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Minimizing Average Risk with Short-Term Futures Hedging

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In this paper, we study the strategies to minimize risk in long-term hedging with short-term futures contracts. We re-evaluate these strategies in Glasserman (2001) by analyzing the average risk.

Introduction

Consider the following market model that has been discussed in Culp and Miller (1995), Glasserman (2001), Larcher and Leobacher (2003) and Wu, Yu, and Zheng (2011):

A firm commits itself to supplying a commodity with fixed quantity q and price a_t at time $t \in [0, T]$. Assume that the market price of the commodity S_t satisfies $dS_t = \mu dt + \sigma dB_t$, where μ is the drift coefficient, σ is the diffusion coefficient, B_t is the standard Brownian motion, and interest rate $r = 0$. If the hedging strategy is to purchase continuously G_t short term futures with life time dt at time t, then the unhedged cash flow and payoff from the hedging strategy over the time interval $[t, t + dt]$ satisfies

$$dC_t = q(a_t - S_t)dt$$

$$dH_t = G_t\mu dt + G_t(\sigma - b_t)dB_t$$

where b_t is the basis of the futures.

If a_t, μ, G_t, b_t are all deterministic, assuming $q = 1, T = 1$ and σ be a constant, we can have:

$$C_t + H_t - E(C_t + H_t) = \int_0^t \sigma(s - t + G_s)dB_s$$

$$Var(C_t + H_t) = \sigma^2 \int_0^t (s - t + G_s)^2 ds$$

where $t \in [0, 1]$.

We call the quantity $Var(C_t+H_t)$ the spot risk of the corresponding hedging strategy G_t. Clearly, if $G_s = 0$, there is no hedge, $Var(C_t + H_t) = \frac{1}{3}\sigma^2 t^3, t \in [0, 1]$. If $G_s = 1 - s$, called the rolling stack hedge, $Var(C_t + H_t) = \sigma^2 t(t - 1)^2, t \in [0, 1]$.

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Glasserman's work

Glasserman studied another two hedging strategies in Glasserman (2001): fixed horizon and fixed fraction hedges. For readers' convenience, we restate his work in details as follows.

In the fixed horizon hedge, a rolling stack strategy is performed, not for the whole time interval$[0, 1]$, but only throughout $[0, \tau]$ for $0 < \tau < 1$. The remaining period of time is left unhedged. Thus, the strategy is

$$G_s = \begin{cases} \tau - s & s \in [0, \tau] \\ 0 & s \in (\tau, 1] \end{cases}$$

Without loss of generality, assume $\sigma = 1$ and let $\sigma_t = \int_0^t (s - t + G_s)^2 ds$.

$$\sigma_t^2 = \begin{cases} f_1(t) & t \in [0, \tau] \\ f_2(t) & t \in (\tau, 1] \end{cases}$$

where $f_1(t) = \int_0^t (\tau - t)^2 ds, f_2(t) = \int_0^\tau (\tau - t)^2 ds + \int_\tau^t (s - t)^2 ds$.

Now,

$$f_1(t) = t(\tau - t)^2 = t^3 - 2\tau t^2 + \tau^2 t$$

$$f_1(0) = f_1(\tau) = 0$$

$$f_1'(t) = 3t^2 - 4\tau t + \tau^2 = (3t - \tau)(t - \tau)$$

$$f_1'(t) = 0 \to t = \tau \quad \text{or} \quad t = \frac{1}{3}\tau$$

Thus, the maximum variance during the hedged portion occurs at $t = \frac{1}{3}\tau$ where $f_1(\frac{1}{3}\tau) = \frac{4}{27}\tau^3$.

We must also consider the variance during the subsequent unhedged portion of the exposure. Since we are conducting a rolling stack with a horizon of τ, having no hedge in the interval $\tau < t \leq 1$ will result in some terminal risk.

$$f_2(t) = \tau(\tau - t)^2 - (\tau - t)^3/3$$

$$f_2'(t) = 0 \to t = \tau$$

Thus, the maximum variance during the unhedged portion is the terminal variance.

$$f_2(1) = \frac{2}{3}\tau^3 - \tau^2 + \frac{1}{3}$$

The relative maximum at $t = \frac{1}{3}\tau$ increases when the horizon is lengthened. Some straightforward calculus shows that the terminal variance decreases when the horizon is lengthened. Therefore, the optimal horizon (the one that minimizes the overall maximum variance) is the one that equates these two values:

$$\frac{2}{3}\tau^3 - \tau^2 + \frac{1}{3} = \frac{4}{27}\tau^3$$

Solving numerically: $\tau^* \approx 0.73340$.

This results in a maximum variance of $f_1(\frac{1}{3}\tau^*) = f_2(1) \approx \frac{4}{27}(0.73340)^3 \approx 0.05844$, a value much lower than that of the elementary rolling stack strategy.

In the fixed fraction hedge, a modified rolling stack strategy is carried out throughout the whole interval $[0, 1]$ for only a fraction of the typical stack. That is, for $0 < \kappa < 1$, the strategy is given by $G_s = \kappa(1 - s)$.

The spot variance is

$$\begin{aligned} \sigma_t^2 = h(t) &= \int_0^t (\kappa(1-s) + s - t)^2 ds \\ &= \frac{1}{3}(\kappa^2 + \kappa + 1)t^3 - (\kappa^2 + \kappa)t^2 + \kappa^2 t \end{aligned}$$

We can use standard methods to figure out where this function might take its maximum.

$$h'(t) = (\kappa^2 + \kappa + 1)t^2 - 2(\kappa^2 + \kappa)t + \kappa^2 = 0$$

The general quadratic formula produces the critical points

$$t_{1,2} = \frac{\kappa^2 + \kappa \pm \kappa^{\frac{3}{2}}}{\kappa^2 + \kappa + 1}$$

The second derivative

$$h''(t) = 2(\kappa^2 + \kappa + 1)t - 2(\kappa^2 + \kappa)$$

allows us to determine where the relative maximum occurs.

$$h''(t_{1,2}) = 2(\kappa^2 + \kappa \pm \kappa^{\frac{3}{2}}) - 2(\kappa^2 + \kappa) = \pm 2\kappa^{\frac{3}{2}}$$

The second derivative is negative (there is a relative maximum) at

$$t_2 = \frac{\kappa^2 + \kappa - \kappa^{\frac{3}{2}}}{\kappa^2 + \kappa + 1}$$

The maximum is

$$h(t_2) = \frac{(\kappa^2 + \kappa - \kappa^{\frac{3}{2}})^3}{3(\kappa^2 + \kappa + 1)^2} - \frac{(\kappa^2 + \kappa)(\kappa^2 + \kappa - \kappa^{\frac{3}{2}})^2}{(\kappa^2 + \kappa + 1)^2} + \frac{\kappa^2(\kappa^2 + \kappa - \kappa^{\frac{3}{2}})}{(\kappa^2 + \kappa + 1)}$$

It is important that we remember that unlike the traditional rolling stack strategy, the fixed fraction hedge does not provide us zero terminal variance. Therefore, if we are looking for the maximum, we must also consider

$$h(1) = \frac{1}{3}\kappa^2 - \frac{2}{3}\kappa + \frac{1}{3}$$

As in the case with the optimal fixed horizon hedge, the fixed fraction that minimizes the overall maximum variance is the one that equates these two values. The complex-looking algebraic expression for the relative maximum at t_2 might point to a numerical solution of this equation. However, an exact solution can actually be found without too many complications. Let us simplify.

$$h(t_2) = \frac{\kappa^2 + \kappa - \kappa^{\frac{3}{2}}}{3(\kappa^2 + \kappa + 1)^2}\left[\kappa^4 + \kappa^{\frac{7}{2}} + \kappa^{\frac{5}{2}} + \kappa^2\right] = \frac{\kappa^6 + 2\kappa^{\frac{9}{2}} + \kappa^3}{3(\kappa^2 + \kappa + 1)^2}$$

Setting them equal to each other:

$$\frac{\kappa^6 + 2\kappa^{\frac{9}{2}} + \kappa^3}{3(\kappa^2 + \kappa + 1)^2} = \frac{1}{3}\kappa^2 - \frac{2}{3}\kappa + \frac{1}{3}$$

Simplify:

$$2\kappa^{\frac{9}{2}} + 3\kappa^3 - 1 = 0$$

We can make the substitution $u = \kappa^{\frac{3}{2}}$ to obtain

$$2u^3 + 3u^2 - 1 = 0$$

which yields the solution $u = \frac{1}{2} \rightarrow \kappa^* = \left(\frac{1}{4}\right)^{\frac{1}{3}} \approx 0.62996$. This results in a maximum variance of

$$h(t_2) = h(1) = \frac{1}{3}\left(\frac{1}{4}\right)^{\frac{2}{3}} - \frac{2}{3}\left(\frac{1}{4}\right)^{\frac{1}{3}} + \frac{1}{3} \approx 0.04564$$

Thus, the optimal fixed fraction hedge leads to lower maximum variance than the optimal fixed horizon hedge. Figure 1 below compares these two strategies to the traditional rolling stack.

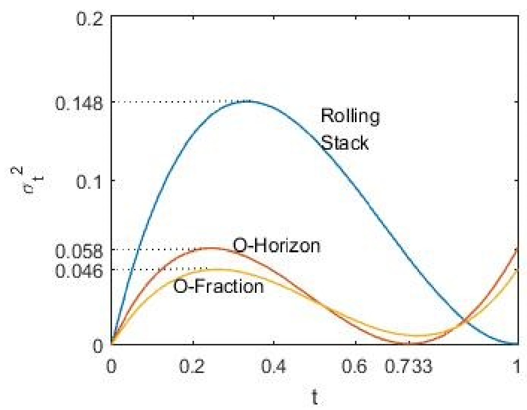

Figure 1. Comparison of the variances of the optimal fixed horizon, the optimal fixed fraction, and the traditional rolling stack strategies. The modified strategies offer much lower maximum variance, with the optimal fraction beating out the optimal horizon.

Minimize the average risk

We also mean to analyze the average risk one is exposed to under these strategies. Indeed, we should expect the minimization of the average risk to give rise to a different optimal horizon and optimal fraction. We will also be interested in comparing this minimal average risk to the one exhibited by the strategies that minimize the maximum variance. As introduced in Larcher and Leobacher (2003), the average risk for the fixed horizon strategy is given by

$$\begin{aligned} A(\tau) &:= \int_0^1 \sigma_t^2 dt = \int_0^\tau f_1(t)dt + \int_\tau^1 f_2(t)dt \\ &= \int_0^t \left(t^3 - 2\tau t^2 + \tau^2 t\right) dt + \int_\tau^1 \left(\frac{1}{3}t^3 - \tau^2 t + \frac{2}{3}\tau^3\right) dt \\ &= -\frac{1}{6}\tau^4 + \frac{2}{3}\tau^3 - \frac{1}{2}\tau^2 + \frac{1}{12} \end{aligned}$$

Using our $\tau^* \approx 0.73340$, we get $A(\tau^*) \approx 0.02916$. However, from the expression we have just derived, standard methods will allow us to determine exactly the horizon that minimizes the average risk.

$$A'(\tau) = -\frac{2}{3}\tau^3 + 2\tau^2 - \tau = 0$$

$$\tau_{1,2} = \frac{3}{2} \pm \frac{\sqrt{3}}{2}$$

The only solution in the interval $[0, 1]$ is $\tau_2 = \frac{3}{2} - \frac{\sqrt{3}}{2} \approx 0.63397$. To confirm it is a minimum we look towards the second derivative.

$$A''(\tau) = -2\tau^2 + 4\tau - 1$$

$$A''(\tau_2) = \sqrt{3} - 1 > 0$$

Therefore, the horizon that minimizes the average risk is at $\tau_2 = \frac{3}{2} - \frac{\sqrt{3}}{2} \approx 0.63397$, with $A(\tau_2) \approx 0.02532$, about 87% of the average risk under the horizon τ^*.

For a more detailed comparison, let us look at the maximal variance that would occur under this horizon. Recall that under a fixed horizon full hedge the candidates for maximal variance are

$$f_1\left(\frac{1}{3}\tau\right) = \frac{4}{27}\tau^3 \quad \text{and} \quad f_2(1) = \frac{2}{3}\tau^3 - \tau^2 + \frac{1}{3}$$

Substituting $\tau = \tau_2$ we get

$$f_1\left(\frac{1}{3}\tau_2\right) \approx 0.03775 \quad \text{and} \quad f_2(1) \approx 0.10128$$

This demonstrates that although this new horizon has a lower relative maximum at the peak of the hedged portion (at $t = \frac{1}{3}\tau$), it almost doubles the overall maximum (at $t = 1$).

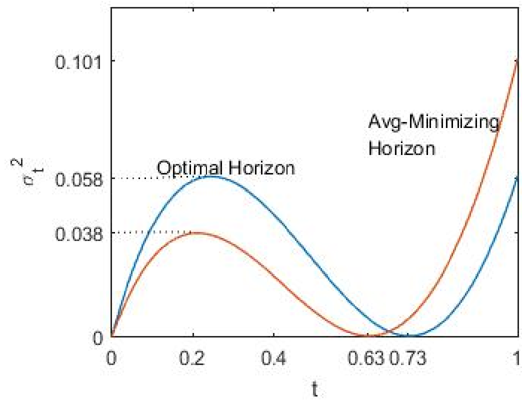

Figure 2. Variances of the optimal fixed horizon and the fixed horizon that minimizes the average risk.

Let us consider the running risk, $R(\sigma_t^2)$, of both strategies, measured by the running maximum variance. i.e.

$$R(\sigma_t^2) = \sup_{0\le s\le t} \sigma_s^2$$

The running risk of the optimal horizon is

$$R_1(\sigma_t^2) = \begin{cases} t\,(\tau^* - t)^2\,, & 0 \le t \le \frac{1}{3}\tau^* \\ \frac{4}{27}\,(\tau^*)^3\,, & \frac{1}{3}\tau^* < t \le 1 \end{cases}$$

where $\tau^* \approx 0.73340$. The running risk of the average-minimizing horizon is

$$R_2(\sigma_t^2) = \begin{cases} t\,(\tau_2 - t)^2\,, & 0 \le t \le \frac{1}{3}\tau_2 \\ \frac{4}{27}\,(\tau_2)^3\,, & \frac{1}{3}\tau_2 < t < \eta \\ \frac{1}{3}t^3 - \tau_2^2 t + \frac{2}{3}\tau_2^3, & \eta \le t \le 1 \end{cases}$$

where $\tau_2 \approx 0.63397$ and $\eta \approx 0.86443$.

Here, η is obtained numerically by solving the equation, that $f_2(\eta)$, under the hedging horizon τ_2, is equal to the value of $\frac{4}{27}\,(\tau_2)^3$.

We see that τ_2 produces lower average risk than τ^* and also that it has a lower value for the relative maximum of the hedged portion of the exposure. In order to compare the running risk between the optimal horizon and the average-minimizing horizon, we need to solve the inequality $R_2(\sigma_t^2) \le R_1(\sigma_t^2)$. Clearly, $R_1(\sigma_t^2)$ and $R_1(\sigma_t^2)$ are non-decreasing functions and when $0 \le t < \eta$, $R_2(\sigma_t^2) \le R_1(\sigma_t^2)$. For $\eta \le t \le 1$, we only need to solve the inequality

$$\frac{1}{3}t^3 - \tau_2^2 t + \frac{2}{3}\tau_2^3 \le \frac{4}{27}\,(\tau^*)^3$$

Numerically, we get $t \le 0.91723$.

Therefore, not only does τ_2 result in lower average risk, it also leads to lower running risk throughout the majority of the life of the exposure (around 92%). However, roughly locking in terminal expected value is an attractive quality in a hedging strategy, so the increased spot variance under τ_2 towards the end of the life of the exposure is a shortcoming.

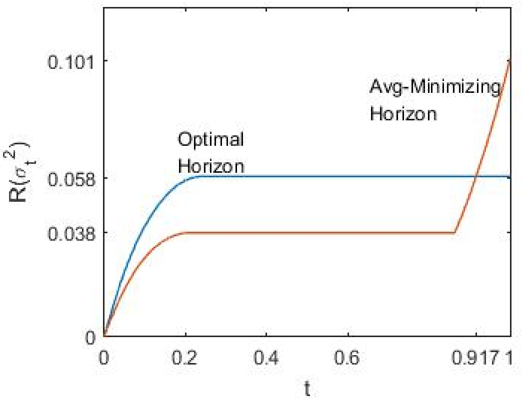

Figure 3. Comparison of the running risk between the optimal horizon and the average-minimizing horizon as measured by the running maximum variance. The average-minimizing horizon only becomes "riskier" after $t \approx 0.917$.

Given its characteristics, the hedging horizon τ_2 could have its place among a certain type of investor. For example, an investor with a particular concern for liquidity issues throughout most of the life of the exposure but who will be expecting significant amounts of revenue near the end of the contract might be comfortable gambling with the increased terminal variance.

We will now go through the same analysis on the fixed fraction hedging strategy. Given that the optimal fixed fraction posted better results than the optimal fixed horizon, it is not unreasonable to expect that the strategy found by minimizing the average risk under a fixed fraction will be a better option for our hypothetical investor than the one found under a fixed horizon.

The average risk for the fixed fraction strategy is given by

$$\begin{aligned} A(\kappa) &:= \int_0^1 \sigma_t^2 dt \\ &= \int_0^1 \left(\frac{1}{3}(\kappa^2+\kappa+1)t^3 - (\kappa^2+\kappa)t^2 + \kappa^2 t\right) dt \\ &= \frac{1}{4}\kappa^2 - \frac{1}{4}\kappa + \frac{1}{12} \end{aligned}$$

Using our $\kappa^* \approx 0.62996$, we get $A(\kappa^*) \approx 0.02506$.

Due to the fact that $A(\kappa)$ is an upwards opening parabola, we can immediately see that it is minimized by $\kappa_2 = \frac{1}{2}$ (where the vertex is located). This gives the minimum average risk $A\left(\frac{1}{2}\right) = \frac{1}{48} \approx 0.02083$, which is approximately equal to 83% of the average risk suffered under the fixed fraction κ^*. Again, for comparison, let us look at the maximal variance that would occur under the fixed fraction hedge with κ_2. Recall that

$$h(t) = \frac{1}{3}(\kappa^2+\kappa+1)t^3 - (\kappa^2+\kappa)t^2 + \kappa^2 t$$

which, under κ_2, becomes

$$h(t) = \frac{7}{12}t^3 - \frac{3}{4}t^2 + \frac{1}{4}t$$

We know from previous calculations that the relative maximum occurs at

$$t_2^* = \frac{\kappa^2+\kappa-\kappa^{\frac{3}{2}}}{\kappa^2+\kappa+1} = \frac{3-\sqrt{2}}{7} \approx 0.22654$$

Then,

$$h\left(t_2^*\right) = \frac{9+4\sqrt{2}}{588} \approx 0.02493$$

The other candidate for maximal variance is the end of the hedge, $h(1) = \frac{1}{12} \approx 0.08333$. Bearing some resemblance to our discussion of horizons, the fixed fraction that minimizes the average risk ($\kappa_2 = \frac{1}{2}$) reduces the maximal variance at the local maximum but almost doubles the overall maximum.

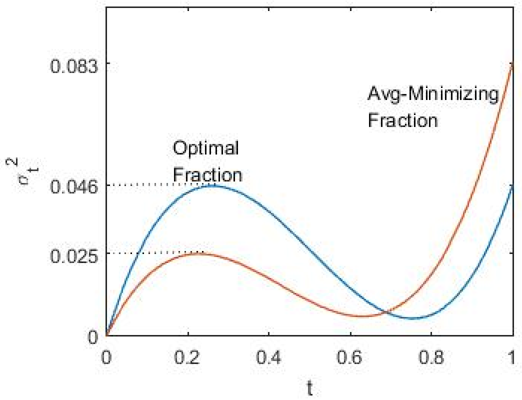

Figure 4. Variances of the optimal fixed fraction and the fixed fraction that minimizes the average risk.

Again, let us consider the running risk of both strategies, as measured by the running maximum variance. The running risk of the optimal fixed fraction is

$$R_3(\sigma_t^2) = \begin{cases} \frac{1}{3}\left((\kappa^*)^2+\kappa^*+1\right)t^3 - \left((\kappa^*)^2+\kappa^*\right)t^2 + (\kappa^*)^2 t, & 0 \le t \le t_2 \\ \frac{1}{3}(\kappa^*)^2 - \frac{2}{3}\kappa^* + \frac{1}{3}, & t_2 < t \le 1 \end{cases}$$

where $\kappa^* \approx 0.62996$ and $t_2 \approx 0.25992$.

The running risk of the average-minimizing fraction is

$$R_4(\sigma_t^2) = \begin{cases} \frac{7}{12}t^3 - \frac{3}{4}t^2 + \frac{1}{4}t, & 0 \le t \le t_2^* \\ \frac{9+4\sqrt{2}}{588}, & t_2^* < t < \eta^* \\ \frac{7}{12}t^3 - \frac{3}{4}t^2 + \frac{1}{4}t, & \eta^* \le t \le 1 \end{cases}$$

where $t_2^* = \frac{3-\sqrt{2}}{7} \approx 0.22654$ and $\eta^* = \frac{3+\sqrt{2}}{7} \approx 0.83263$.

We see that κ_2 produces lower average risk than κ^* and also that it has a lower value for the local maximum. Just like the case with fixed horizons, we notice that $R_3(\sigma_t^2)$ and

$R_4(\sigma_t^2)$ are non-decreasing functions. Solve the inequality $R_4(\sigma_t^2) \leq R_3(\sigma_t^2)$ numerically to get $t \leq 0.90890$.

Therefore, we arrive at a result that mirrors the case with fixed horizons. Not only does κ_2 result in lower average risk, it also leads to lower running risk throughout the majority of the life of the exposure (around 91%, as opposed to 92% in the case of horizons). The main disadvantage of the average-minimizing strategy, the increased terminal variance, is also emulated.

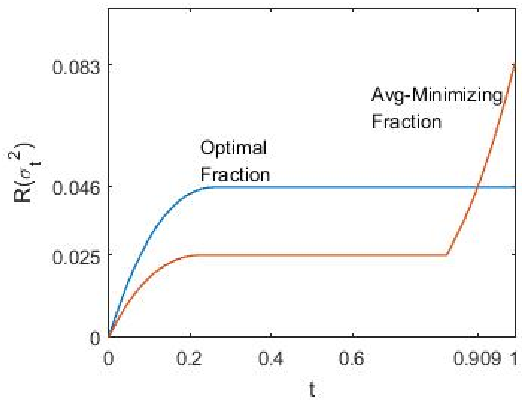

Figure 5. above compares the running risk between the optimal fraction and the average-minimizing fraction as measured by the running maximum variance. The average-minimizing fraction only becomes "riskier" after $t \approx 0.909$.

The average-minimizing fixed fraction strategy would definitely be useful for our hypothetical investors. Moreover, as expected, it performs better than the average-minimizing fixed horizon. It provides both lower average risk (0.02083 as opposed to 0.02532) and lower running risk throughout the entire life of the exposure.

Conclusion

By analyzing the average risk, we re-evaluate hedging strategies in Glasserman (2001). We introduce two new hedging strategies, avg-minimizing horizon strategy and avg-minimizing fixed fraction strategy. After comparing the running risk and spot variance, we suggest that the average-minimizing fixed fraction strategy would be useful for a hypothetical investor with a particular concern for liquidity issues throughout most of the life of the exposure but who will be expecting significant amounts of revenue near the end of the contract.

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