 (Brice and Huang (2015))**.** *If* $\mathcal{L}$ *is upper triangular, then* $M_{\mathcal{L}}$ *under matrix multiplication is zero product determined.*

Recall that $A \otimes A = \text{Span}\{x \otimes y | x, y \in A\}$. Members of $A \otimes A$ of the form $x \otimes y$ with $x, y \in A$ are called *rank-one* tensors. We will make extensive use of the following theorem.

Theorem 13 (Brice and Huang (2015))**.** *An algebra* (A, μ) *is zero product determined if and only if* $\text{Ker}\,\mu$ *is generated by rank-one tensors.*

We note that while $A \otimes A$ is generated by rank-one tensors by definition, an arbitrary subspace of $A \otimes A$ need not be generated by the rank-one tensors it contains.

Main Result

We state and prove our main result.

Proposition 14. *Let* $\mathcal{L}$ *be a* 1*-step ladder on n. The ladder matrix Lie algebra* $M_{\mathcal{L}}$ *is zero product determined.*

Proof. Let $\mathcal{L} = \{(i_1, j_1)\}$. If $i_1 < j_1$, then $M_{\mathcal{L}}$ is abelien and is zero product determined by Theorem 11. We assume without loss of generality that $i_1 \ge j_1$.

Let $\mu : \sum_t x_t \otimes y_t \mapsto \sum_t [x_t, y_t]$. In light of Theorem 13, our task is to construct a basis of $\text{Ker}\,\mu$ consisting of elements of $M_{\mathcal{L}} \otimes M_{\mathcal{L}}$ of the form $x \otimes y$ with $x, y \in M_{\mathcal{L}}$.

We partition $M_{\mathcal{L}}$ into blocks of size

$$n_1 = j_1 - 1 \ge 0,$$
$$n_2 = i_1 - j_1 + 1 > 0, \text{ and}$$
$$n_3 = n - i_1 \ge 0$$

so that $n_1 + n_2 + n_3 = n$. Under this block scheme, $M_{\mathcal{L}}$ has the form

$$M_{\mathcal{L}} = \begin{matrix} & n_1 & n_2 & n_3 \\ n_1 & 0 & \mathfrak{l} & \mathfrak{a} \\ n_2 & 0 & \mathfrak{h} & \mathfrak{r} \\ n_3 & 0 & 0 & 0 \end{matrix},$$

or in case $n_1 = 0$

$$M_{\mathcal{L}} = \begin{matrix} & n_2 & n_3 \\ n_2 & \mathfrak{h} & \mathfrak{r} \\ n_3 & 0 & 0 \end{matrix},$$

or in case $n_3 = 0$

$$M_{\mathcal{L}} = \begin{matrix} & n_1 & n_2 \\ n_1 & 0 & \mathfrak{l} \\ n_2 & 0 & \mathfrak{h} \end{matrix},$$

where each of $\mathfrak{h}$, $\mathfrak{l}$, $\mathfrak{r}$, and $\mathfrak{a}$ is a subalgebra consisting of the full matrix subspace of the appropriate size. All three cases are treated simultaneously by the below argument.

$M_{\mathcal{L}}$ admits the structural decomposition

$$M_{\mathcal{L}} = \mathfrak{h} \ltimes ((\mathfrak{l} \dot{+} \mathfrak{r}) \ltimes \mathfrak{a})$$

obeying multiplication containment relations below.

$[\cdot,\cdot]$	$\mathfrak{h}$	$\mathfrak{l}$	$\mathfrak{r}$	$\mathfrak{a}$
$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{l}$	$\mathfrak{r}$	0
$\mathfrak{l}$	$\mathfrak{l}$	0	$\mathfrak{a}$	0
$\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{a}$	0	0
$\mathfrak{a}$	0	0	0	0

(where $\mathfrak{l} = \mathfrak{a} = 0$ in case $n_1 = 0$ and $\mathfrak{r} = \mathfrak{a} = 0$ in case $n_3 = 0$.)

We require the dimension of $\operatorname{Ker}\mu$.

We see that for $h \in \mathfrak{h}$ and $r \in \mathfrak{r}$ we have $[h, r] = hr$, since $rh = 0$, and similarly with $l \in \mathfrak{l}$ we have $[h, l] = -lh$. Thus $[\mathfrak{h}, \mathfrak{r}] = \mathfrak{r}$ and $[\mathfrak{h}, \mathfrak{l}] = \mathfrak{l}$. Furthermore, for $l \in \mathfrak{l}$ and $r \in \mathfrak{r}$, we have $[l, r] = lr$, whereby $[\mathfrak{l}, \mathfrak{r}] = \mathfrak{a}$. Finally, $[\mathfrak{h}, \mathfrak{h}]$ produces only the traceless matrices, thus $\dim[\mathfrak{h}, \mathfrak{h}] = \dim \mathfrak{h} - 1$.

In light of these observations, we find that $\operatorname{Ker}\mu$ has dimension

$$n_1^2 n_2^2 + 2n_1^2 n_2 n_3 + n_1^2 n_3^2 + 2n_1 n_2^3 + 4n_1 n_2^2 n_3 + 2n_1 n_2 n_3^2 \\ -n_1 n_2 - n_1 n_3 + n_2^4 + 2n_2^3 n_3 + n_2^2 n_3^2 - n_2^2 - n_2 n_3 + 1.$$

Each pairing of subspaces that is killed by the bracket yields its full basis of rank-one tensors to $\operatorname{Ker}\mu$. We have:

Subspace pair	Rank-one tensors contributed
$\mu(\mathfrak{h} \otimes \mathfrak{a}) = 0 = \mu(\mathfrak{a} \otimes \mathfrak{h})$	$2n_1 n_2^2 n_3$
$\mu(\mathfrak{l} \otimes \mathfrak{a}) = 0 = \mu(\mathfrak{a} \otimes \mathfrak{l})$	$2n_1^2 n_2 n_3$
$\mu(\mathfrak{r} \otimes \mathfrak{a}) = 0 = \mu(\mathfrak{a} \otimes \mathfrak{r})$	$2n_1 n_2 n_3^2$
$\mu(\mathfrak{a} \otimes \mathfrak{a}) = 0$	$n_1^2 n_3^2$
$\mu(\mathfrak{l} \otimes \mathfrak{l}) = 0$	$n_1^2 n_2^2$
$\mu(\mathfrak{r} \otimes \mathfrak{r}) = 0$	$n_2^2 n_3^2$

Further, $\mathfrak{h}$ is isomorphic to $M_F^{n_2 \times n_2}$, which is zero product determined as a Lie algebra by Theorem 8. By Theorem 13 there are $n_2^4 - n_2^2 + 1$ rank-one tensors in $\mathfrak{h} \otimes \mathfrak{h}$ that μ kills. The above listed rank-one tensors in $\operatorname{Ker}\mu$ are linearly independent by construction from block pairings. This leaves

$$2n_1 n_2^3 + 2n_2^3 n_3 + 2n_1 n_2^2 n_3 - n_1 n_2 - n_1 n_3 - n_2 n_3$$

rank-one tensors in $\operatorname{Ker}\mu$ we have left to construct.

We examine $\mathfrak{h} \otimes \mathfrak{r}$, $\mathfrak{r} \otimes \mathfrak{h}$, and $(\mathfrak{h} \dot{+} \mathfrak{r}) \otimes (\mathfrak{h} \dot{+} \mathfrak{r})$. We will find that these subspaces contribute $2n_2^3 n_3 - n_2 n_3$ tensors to our basis.

Consider the $2n_2^3 n_3 - 2n_2^2 n_3$ tensors

$$T_{i,j,l,q} = e_{i,j} \otimes e_{l,q} \in \mathfrak{h} \otimes \mathfrak{r}$$

and

$$T^{i,j,l,q} = e_{l,q} \otimes e_{i,j} \in \mathfrak{r} \otimes \mathfrak{h}$$

for $i, j, l \in (n_1, n_1 + n_2]$ and $q \in (n_1 + n_2, n_1 + n_2 + n_3]$ with $j \neq l$.

Additionally, we have $2n_2^2 n_3 - 2n_2 n_3$ tensors

$$S_{i,j,q} = \left(e_{i,j} - e_{i,j+1}\right) \otimes \left(e_{j,q} + e_{j+1,q}\right) \in \mathfrak{h} \otimes \mathfrak{r}$$

and

$$S^{i,j,q} = \left(e_{j,q} + e_{j+1,q}\right) \otimes \left(e_{i,j} - e_{i,j+1}\right) \in \mathfrak{r} \otimes \mathfrak{h}$$

with $i \in (n_1, n_1 + n_2]$, $j \in (n_1, n_1 + n_2 - 1]$, and $q \in (n_1 + n_2, n_1 + n_2 + n_3]$.

Finally, we have $n_2 n_3$ tensors of the form

$$R(i, q) = \left(e_{i,i} + e_{i,q}\right) \otimes \left(e_{i,i} + e_{i,q}\right) \in (\mathfrak{h} \dot{+} \mathfrak{r}) \otimes (\mathfrak{h} \dot{+} \mathfrak{r})$$

for $i \in (n_1, n_1 + n_2]$ and $q \in (n_1 + n_2, n_1 + n_2 + n_3]$, giving the desired $2n_2^3 n_3 - n_2 n_3$ rank-one tensors. By applying $\mu(x \otimes y) = [x, y]$, we see that each tensor above is in $\operatorname{Ker}\mu$. We must show that these tensors are linearly independent.

Expanding $S_{i,j,q}$ we see that

$$S_{i,j,q} = \underbrace{e_{i,j} \otimes e_{j,q} - e_{i,j+1} \otimes e_{j+1,q}}_{\notin \operatorname{Span}\{T_{i,j,l,q}\}} + \underbrace{e_{i,j} \otimes e_{j+1,q} - e_{i,j+1} \otimes e_{j,q}}_{\in \operatorname{Span}\{T_{i,j,l,q}\}}$$

is not in the span of the $T_{i,j,l,q}$ tensors. A similar observation shows that $S^{i,j,q}$ is not in the span of the $T^{i,j,l,q}$ tensors.

Expanding $R(i, q)$ we have

$$R(i, q) = \underbrace{e_{i,i} \otimes e_{i,i}}_{\in \mathfrak{h} \otimes \mathfrak{h}} + \underbrace{e_{i,q} \otimes e_{i,q}}_{\in \mathfrak{r} \otimes \mathfrak{r}} + \underbrace{e_{i,i} \otimes e_{i,q} + e_{i,q} \otimes e_{i,i}}_{\in \mathfrak{h} \otimes \mathfrak{r} \dot{+} \mathfrak{r} \otimes \mathfrak{h}}.$$

Since $e_{i,i} \otimes e_{i,i}$ and $e_{i,q} \otimes e_{i,q}$ are in $\mathfrak{h} \otimes \mathfrak{h}$ and $\mathfrak{r} \otimes \mathfrak{r}$, respectively, and since tensors from those blocks have been accounted for above, we may subtract those terms, leaving $R'(i, q) = e_{i,i} \otimes e_{i,q} + e_{i,q} \otimes e_{i,i}$. $R'(i, q)$ is not in the span of $\left\{T_{i,j,l,q}, T^{i,j,l,q}\right\}$ since we require $j \neq l$ in $T_{i,j,l,q}$ and $T^{i,j,l,q}$.

$R'(i,j)$ is linearly independent of the $S_{i,j,q}$ and $S^{i,j,q}$ tensors in case $i = n_1 + n_2$, since we require $j \leq n_1 + n_2 - 1$ in $S_{i,j,q}$ and $S^{i,j,q}$. Now, consider $S_{i,i,q} + S^{i,i,q}$ where $i < n_1 + n_2$

We have

$$S_{i,i,q} + S^{i,i,q} = R'(i,q) + T - \left(e_{i,i+1} \otimes e_{i+1,q} + e_{i+1,q} \otimes e_{i,i+1}\right)$$

with $T \in \operatorname{Span}\left\{T_{i,j,l,q}, T^{i,j,l,q}\right\}$, so we have

$$R'(i,q) = S_{i,i,q} + S^{i,i,q} - T + R''(i,q)$$

where $R''(i,q) = e_{i,i+1} \otimes e_{i+1,q} + e_{i+1,q} \otimes e_{i,i+1}$.

Now, if $i = n_1 + n_2 - 1$ we are done (as above). If $i < n_1 + n_2 - 1$ we may reduce $R''(i,q)$ using the same method just employed, and so by induction we are done. That is to say that $T_{i,j,l,q}$, $T^{i,j,l,q}$, $S_{i,j,q}$, $S^{i,j,q}$, and $R(i,j)$ are linearly independent.

Next, we examine $\mathfrak{h} \otimes \mathfrak{l}$, $\mathfrak{l} \otimes \mathfrak{h}$, and $(\mathfrak{h} \dotplus \mathfrak{l}) \otimes (\mathfrak{h} \dotplus \mathfrak{l})$. The consideration of these subspaces is symmetric with the subspaces considered above, and so we will find that these subspaces contribute $2n_1n_2^3 - n_1n_2$ tensors to our basis of $\operatorname{Ker}\mu$.

Finally, we examine $\mathfrak{l}\otimes\mathfrak{r}$, $\mathfrak{r}\otimes\mathfrak{l}$, and $(\mathfrak{l}\dotplus\mathfrak{r})\otimes(\mathfrak{l}\dotplus\mathfrak{r})$. We proceed similarly to the discussion of $\mathfrak{h}$ and $\mathfrak{r}$ above, and we will find that $\mathfrak{l}$ and $\mathfrak{r}$ contribute the remaining $2n_1n_2^2n_3 - n_1n_3$ rank-one tensors needed to span $\operatorname{Ker}\mu$.

Consider the $2n_1n_2^2n_3 - 2n_1n_2n_3$ tensors

$$U_{i,j,l,q} = e_{i,j} \otimes e_{l,q} \in \mathfrak{l} \otimes \mathfrak{r}$$

and

$$U^{i,j,l,q} = e_{l,q} \otimes e_{i,j} \in \mathfrak{r} \otimes \mathfrak{l}$$

for $i \in (0, n_1]$, $j, l \in (n_1, n_1 + n_2]$, and $q \in (n_1 + n_2, n_1 + n_2 + n_3]$ with $j \neq l$.

Additionally, we have $2n_1n_2n_3 - 2n_1n_3$ tensors

$$V_{i,j,q} = \left(e_{i,j} - e_{i,j+1}\right) \otimes \left(e_{j,q} + e_{j+1,q}\right) \in \mathfrak{l} \otimes \mathfrak{r}$$

and

$$V^{i,j,q} = \left(e_{j,q} + e_{j+1,q}\right) \otimes \left(e_{i,j} - e_{i,j+1}\right) \in \mathfrak{r} \otimes \mathfrak{l}$$

with $i \in (0, n_1]$, $j \in (n_1, n_1 + n_2 - 1]$, and $q \in (n_1 + n_2, n_1 + n_2 + n_3]$.

Finally, we have n_1n_3 tensors of the form

$$\begin{aligned} W(i,q) &= \left(e_{i,n_1+n_2} + e_{n_1+n_2,q}\right) \otimes \left(e_{i,n_1+n_2} + e_{n_1+n_2,q}\right) \\ &\in (\mathfrak{l} \dotplus \mathfrak{r}) \otimes (\mathfrak{l} \dotplus \mathfrak{r}) \end{aligned}$$

for $i \in (0, n_1]$ and $q \in (n_1 + n_2, n_1 + n_2 + n_3]$, giving the remaining $2n_1n_2^2n_3 - n_1n_3$ rank-one tensors. Again, the above tensors were chosen so that applying $\mu(x \otimes y) = [x, y]$ results in 0. Below we verify that they are linearly independent.

Expanding $V_{i,j,q}$ we see that

$$V_{i,j,q} = \underbrace{e_{i,j} \otimes e_{j,q} - e_{i,j+1} \otimes e_{j+1,q}}_{\notin \operatorname{Span}\{U_{i,j,l,q}\}} + \underbrace{e_{i,j} \otimes e_{j+1,q} - e_{i,j+1} \otimes e_{j,q}}_{\in \operatorname{Span}\{U_{i,j,l,q}\}}$$

is not in the span of the $U_{i,j,l,q}$ tensors. A similar observation shows that $V^{i,j,q}$ is not in the span of the $U^{i,j,l,q}$ tensors.

Expanding $W(i,q)$ we have

$$\begin{aligned} W(i,q) = {} & \underbrace{e_{i,n_1+n_2} \otimes e_{i,n_1+n_2}}_{\in \mathfrak{l}\otimes\mathfrak{l}} + \underbrace{e_{n_1+n_2,q} \otimes e_{n_1+n_2,q}}_{\in \mathfrak{r}\otimes\mathfrak{r}} \\ & + \underbrace{e_{i,n_1+n_2} \otimes e_{n_1+n_2,q} + e_{n_1+n_2,q} \otimes e_{i,n_1+n_2}}_{\in \mathfrak{l}\otimes\mathfrak{r} \dotplus \mathfrak{r}\otimes\mathfrak{l}}. \end{aligned}$$

$\mathfrak{l} \otimes \mathfrak{l}$ and $\mathfrak{r} \otimes \mathfrak{r}$ are accounted for above, so we may subtract their terms, leaving

$$W'(i,q) = e_{i,n_1+n_2} \otimes e_{n_1+n_2,q} + e_{n_1+n_2,q} \otimes e_{i,n_1+n_2}.$$

$W'(i,q)$ is not in the span of $\left\{U_{i,j,l,q}, U^{i,j,l,q}\right\}$ since we require $j \neq l$ in $U_{i,j,l,q}$ and $U^{i,j,l,q}$. We also see immediately that $W'(i,q)$ is not in the span of $\left\{V_{i,j,q}, V^{i,j,q}\right\}$ since we require $j < n_1 + n_2$ in $V_{i,j,q}$ and $V^{i,j,q}$. Thus we have that $U_{i,j,l,q}$, $U^{i,j,l,q}$, $V_{i,j,q}$, $V^{i,j,q}$, and $W(i,j)$ are linearly independent.

Having explicitly constructed a basis for $\operatorname{Ker}\mu$ consisting of rank-one tensors, the proof is complete. □

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