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On $h(x)$-Fibonacci octonion polynomials

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In this paper, we introduce $h(x)$-Fibonacci octonion polynomials that generalize both Catalan's Fibonacci octonion polynomials and Byrd's Fibonacci octonion polynomials and also k-Fibonacci octonion numbers that generalize Fibonacci octonion numbers. Also we derive the Binet formula and and generating function of $h(x)$-Fibonacci octonion polynomial sequence.

Introduction

To investigate the normed division algebras is greatly a important topic today. It is well known that the octonions **O** are the nonassociative, noncommutative, normed division algebra over the real numbers $\mathbb{R}$. Due to the nonassociative and noncommutativity, one cannot directly extend various results on real, complex and quaternion numbers to octonions. The book by Conway and Smith (n.d.) gives a great deal of useful background on octonions, much of it based on the paper of Coxeter et al. (1946). Octonions made further appearance ever since in associative algebras, analysis, topology, and physics. Nowadays octonions play an important role in computer science, quantum physics, signal and color image processing, and so on (e.g. Adler and Finkelstein (2008)).

Many references may be given for Fibonacci-like sequences and numbers and related issues. The readers are referred to Koshy (n.d.) and Vajda (1989) for some other sequences which can be obtained from the Fibonacci numbers. One may find many applications of this type sequences of numbers in various branches of science like pure and applied mathematics, in biology or in phyllotaxies, among many others. Hence, many researches are performed on the various type of Fibonacci sequence; for example see Cureg and Mukherjea (2010), Falcon and Plaza (2007), A. Horadam (1961), Kilic (2008), Öcal, Tuglu, and Altinişik (2005), Yang (2008)-Yilmaz and Taskara (2013).

The connection between the Fibonacci-type numbers and the Golden Section is expressed by the well-known mathematical formula, so-called Binet's formulas. Many references may be given for Fibonacci-like sequences and numbers and Binet's formulas. By using a generating function Levesque (1985) gave a Binet formula for the Fibonacci sequence. Kilic and Tasci (2006) gave the generalized Binet formulas for the generalized order-k Fibonacci (Er (1984)) and Lucas numbers (Tasci and Kilic (2004)). Kiliç et al. (2006) gave the Binet formula for the generalized Pell sequence.

The investigation of special number sequences over **H** and **O** which are not analogs of ones over $\mathbb{R}$ and $\mathbb{C}$ has attracted some recent attention (see, e.g.,Akyiğit, Kösal, and Tosun (2013), Halici (2012), Halici (2013), Iyer (1969), A. F. Horadam (1963) and Keçilioğlu and Akkus (2015)). While majority of papers in the area are devoted to some Fibonacci-type special number sequences over $\mathbb{R}$ and $\mathbb{C}$, only few of them deal with Fibonacci-type special number sequences over **H** and **O** (see, e.g., Akyiğit et al. (2013), Halici (2012), Halici (2013), Iyer (1969), A. F. Horadam (1963) and Keçilioğlu and Akkus (2015)), notwithstanding the fact that there are a lot of papers on various types of Fibonacci-type special number sequences over $\mathbb{R}$ and $\mathbb{C}$ (see, e.g., Cureg and Mukherjea (2010), Falcon and Plaza (2007), A. Horadam (1961), Kilic (2008), Öcal et al. (2005), Yang (2008)-Yilmaz and Taskara (2013)).

In this paper, we introduce $h(x)$-Fibonacci octonion polynomials that generalize both Catalan's Fibonacci octonion polynomials $\Psi_n(x)$ and Byrd's Fibonacci octonion polynomials and also k-Fibonacci octonion numbers that generalize Fibonacci octonion numbers. Also we derive the Binet formula and and generating function of $h(x)$-Fibonacci octonion polynomial sequence.

The rest of the paper is structured as follows. Some preliminaries which are required are given in Section 2. In Section 3 we introduce Catalan, Byrid, Fibonacci octonion polynomials and numbers and provide their some properties. Section 4 ends with our conclusion.

Some Preliminaries

We start this section by introducing some definitions and notations that will help us greatly in the statement of the results.

We begin this section by giving the following definition for the classic Fibonacci sequence. The classic Fibonacci

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$\{F_n\}_{n\in\mathbb{N}}$ sequence is a very popular integer sequence.

Definition 1. *The classic Fibonacci $\{F_n\}_{n\in\mathbb{N}}$ sequence is defined by*

$$F_0 = 0,\ F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2, \tag{1}$$

where F_n denotes the n^{th} classic Fibonacci number (Dunlap (n.d.), Koshy (n.d.), Vajda (1989)).

The books written by Hoggat (1969), Koshy (n.d.) and Vajda (1989) collects and classifies many results dealing with the these number sequences , most of them are obtained quite recently.

Nalli and Haukkanen (2009) introduced the $h(x)$-Fibonacci polynomials. Then, Ramírez (2014) introduced the convolved $h(x)$-Fibonacci polynomials and obtain new identities.

Definition 2. *Let $h(x)$ be a polynomial with real coefficients. The $h(x)$-Fibonacci polynomials $\{F_{h,n}(x)\}_{n=0}^{\infty}$ are defined by the recurrence relation*

$$F_{h,n+1}(x) = h(x)F_{h,n}(x) + F_{h,n-1}(x),\ n \geq 1, \tag{2}$$

with initial conditions $F_{h,0}(x) = 0$, $F_{h,1}(x) = 1$ (Nalli and Haukkanen (2009)).

For $h(x) = x$, it is obtained Catalan's Fibonacci polynomials $\psi_n(x)$, and for $h(x) = 2x$ it is obtained Byrd's Fibonacci polynomials $\phi_n(x)$. If for k real number $h(x) = k$, it is obtained the k-Fibonacci numbers $F_{k,n}$. For $k = 1$ and $k = 2$ it is obtained the usual Fibonacci numbers F_n and the Pell numbers P_n, respectively.

The quaternion, which is a type of hypercomplex numbers, was formally introduced by Hamilton in 1843.

Definition 3. *The real quaternion is defined by*

$$q = q_r + q_i i + q_j j + q_k k$$

where q_r, q_i, q_j and q_k are real numbers and i, j and k are complex operators obeying the following rules

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

In recent years there has been a flurry of activity for doing research with Fibonacci quaternion and octonion numbers. We can start from the Fibonacci quaternion numbers introduced by Horadam in 1963.

Definition 4. *The Fibonacci quaternion numbers that are given for the n^{th} classic Fibonacci F_n number are defined by the following recurrence relations:*

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} \tag{3}$$

where $n = 0, \pm 1, \pm 2, ...$(A. F. Horadam (1963)).

Catarino (2015) introduced $h(x)$-Fibonacci quaternion polynomials. The basic properties of Fibonacci quaternion numbers can be found in Akyiğit et al. (2013), Halici (2012), Halici (2013), Iyer (1969) and A. F. Horadam (1963).

The octonions in Clifford algebra $\mathbf{C}$ are a normed division algebra with eight dimensions over the real numbers larger than the quaternions. The field $\mathbf{O} \cong \mathbf{C}^4$ of octonions

$$\alpha = \sum_{s=0}^{7} \alpha_s e_s,\ a_i(i = 0, 1, ..., 7) \in \mathbb{R} \tag{4}$$

is an eight-dimensional non-commutative and non-associative $\mathbb{R}$-field generated by eight base elements $e_0, e_1, ..., e_6$ and e_7. The multiplication rules for the basis of $\mathbf{O}$ are listed in the following table

$\times$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Table 1. The multiplication table for the basis of **O.**

(5)

The scalar and the vector part of a octonion $\alpha = \sum\limits_{s=0}^{7} \alpha_s e_s \in \mathbf{O}$ are denoted by $S_\alpha = \alpha_0$ and $\overrightarrow{V_\alpha} = \sum\limits_{s=1}^{7} \alpha_s e_s$, respectively.

Let α and β be two octonions such that $\alpha = \sum\limits_{s=0}^{7} \alpha_s e_s$ and $\beta = \sum\limits_{s=0}^{7} \beta_s e_s$. We now are going to list some properties, without proofs, for α and β octonions.

- Property 1. $\alpha \pm \beta = \sum\limits_{s=0}^{7} (\alpha_s \pm \beta_s) e_s$
- Property 2. $\alpha\beta = S_\alpha S_\beta + S_\alpha V_\beta + V_\alpha S_\beta - V_\alpha \cdot V_\beta + V_\alpha \times V_\beta$, where $S_\alpha = \alpha_0$ and $\overrightarrow{V_\alpha} = \sum\limits_{s=1}^{7} \alpha_s e_s$.
- Property 3. $\overline{\alpha} = \alpha_0 e_0 - \sum\limits_{s=1}^{7} \alpha_s e_s$
- Property 4. $\overline{(\alpha\beta)} = \overline{\beta}\overline{\alpha}$.
- Property 5. $\frac{\alpha+\overline{\alpha}}{2} = \alpha_0 e_0$ and $\frac{\alpha-\overline{\alpha}}{2} = \sum\limits_{k=1}^{7} \alpha_k e_k$.
- Property 6. $\overline{\alpha}\alpha = \alpha\overline{\alpha}$
- Property 7. $\overline{\alpha}\alpha = \sum\limits_{k=0}^{7} \alpha_k^2$.

The norm of an octonion can be defined as $\|\alpha\| = \sqrt{\overline{\alpha}\alpha}$.

- Property 8. $\|\alpha\beta\| = \|\alpha\| \, \|\beta\|$.
- Property 9. $k.\alpha = \sum\limits_{i=0}^{7} (k\alpha_i)\, e_i$.
- Property 10. $\langle \alpha,\beta \rangle = \sum\limits_{i=0}^{7} \alpha_i\beta_i$.

The Fibonacci octonion numbers are given by the following definition.

Definition 5. *For $n \geq 0$, the Fibonacci octonion numbers that are given for the n^{th} classic Fibonacci F_n number are defined by the following recurrence relations:*

$$Q_n = \sum_{s=0}^{7} F_{n+s} e_s.$$

The basic properties of Fibonacci octonion numbers can be found in Keçilioğlu and Akkus (2015).

The $h(x)$-Fibonacci octonion polynomials

In this section, we introduce $h(x)$-Fibonacci octonion polynomials that generalize both Catalan's Fibonacci octonion polynomials $\Psi_n(x)$ and Byrd's Fibonacci octonion polynomials and also k-Fibonacci octonion numbers. Also we derive the Binet formula and and generating function of $h(x)$-Fibonacci octonion polynomial sequence.

Let e_i, $i = 0, 1, 2, 3, 4, 5, 6, 7$, be a basis of $\mathbf{O}$ which satisfy the non-commutative and non-associative multiplication rules are listed in Table 1 in (5). Let $h(x)$ be a polynomial with real coefficients.

Definition 6. *The $h(x)$-Fibonacci octonion polynomials $\{Q_{h,n}(x)\}_{n=0}^{\infty}$ are defined by the recurrence relation*

$$Q_{h,n}(x) = \sum_{s=0}^{7} F_{h,n+s}(x) e_s \tag{6}$$

where $F_{h,n}(x)$ is the n^{th} $h(x)$-Fibonacci polynomial.

Particular cases of the previous definition are:

- For $h(x) = x$, it is obtained Catalan's Fibonacci polynomials $\psi_n(x)$ from the $h(x)$-Fibonacci polynomials $F_{h,n}(x)$, and thus for $h(x) = x$ we have Catalan's Fibonacci octonion polynomials $\Psi_n(x)$ from the $h(x)$-Fibonacci octonion polynomials $Q_{h,n}(x)$.
- For $h(x) = 2x$, it is obtained Byrd's Fibonacci polynomials $\phi_n(x)$ from the $h(x)$-Fibonacci polynomials $F_{h,n}(x)$, and thus for $h(x) = 2x$ we get Byrd's Fibonacci octonion polynomials $\Phi_n(x)$ from the $h(x)$-Fibonacci octonion polynomials $Q_{h,n}(x)$.
- If for k real number $h(x) = k$, it is obtained the k-Fibonacci numbers $F_{k,n}$ from the $h(x)$-Fibonacci polynomials $F_{h,n}(x)$, and thus for $h(x) = k$ we obtain k-Fibonacci octonion numbers $Q_{k,n}$ from the $h(x)$-Fibonacci octonion polynomials $Q_{h,n}(x)$. Also, for $k = 1$ and $k = 2$ we have the Fibonacci octonion numbers Q_n and the Pell octonion numbers Υ_n, respectively.

The ordinary generating function (OGF) of the sequence $\{a_n\}_{n=0}^{\infty}$ is defined by $g(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ (Rosen (1999)) and the exponential generating function (EGF) of a sequence $\{a_n\}_{n=0}^{\infty}$ is defined by $g(x) = \sum\limits_{n=0}^{\infty} a_n \frac{x^n}{n!}$ (Kincaid and Cheney (2002)). For general material on generating functions the interested reader may refer to the books written by Lando (2003) and **?**.

The generating function $g_Q(t)$ of the sequence $\{Q_{h,n}(x)\}_{n=0}^{\infty}$ is defined by

$$g_Q(t) = \sum_{n=0}^{\infty} Q_{h,n}(x) t^n. \tag{7}$$

We know that the power series (7) converges if and only if $|t| < \min(1/\,|\alpha|\,, 1/\,|\beta|)$ for α and β given (22). We consider $g_Q(t)$ a formal power series which is needed not take care of the convergence.

For convenience, we use the following notations: $F_{h,n} = F_{h,n}(x)$ and $Q_{h,n} = Q_{h,n}(x)$.

Equipped with the definitions and properties above, we can present the fundamental theorems of this paper as follows.

Theorem 1. *The generating function for the $h(x)$-Fibonacci octonion polynomials $Q_{h,n}(x)$ is*

$$g_Q(t) = \frac{Q_{h,0} + (Q_{h,1} - h(x)Q_{h,0})\, t}{1 - h(x)t - t^2} \tag{8}$$

and

$$g_Q(t) = \frac{1}{1 - h(x)t - t^2} \sum_{s=0}^{7} (F_{h,s} + F_{h,s-1} t)\, e_s. \tag{9}$$

Proof. The generating function $g_Q(t)$ of the $h(x)$-Fibonacci octonion polynomials is

$$g_Q(t) = Q_{h,0} + Q_{h,1}t + Q_{h,2}t^2 + ... + Q_{h,n}t^n + ... \tag{10}$$

The orders of $Q_{h,n-1}$ and $Q_{h,n-2}$, respectively, are 1 and 2 less than the order of $Q_{h,n}$. Thus, we obtain

$$\begin{aligned} h(x)g_Q(t)t &= h(x)Q_{h,0}t + h(x)Q_{h,1}t^2 \\ &\quad + h(x)Q_{h,2}t^3 + ... + h(x)Q_{h,n-1}t^n + ... \end{aligned} \tag{11}$$

and

$$g_Q(t)t^2 = Q_{h,0}t^2 + Q_{h,1}t^3 + Q_{h,2}t^4 + ... + Q_{h,n-2}t^n + ... \tag{12}$$

From Definition 6, (10) , (11) and (12), we have

$$\left(1 - h(x)t - t^2\right) g_Q(t) = Q_{h,0} + \left(Q_{h,1} - h(x)Q_{h,0}\right) t \qquad (13)$$

and we thus obtain (8). From Definition 6, it follows that

$$Q_{h,1} - h(x)Q_{h,0} = \sum_{s=0}^{7} F_{h,s-1}e_s. \qquad (14)$$

Combining (13) and (14), then (9) is evident. □

Similarly, we have the following result.

Theorem 2. *Suppose that $h(x)$ is an odd polynomial. Then for $g^{'}_Q(t) = \sum_{n=0}^{\infty} Q_{h,n}(-x)(-t)^n$ we have*

$$g^{'}_Q(t) = \frac{Q_{h,0}(-x) - \left(Q_{h,1}(-x) + h(x)Q_{h,0}(-x)\right) t}{1 - h(x)t - t^2} \qquad (15)$$

and

$$g^{'}_Q(t) = \frac{1}{1 - h(x)t - t^2} \sum_{s=0}^{7} (-1)^{s+1} \left(F_{h,s}(x) + F_{h,s-1}(x)t\right) e_s. \qquad (16)$$

Proof. We now consider

$$g^{'}_Q(t) = \sum_{n=0}^{\infty} Q_{h,n}(-x)(-t)^n.$$

$g^{'}_Q(t)$ is a formal power series. Therefore, we need not take care of the convergence of the series. Thus, we write

$$\begin{aligned} g^{'}_Q(t) &= Q_{h,0}(-x) - Q_{h,1}(-x)t + Q_{h,2}(-x)t^2 \qquad (17)\\ &\quad -\ldots + Q_{h,n}(-x)(-t)^n + \ldots. \end{aligned}$$

The orders of $Q_{h,n-1}(-x)$ and $Q_{h,n-2}(-x)$, respectively, are 1 and 2 less than the order of $Q_{h,n}(-x)$. Thus, since $h(-x) = -h(x)$ we obtain

$$\begin{aligned} -h(x)g^{'}_Q(t)t &= -h(x)Q_{h,0}(-x)t + h(x)Q_{h,1}(-x)t^2 \qquad (18)\\ &\quad -h(x)Q_{h,2}(-x)t^3 + \ldots\\ &\quad -(-1)^{n-1}h(x)Q_{h,n-1}(-x)t^n + \ldots \end{aligned}$$

and

$$\begin{aligned} g^{'}_Q(t)t^2 &= Q_{h,0}(-x)t^2 - Q_{h,1}(-x)t^3 + Q_{h,2}(-x)t^4 \qquad (19)\\ &\quad -\ldots + (-1)^{n-2}Q_{h,n-2}(-x)(t)^n + \ldots. \end{aligned}$$

From (17) , (18) and (19) we get (15). On the other hand, by using Definition (6), we can compute $LHS = Q_{h,1}(-x) + h(x)Q_{h,0}(-x)$

$$LHS = \sum_{s=0}^{7} \left(F_{h,s+1}(-x) + h(x)F_{h,s}(-x)\right) e_s. \qquad (20)$$

Combining Definition (2) and (20) gives the following equality

$$Q_{h,1}(-x) + h(x)Q_{h,0}(-x) = \sum_{s=0}^{7} F_{h,s-1}(-x)e_s. \qquad (21)$$

Since $F_{h,n}(-x) = (-1)^{n+1}F_{h,n}(x)$ from Theorem 2.2 in Nalli and Haukkanen (2009) from (15) and (21) we have (16). □

Binet's formulas are well known in the theory of the Fibonacci numbers. These formulas can also be carried out for the $h(x)$-Fibonacci octonion polynomials.

The characteristic equation associated with the recurrence relation (2) is $v^2 = h(x)v + 1$. The roots of this equation are

$$\alpha(x) = \frac{h(x) + \sqrt{h(x)^2 + 4}}{2}, \beta(x) = \frac{h(x) - \sqrt{h(x)^2 + 4}}{2}. \qquad (22)$$

The following basic identities is needed for our purpose in proving.

$$\left.\begin{aligned} \alpha(x) + \beta(x) &= h(x),\\ \alpha(x) - \beta(x) &= \sqrt{h(x)^2 + 4},\\ \alpha(x).\beta(x) &= -1. \end{aligned}\right\} \qquad (23)$$

For convenience of representation, we adopt the following notations: $\alpha = \alpha(x)$ and $\beta = \beta(x)$.

The following lemma is directly useful for stating our next main results.

Lemma 1. *For the generating function $g_Q(t)$ in (7) of the $h(x)$-Fibonacci octonion polynomials $Q_{h,n}(x)$, we have*

$$g_Q(t) = \frac{1}{\alpha - \beta}\left[\frac{Q_{h,1} - \beta Q_{h,0}}{1 - \alpha t} - \frac{Q_{h,1} - \alpha Q_{h,0}}{1 - \beta t}\right].$$

Proof. The proof can be obtained easily from (23). □

We obtain following Binet's formula for $Q_{h,n}(x)$.

Theorem 3. *For $n \geq 0$, the Binet's formula for the $h(x)$-Fibonacci octonion polynomials $Q_{h,n}(x)$ is as follows*

$$Q_{h,n}(x) = \frac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha - \beta} \qquad (24)$$

where $\alpha^* = \sum_{s=0}^{7} \alpha^s e_s$ and $\beta^* = \sum_{s=0}^{7} \beta^s e_s$.

Proof. From Lemma 1, we obtain

$$\begin{aligned} g_Q(t) &= \frac{1}{\alpha - \beta}\left[\frac{Q_{h,1} - \beta Q_{h,0}}{1 - \alpha t} - \frac{Q_{h,1} - \alpha Q_{h,0}}{1 - \beta t}\right]\\ &= \frac{1}{\alpha - \beta}\left[(Q_{h,1} - \beta Q_{h,0}) \sum_{n=0}^{\infty} \alpha^n t^n \right. \qquad (25)\\ &\quad \left. -(Q_{h,1} - \alpha Q_{h,0}) \sum_{n=0}^{\infty} \beta^n t^n\right]. \end{aligned}$$

By taking (6) into (25), we can get

$$g_Q(t) = \frac{1}{\alpha-\beta}\left[\sum_{s=0}^{7}(F_{h,s+1}-\beta F_{h,s})\,e_s\sum_{n=0}^{\infty}\alpha^n t^n \right. \quad (26)$$

$$\left. -\sum_{s=0}^{7}(F_{h,s+1}-\alpha F_{h,s})\,e_s\sum_{n=0}^{\infty}\beta^n t^n\right].$$

Since $F_{h,s+1} - \beta F_{h,s} = \alpha^s$ and $F_{h,s+1} - \alpha F_{h,s} = \beta^s$ and from (26) we have

$$g_Q(t) = \frac{1}{\alpha-\beta}\left[\sum_{s=0}^{7}\alpha^s e_s\sum_{n=0}^{\infty}\alpha^n t^n - \sum_{s=0}^{7}\beta^s e_s\sum_{n=0}^{\infty}\beta^n t^n\right]. \quad (27)$$

For $\alpha^* = \sum_{s=0}^{7}\alpha^s e_s$ and $\beta^* = \sum_{s=0}^{7}\beta^s e_s$, from (27) we obtain

$$g_Q(t) = \sum_{n=0}^{\infty}\frac{(\alpha^*\alpha^n - \beta^*\beta^n)}{\alpha-\beta}t^n. \quad (28)$$

Consequently, by the equality of generating function in (7) and (28), we have Binet's formula for $Q_{h,n}(x)$ in (24) □

Theorem 4. *For $m \in \mathbb{Z}$, $n \in \mathbb{N}$, the generating function of the sequence $\{Q_{h,m+n}(x)\}$ is as follows*

$$\sum_{n=0}^{\infty} Q_{h,m+n}(x)t^n = \frac{Q_{h,m}(x) + Q_{h,m-1}(x)t}{1-h(x)t-t^2}.$$

Proof. By using the Binet formula for $Q_{h,n}(x)$, we write

$$\sum_{n=0}^{\infty} Q_{h,m+n}(x)t^n = \sum_{n=0}^{\infty}\frac{(\alpha^*\alpha^{m+n} - \beta^*\beta^{m+n})}{\alpha-\beta}t^n.$$

Therefore, we get

$$\sum_{n=0}^{\infty} Q_{h,m+n}(x)t^n = \frac{1}{\alpha-\beta}\left[\alpha^*\alpha^m\sum_{n=0}^{\infty}\alpha^n t^n - \beta^*\beta^m\sum_{n=0}^{\infty}\beta^n t^n\right]$$

$$= \frac{1}{\alpha-\beta}\left[\alpha^*\alpha^m\frac{1}{1-\alpha t} - \beta^*\beta^m\frac{1}{1-\beta t}\right].$$

So, from this and (23) we obtain

$$\sum_{n=0}^{\infty} Q_{h,m+n}(x)t^n = \frac{1}{\alpha-\beta}\left[\frac{\alpha^*\alpha^m - \beta^*\beta^m + \left(\alpha^*\alpha^{m-1} - \beta^*\beta^{m-1}\right)t}{1-h(x)t-t^2}\right].$$

Consequently, if we recall the Binet's formula for $Q_{h,n}(x)$, we get the result. □

Conclusions

In this paper, we consider the $h(x)$-Fibonacci octonion polynomials that generalize both Catalan's Fibonacci octonion polynomials $\Psi_n(x)$ and Byrd's Fibonacci octonion polynomials and also k-Fibonacci octonion numbers that generalize Fibonacci octonion numbers. Then we give the generating function and Binet formula of the $h(x)$-Fibonacci octonion polynomials. We predict that in which part of science the above-introduced generating function and Binet formula for the $h(x)$-Fibonacci octonion polynomials and numbers will have the most effective application.

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