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Classroom Notes: Why is the 4-term polynomial arising from the *ac-method* of factoring always factorable?

Noel Sagullo
University of Arkansas - Fort Smith
Department of Mathematics

The procedure that is often taught for factoring a quadratic polynomial ax^2+bx+c with nonzero integer coefficients has been called the *ac-method* of factoring (Dugopolski, 2011). Using the notion of multisets (Bender, 1974; Blizard, 1989), we will prove that the 4-term polynomial that arises when the *ac-method* applies always factors by grouping. As a consequence, we will be able to provide a justification for this factoring procedure.

Introduction

We give the procedure for factoring the quadratic polynomials with integer coefficients often found in beginning mathematics textbooks.

Procedure for Factoring $ax^2 + bx + c$

1. Find two numbers whose sum is b and whose product is ac.
2. Replace b by the sum of these two numbers.
3. Factor the resulting 4-term polynomial by grouping.

Using this procedure on $18x^2 + 57x + 17$, we get $ac = 18 \cdot 17 = 2 \cdot 3^2 \cdot 17$. A pair of divisors of ac, 6 and 51, whose sum equals the coefficient of the middle term, 57, gives us the 4-term polynomial $18x^2 + 6x + 51x + 17$. Factor by grouping then gives us the factorization $(6x + 17)(3x + 1)$. The question a student might be interested in is why any 4-term polynomial that arises in this way is factorable by grouping. An additional question might be whether the resulting 4-term polynomial is still factorable by grouping had $6x$ and $51x$ been commuted. Yet another question might be whether the absence of a divisor pair adding to the middle coefficient precludes the existence of linear factors. We hope that in the course of our exposition, the answers to these questions will become clear to the student.

Complementary Subsets

Definition 1 (Complementary Subsets)**.** *Let U be a set. If A and C are disjoint subsets of U whose union is U, we say that A, C is a pair of complementary subsets of U.*

Author Email: noel.sagullo@uafs.edu

Theorem 1. *Let U be a set. If A, C and D_1, D_2 are pairs of complementary subsets of U, then $A \setminus D_1 = D_2 \setminus C$.*

The proof of Theorem 1 requires a straightforward use of the definition of set complement and is omitted. As an application, we prove that the quadratic factoring procedure works when ac has no repeated prime factor and all the coefficients are positive.

Application 1. *Let ax^2+bx+c be a polynomial with positive integer coefficients. Suppose ac has no repeated prime factor. If there exist two integers d_1 and d_2 such that $d_1d_2 = ac$ and $d_1+d_2 = b$, then ax^2+bx+c is a product of two linear factors both with integer coefficients.*

Proof. Let $ac = p_1p_2p_3 \dots p_n$ where the p_i's are distinct primes. If $d_1d_2 = ac$ and $d_1 + d_2 = b$, then $ax^2 + bx + c = ax^2 + d_1x + d_2x + c$.

Factor by grouping then gives us:

$$\begin{aligned} ax^2 + d_1x + d_2x + c &= \gcd(a, d_1)x\Big(\frac{a}{\gcd(a, d_1)}x + \frac{d_1}{\gcd(a, d_1)}\Big) \\ &\quad + \gcd(c, d_2)\Big(\frac{d_2}{\gcd(c, d_2)}x + \frac{c}{\gcd(c, d_2)}\Big). \end{aligned} \tag{1}$$

The notation we use for the greatest common divisor of two integers, say a and b, is $\gcd(a, b)$. Note that each of the four "fractions" in (1) is an integer. Let set A consist of the prime factors of a; C the prime factors of c; D_1 the prime factors of d_1; and D_2 the prime factors of d_2. We have:

$$\frac{a}{\gcd(a, d_1)} = \prod_{x \in A \setminus D_1} x \tag{2}$$

and

$$\frac{d_2}{\gcd(c, d_2)} = \prod_{x \in D_2 \setminus C} x. \tag{3}$$

Since $A \setminus D_1 = D_2 \setminus C$, we have

$$\frac{a}{\gcd(a,d_1)} = \frac{d_2}{\gcd(c,d_2)}. \tag{4}$$

By a similar use of Theorem 1, we also have:

$$\frac{d_1}{\gcd(a,d_1)} = \frac{c}{\gcd(c,d_2)}. \tag{5}$$

Hence

$$\begin{aligned} ax^2+bx+c &= ax^2+d_1x+d_2x+c \\ &= (\gcd(a,d_1)x+\gcd(c,d_2)) \\ &\quad \left(\frac{a}{\gcd(a,d_1)}x+\frac{d_1}{\gcd(a,d_1)}\right). \end{aligned}$$

□

Remark 1. *If the product is taken over an empty set, then we take the product to be equal to 1.*

The Case of Negative Coefficients

The integers d_1 and d_2 satisfy $d_1d_2 = ac$ and $d_1 + d_2 = b$. These requirements determine the signs of d_1 and d_2 according to the signs of the coefficients a, b, and c. The following table lists each of the cases where one or more of the coefficients is negative along with the signs, necessarily, of d_1 and d_2.

Case	a	b	c	Signs of d_1 and d_2
1)	+	+	−	Exactly one of d_1 or d_2 is negative
2)	+	−	+	Both d_1 and d_2 are negative
3)	+	−	−	Exactly one of d_1 or d_2 is negative
4)	−	+	+	Exactly one of d_1 or d_2 is negative
5)	−	+	−	Both d_1 and d_2 are positive
6)	−	−	+	Exactly one of d_1 or d_2 is negative
7)	−	−	−	Both d_1 and d_2 are negative

In cases 1, 3, 4, and 6, since exactly one of d_1 and d_2 is negative (and the other positive) and since a and c have different signs, we may arrange a to have the same sign as d_2 and d_1 to have the same sign as c, by switching the values referred to by d_1 and d_2 as needed, so that equalities (4) and (5) in Application 1 hold. Thus the factorization given at the conclusion of Application 1 holds for these cases.

In case 2, since the pair a, c has the same sign (both positive) and the pair d_1, d_2 has the same sign (both negative), and the signs of these pairs are opposite, we cannot simply switch the values referred to by d_1 and d_2 to have the signs agree with a and c. We need to modify equalities (4) and (5) to

$$\frac{a}{\gcd(a,d_1)} = \frac{-d_2}{\gcd(c,d_2)}$$

and

$$\frac{-d_1}{\gcd(a,d_1)} = \frac{c}{\gcd(c,d_2)}$$

respectively. In which case the factorization becomes

$$\begin{aligned} ax^2+bx+c &= ax^2+d_1x+d_2x+c \\ &= (\gcd(a,d_1)x-\gcd(c,d_2)) \\ &\quad \left(\frac{a}{\gcd(a,d_1)}x+\frac{d_1}{\gcd(a,d_1)}\right). \end{aligned} \tag{6}$$

Case 5 is similar to case 2. Since the pair a, c has the same sign (both negative) and the pair d_1, d_2 have the same sign (both positive), and the signs of these pairs are opposite, we need to modify equalities (4) and (5) to

$$\frac{-a}{\gcd(a,d_1)} = \frac{d_2}{\gcd(c,d_2)}$$

and

$$\frac{d_1}{\gcd(a,d_1)} = \frac{-c}{\gcd(c,d_2)}$$

respectively. In which case the factorization becomes as in (6).

Finally, in case 7, equalities (4) and (5) hold and so the factorization given at the conclusion of Application 1 holds.

Repeated Prime Factors and Multisets

If $a = 3^2$ and $d_1 = 3$ then letting A be the set of prime divisors of a and D_1 the set of prime divisors of d_1, would give us $A = D_1$. In which case, the equality in (2) no longer holds. The main problem is that repeated prime factors are lost when the sets A and D_1 are formed.

Definition 2 (Multiset)**.** *A multiset is an unordered collection of objects in which elements are allowed to repeat. Formally, if U is a set and $A \subseteq U$, then a multiset is a function, m_A, from U to the set of nonnegative integers. The number $m_A(u)$ is the number of occurrences of u. If $u \in A$, then $m_A(u) \geq 1$ and $m_A(u) = 0$ otherwise.*

Remark 2. *For example, if $a = 90$, then the multiset consisting of the prime divisors of a is $\{(2,1),(3,2),(5,1)\} \cup \{(p,0) \mid p \text{ is a prime not dividing } 90\}$ or more succinctly, $[2,3,3,5]$. We use square brackets for multisets to distinguish them from sets.*

Definition 3 (Multiset Difference and Sum)**.** *Let m_A and m_B be multisets with $A, B \subseteq U$. The multiset difference $m_A - m_B$ is defined to be $max(0, m_A(u) - m_B(u))$ for all u in U. The multiset sum $m_A \uplus m_B$ is defined to be $m_A(u) + m_B(u)$ for all u in U.*

Remark 3. *For example,* $[2, 2, 3, 3, 5] - [3, 7] = [2, 2, 3, 5]$ *and* $[2, 2, 3, 3, 5] \uplus [3, 7] = [2, 2, 3, 3, 3, 5, 7]$.

Definition 4 (Partition of a Multiset)**.** *Let m_A be a multiset. A partition of m_A is a collection of multisets whose multiset sum is m_A.*

Remark 4. *For example, the collection consisting of* $[2, 3]$ *and* $[3, 5]$ *is a partition of* $[2, 3, 3, 5]$.

Theorem 2. *Let m_U be a multiset. If m_A, m_C and m_{D_1}, m_{D_2} are partitions of m_U, then $m_A - m_{D_1} = m_{D_2} - m_C$.*

Proof. Since

$$(m_A - m_{D_1})(u) = \max(0, m_A(u) - m_{D_1}(u))$$

and

$$(m_{D_2} - m_C)(u) = \max(0, m_{D_2}(u) - m_C(u)),$$

it suffices to show

$$m_A(u) - m_{D_1}(u) = m_{D_2}(u) - m_C(u)$$

for all $u \in U$. Indeed, using the definition of multiset sum along with the hypotheses, we have:

$$\begin{aligned} m_A(u) - m_{D_1}(u) &= (m_U(u) - m_C(u)) - (m_U(u) - m_{D_2}(u)) \\ &= m_{D_2}(u) - m_C(u). \end{aligned}$$

□

Application 2. *The case when ac has repeated prime factors may now be handled using Theorem 2. Let $ax^2 + bx + c$ be a polynomial with positive integer coefficients. If there exist two integers d_1 and d_2 such that $d_1 d_2 = ac$ and $d_1 + d_2 = b$, then $ax^2 + bx + c$ is a product of two linear factors both with integer coefficients.*

Remark 5. *The interested student may supply his or her proof. Mutatis mutandis, the proof is the same as the proof of Application 1. Likewise, the case of negative coefficients is handled in the same way as it was handled in Application 1.*

Finally, we may now state a formal version of the *ac-method* factoring procedure.

Theorem 3. *Let a, b, and c be nonzero integers. The polynomial $ax^2 + bx + c$ is a product of two linear factors, both with integer coefficients, if and only if there exist two integers d_1 and d_2 such that $d_1 d_2 = ac$ and $d_1 + d_2 = b$.*

Remark 6. *If $ax^2 + bx + c = (mx + n)(sx + t)$, with m, n, s, and t, all integers, then multiplying out and equating coefficients, we see that mt and sn are two integers such that $mtsn = ac$ and $mt + sn = b$. The converse is essentially the result of Application 2.*

Summary of Factorization Formulas

The following table gives the factorization of $ax^2 + bx + c$, with a, b, and c, nonzero integers and for which there exist integers d_1 and d_2 satisfying $d_1 d_2 = ac$ and $d_1 + d_2 = b$. As an example, we use the table to factor $-18x^2 + 171x - 253$.

a	b	c	Factorization
+	+	+	$(\gcd(a, d_1)x + \gcd(c, d_2))\left(\frac{a}{\gcd(a,d_1)}x + \frac{d_1}{\gcd(a,d_1)}\right)$
+	+	−	$(\gcd(a, d_1)x + \gcd(c, d_2))\left(\frac{a}{\gcd(a,d_1)}x + \frac{d_1}{\gcd(a,d_1)}\right)$
+	−	+	$(\gcd(a, d_1)x - \gcd(c, d_2))\left(\frac{a}{\gcd(a,d_1)}x + \frac{d_1}{\gcd(a,d_1)}\right)$
+	−	−	$(\gcd(a, d_1)x + \gcd(c, d_2))\left(\frac{a}{\gcd(a,d_1)}x + \frac{d_1}{\gcd(a,d_1)}\right)$
−	+	+	$(\gcd(a, d_1)x + \gcd(c, d_2))\left(\frac{a}{\gcd(a,d_1)}x + \frac{d_1}{\gcd(a,d_1)}\right)$
−	+	−	$(\gcd(a, d_1)x - \gcd(c, d_2))\left(\frac{a}{\gcd(a,d_1)}x + \frac{d_1}{\gcd(a,d_1)}\right)$
−	−	+	$(\gcd(a, d_1)x + \gcd(c, d_2))\left(\frac{a}{\gcd(a,d_1)}x + \frac{d_1}{\gcd(a,d_1)}\right)$
−	−	−	$(\gcd(a, d_1)x + \gcd(c, d_2))\left(\frac{a}{\gcd(a,d_1)}x + \frac{d_1}{\gcd(a,d_1)}\right)$

Example. For the quadratic polynomial $-18x^2 + 171x - 253$, we have the following:

$$\begin{aligned} a &= -18 = (-1) \cdot 2 \cdot 3^2 \\ b &= 171 \\ c &= -253 = (-1) \cdot 11 \cdot 23 \\ d_1 &= 33 = 3 \cdot 11 \\ d_2 &= 138 = 2 \cdot 3 \cdot 23 \end{aligned}$$

and

$$\begin{aligned} \gcd(a, d_1) &= 3 \\ \gcd(c, d_2) &= 23. \end{aligned}$$

Hence, using the sixth line in the table above, we obtain the factorization:

$$\begin{aligned} -18x^2 + 171x - 253 &= (\gcd(a, d_1)x - \gcd(c, d_2)) \\ &\quad \left(\frac{a}{\gcd(a, d_1)}x + \frac{d_1}{\gcd(a, d_1)}\right) \\ &= (3x - 23)(-6x + 11). \end{aligned}$$

References

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