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On Semistability of Perfect Lattices

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We present the finding that all perfect lattices of dimension at most 7 are semistable, and that all but one 8-dimensional perfect lattices are semistable. The Coxeter-Barnes lattice $\mathbb{A}_8^2$ is the unique 8-dimensional perfect lattice that is not semistable.

Introduction

In this paper, we study the semistability of perfect lattices. A lattice (discrete subgroup of a Euclidean space) is said to be *perfect* if it can be determined (up to isometry) by the length of shortest vectors and the components of those vectors on some (unknown) basis (Martinet, 2003; Voronoï, 1908). Perfect lattices play an interesting role in the classical reduction theory. In particular, extreme lattices, the lattices whose density is a local maximum, are among perfect lattices (Martinet, 2003, Theorem 3.4.6). It is well-known that there are finitely many perfect lattices in each dimension (up to similarity). All perfect lattices of dimension up to 8 have been found (Schürmann & Vallentin, 2005a; Sikirić, Schürmann, & Vallentin, 2007), but the search for perfect lattices of dimension 9 is not complete yet (Schürmann & Vallentin, 2005b).

The classical reduction theory developed by Minkowski (1968) and others is concerned with the upper bound on lengths of shortest vectors. Another, more recent reduction theory using the notion of semistability (Grayson, 1984) is related to the lower bounds on lengths of vectors. The notion of semistability of lattices in Euclidean spaces was first introduced by Stuhler (1976). He defined the *canonical filtration* of a lattice in analogy with a similar filtration for vector bundles over algebraic curves. Then he defined a lattice to be *semistable* if its canonical filtration is trivial. Grayson (1984) developed the idea further and produced an alternative method of proving a result of Borel and Serre (1973) on aritheoremetic groups. He did it by studying the manifold of semistable lattices.

The main result of this paper is that the notions of perfectness and semistability from two reduction theories are not related but that they overlap significantly. To be precise, among 10916 perfect lattice of dimension 8, all but one are semistable. In dimension 9, among over 500000 known perfect lattices, only one is not semistable.

In the following sections, we review the definition and some properties of canonical filtrations of lattices. Then we justify the method (algoritheorem) to compute the canonical filtration of a lattice, then in the last section we present the computational result.

The paper stems from another research project of finding the CW-structure of Grayson's manifold described by the number of shortest nonzero vectors of lattices. For example, in dimension 2, the set of semistable lattices modulo isometries (minus the boundary) is a CW-complex (Grayson, 1984, Figure 1.26). It has one 0-cell that is the perfect lattice of dimension 2 with three pairs of shortest vectors. The 1-cells are the lattices with two pairs of shortest vectors, and other lattices with only one pair of shortest vectors constitute 2-cells. Unfortunately, this idea could not be generalized to higher dimensions. The author's conjecture was that perfect lattices would constitute 0-cells because of the abundance of shortest vectors, but it is proved wrong in this paper by the existence of unstable perfect lattices in dimension 8 or higher. It still remains as an interesting problem to find a CW-structure of the manifold. I thank Daniel R. Grayson for helpful discussions and ideas.

Canonical Filtrations

We review the definition and some properties of the canonical filtration of a lattice necessary for this paper from Stuhler (1976) and Grayson (1984). Let L be a lattice in a Euclidean space E. We denote the inner product of $x, y \in E$ by $x \cdot y$. A subgroup M of L with the inherited inner product is called a *sublattice* of L. If the quotient group L/M has no torsion, then M is called a *saturated* sublattice. We will consider only saturated sublattices in this paper and will simply call them sublattices. For each sublattice M of L, we define the volume $\operatorname{vol} M$ to be the (nonzero) covolume of the fundamental domain of M in the real span of the generators $v_1, \ldots, v_k$ of M. Then $(\operatorname{vol} M)^2$ is the determinant of the Gram matrix $(v_i \cdot v_j)_{1 \le i,j \le k}$, which is denoted by $\det M$. In other words, $\operatorname{vol} M = \sqrt{\det M}$.

Suppose M is a sublattice. The quotient group L/M can be given the structure of a lattice if it is identified with the projection of L onto the orthogonal complement of the subspace of E generated by M. The lattice structure is defined in such a way so that

$$\operatorname{vol} L = \operatorname{vol} M \ \operatorname{vol}(L/M).$$

If L_1 and L_2 are sublattices, then so are $L_1 \cap L_2$ and $L_1 + L_2$.

The following theorem compares their volumes.

Theorem 1 (Grayson (1984), Theorem 1.12; Stuhler (1976), Proposition 2)**.** *Suppose L_1 and L_2 are sublattices of L. Then*

$$\dim(L_1 \cap L_2) + \dim(L_1 + L_2) = \dim L_1 + \dim L_2,$$
$$\operatorname{vol}(L_1 \cap L_2)\operatorname{vol}(L_1 + L_2) \leq \operatorname{vol} L_1 \ \operatorname{vol} L_2.$$

The theorem can be interpreted geometrically as follows. Consider the plot of points $(\dim M, \log\operatorname{vol} M)$ in the (x, y)-plane for all sublattices M of L. (The log is added to turn multiplicative relations to additive relations.) This plot is called the *canonical plot* of L. The above theorem describes the relative positions of points corresponding to L_1, L_2, $L_1 \cap L_2$, and $L_1 + L_2$. If three of them are given, we can draw a parallelogram after plotting three points as vertices. If the fourth point comes from L_1 or L_2, then it lies at or above the fourth vertex of that parallelogram. On the other hand, if the fourth point comes from $L_1 \cap L_2$ or $L_1 + L_2$, then it lies at or below the fourth vertex of that parallelogram. This is called *the parallelogram constraint* (Grayson, 1984, Discussion 1.13).

The canonical plot of L is bounded below since there are only finitely many sublattices of volume less than a specified upper bound. Thus the convex hull of the plot is bounded below by a convex polygon. This polygon is called the *canonical polygon* of L. Using the parallelogram constraint, Grayson (1984) in Discussion 1.16 proves that the vertices of the canonical polygon are represented by unique sublattices of L, and that they form a chain $0 = L_0 \subset L_1 \subset \cdots \subset L_r = L$. This chain is called the *canonical filtration* of L.

Definition 1 (Grayson (1984), Definition 1.20)**.** *A lattice L is called* stable *if all nonzero proper sublattices are plotted above the line segment from* 0 *to L. It is called* semistable *if they are plotted on or above the line. Otherwise, the lattice is called* unstable.

If L is stable or semistable, its canonical filtration is $0 \subset L$.

Definition 2. *For a nonzero lattice M of dimension k, we define*

$$\operatorname{slope}(M) = \frac{\log(\operatorname{vol} M)}{k} = \log\left((\operatorname{vol} M)^{1/k}\right).$$

If $L_1 \subsetneq L_2$ are sublattices of L, then $\operatorname{slope}(L_2/L_1)$ is the slope of the line joining the plots of L_1 and L_2 as

$$\operatorname{slope}(L_2/L_1) = \frac{\log\operatorname{vol} L_2 - \log\operatorname{vol} L_1}{\dim L_2 - \dim L_1}.$$

Definition 3. *For each $1 \leq k \leq n$, we define the* minimum average length *of k-dimensional sublattices of L to be the minimum of* $(\operatorname{vol} M)^{1/k}$ *as M ranges over all k-dimensional sublattices of L. We denote it by $\lambda_k(L)$.*

Remark 1. *The invariant $\lambda_k(L)$ is related to Rankin k-invariant $\gamma_k(L)$ (Coulangeon, 1996; Rankin, 1953) as*

$$\gamma_k(L) = \lambda_k(L)^{2k}/(\det L)^{k/n}. \tag{1}$$

Rankin's constants are defined by

$$\gamma_{n,k} = \sup\{\gamma_k(L) \mid L \text{ is an } n\text{-dimensional lattice.}\}.$$

They generalize Hermite's constants $\gamma_n = \gamma_{n,1}$. The values of some low-dimensional Rankin's constants have been computed by Rankin (1953) and Sawatani, Watanabe, and Okuda (2010), but not much is known about methods of computing Rankin's invariants of an arbitrary lattice. In the next section, we describe a brute force method to find a k-dimensional sublattice of the smallest volume, which leads to the computation of $\lambda_k(L)$ or $\gamma_k(L)$.

Proposition 1. *Suppose L is a lattice. The following are equivalent.*

1. *L is semistable.*
2. *$\gamma_k(L) \geq 1$ for all $1 \leq k \leq n$.*
3. *$\lambda_k(L) \geq \lambda_n(L)$ for all $1 \leq k \leq n$.*
4. *$(\operatorname{vol} M)^{1/k} \geq (\operatorname{vol} L)^{1/n}$ for all $1 \leq k \leq n$ and all k-dimensional sublattices M.*

Proof. The second and the third conditions are equivalent by equation (1) in Remark 1 and the identity $\gamma_n(L) = 1$. The third and the fourth conditions are equivalent by the definition of $\lambda_k(L)$ and the identity $\lambda_n(L) = (\operatorname{vol} L)^{1/n}$. We only need to prove the equivalence of the first and the fourth. A lattice L is semistable if and only if all sublattices are plotted on or above the line joining zero lattice and the whole lattice L. An equivalent condition is that $\operatorname{slope} M \geq \operatorname{slope} L$ for every nonzero sublattice M of L as $\operatorname{slope} M$ is the slope of the line passing through the lattices 0 and M. The inequality implies, by definition,

$$\log\left((\operatorname{vol} M)^{1/k}\right) \geq \log\left((\operatorname{vol} L)^{1/n}\right)$$

or by dropping log,

$$(\operatorname{vol} M)^{1/k} \geq (\operatorname{vol} L)^{1/n}.$$

□

We will use Proposition 1 and the following theorem for the recursive algoritheorem in the next section to compute the canonical filtration of a lattice.

Definition 4. *For a lattice L, let* $\min L$ *denote the smallest slope of the canonical polygon of L, and let* $\max L$ *denote the largest.*

If $0 = L_0 \subset L_1 \subset \cdots \subset L_r = L$ is the canonical filtration of L, then $\min L = \operatorname{slope} L_1$ and $\max L = \operatorname{slope}(L/L_{r-1})$.

Theorem 2 (Grayson (1984), Corollary 1.29)**.** *Suppose L has a filtration $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_r = L$ by sublattices such that $\max L_i/L_{i-1} \leq \min L_{i+1}/L_i$. Then*

1. *The canonical polygon of L is formed by laying the canonical polygons of the subquotients L_i/L_{i-1} end to end.*
2. *Each L_i lies on the canonical polygon of L.*
3. *If $\max L_i/L_{i-1} < \min L_{i+1}/L_i$, then L_i is in the canonical filtration of L.*
4. *If $L_i \subset L' \subset L_{i+1}$ and L' is in the canonical filtration of L_{i+1}/L_i, then L' is in the canonical filtration of L.*
5. *The canonical filtration of L consists solely of sublattices arising as in 3 and 4.*

Computing the Canonical Filtration of a Lattice

In this section, we describe a recursive algoritheorem to compute the canonical filtration of a lattice. Suppose L is a lattice of dimension n, and M is a sublattice of dimension k for $1 \leq k \leq n-1$. If $\max M \leq \min L/M$, then by Theorem 2, M lies on the canonical polygon of L, and the canonical filtration of L is obtained by combining the canonical filtration of M and the canonical filtration of L/M. Such a sublattice M should satisfy the following conditions: $\operatorname{vol} M$ is minimal among k-dimensional sublattices, and $\operatorname{vol} M^{1/k} \leq \operatorname{vol} L^{1/n}$. (c.f. Proposition 1). The algoritheorem first tries to find such a lattice M, and if it succeeds, then it recursively finds the canonical filtrations of M and L/M, and they are combined together. The worst computational case occurs for semistable lattices where no such M exists.

A sublattice of L of dimension k corresponds to a one dimensional sublattice (or a vector) of the kth exterior power $\bigwedge^k L$. So finding k-dimensional sublattices of L bounded by a constant has the same computational complexity as finding vectors in $\bigwedge^k L$ whose lengths are bounded by the constant. It is the most time-consuming when k is close to $n/2$ as the dimension of $\bigwedge^k L$ is $\binom{n}{k}$, and it is faster when k is close to 1 or n. When the dimension of M is larger than $n/2$, we use the dual lattice L^* of L. The relationship between the canonical plot of L and the canonical plot of L^* is described in (Grayson, 1984, Section 7). The transformation

$$(x, y) \mapsto (n - x, y - \log \operatorname{vol} L) \tag{2}$$

of the xy-plane transforms the canonical plot of L to the canonical plot of L^*.

The dual lattice L^* is defined to be $\operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ considered as a subgroup of $E^* = \operatorname{Hom}_{\mathbb{R}}(E, \mathbb{R})$ where E is the Euclidean space in which L is embedded. The isomorphism $E \to E^*$ defined by $v \mapsto (w \mapsto v \cdot w)$ transports the inner product on E to the inner product on E^*. The dual basis of the orthonormal basis becomes an orthonormal basis, and E^{**} is isometric to E via the canonical isomorphism $E \to E^{**}$. If A is the Gram matrix of L with respect to the basis $\mathcal{E} = \{e_1, \ldots, e_n\}$, then the matrix of the map $E \to E^*$ with respect to $\mathcal{E}$ and the dual basis $\mathcal{E}^*$ is the same as A. Therefore, the Gram matrix of L^* with respect to $\mathcal{E}^*$ is $(A^{-1})^t A (A^{-1}) = (A^{-1})^t = A^{-1}$. This implies that $\operatorname{vol} L^* = (\operatorname{vol} L)^{-1}$.

Suppose M is a sublattice of L and $\{b_1, \ldots, b_k\}$ is a basis of M. The basis can be extended to a basis $\{b_1, \ldots, b_k, b_{k+1}, \ldots, b_n\}$ of L. The additional vectors can be computed explicitly by reversing the row and column operations for the Smith normal form of the inclusion map $M \to L$. (Every diagonal entry of the Smith normal form is 1 since L/M is torsion-free.) Dualizing the exact sequence $0 \to M \to L \to L/M \to 0$, we get an exact sequence $0 \to (L/M)^* \to L^* \to M^* \to 0$. The image of $(L/M)^*$ in L^* is denoted by $M^\#$. The lattice $(L/M)^*$ is generated by the dual basis of $\left\{\overline{b_{k+1}}, \ldots, \overline{b_n}\right\}$, and the image of $\overline{b_i}^*$ in L^* is b_i^*. Therefore, $M^\#$ is generated by $\left\{b_{k+1}^*, \ldots, b_n^*\right\}$. The assignment $M \mapsto M^\#$ defines a one-to-one correspondence between the sublattices of L and the sublattices of L^* since $M^{\#\#}$ is identified with M via the identification of L^{**} with L.

Let `CanonicalFiltration`(L) denote the program returning the canonical filtration of L, and let `SmallestSub`(L,k,V) be the program that returns `fail` if there is no sublattice of L of volume $\leq V$ and returns a sublattice of the smallest volume in dimension k otherwise. Table 1 describes the algoritheorem for `CanonicalFiltration`.

The function `SmallestSub` requires more explanation. In dimension 1, the algoritheorem solves the problem of finding the shortest vector, which is known as the shortest vector problem (SVP). In higher dimensions, it finds a basis of a sublattice of the smallest volume. If the lattice is semistable, the function returns the unique sublattice of the smallest volume. Otherwise, it returns an arbitrary sublattice of the smallest volume. By Proposition 2 below, the basis can be formed by vectors from a finite set of short vectors bounded by a constant. We use the brute force method of examining all possible combinations of short vectors satisfying the bound condition on norms to find a sublattice with the smallest volume. The method may not be theoretically refined, but it is practial enough for the purpose of this paper. With this method, it takes less than a day on an average personal computer to find the canonical filtrations of all known 9 dimensional perfect lattices.

In order to derive the bound for short vectors we will use Korkine-Zolotareff reduced basis. Let M be a lattice and suppose it has basis $B = \{b_1, b_2, \ldots, b_k\}$ and let $\widetilde{B} = \left\{\widetilde{b_1}, \widetilde{b_2}, \ldots, \widetilde{b_k}\right\}$ be its Gram-Schmidt orthogonal basis. They are defined by $\widetilde{b_i} = b_i - \sum_{j=1}^{i-1} \mu_{ij} \widetilde{b_j}$ and $\mu_{ij} = \left(b_i, \widetilde{b_j}\right) / \left(\widetilde{b_j}, \widetilde{b_j}\right)$.

Table 1
Canonical Filtration

	CanonicalFiltration(L)
(1)	If $\dim L = 1$, then return the trivial filtration $0 \subset L$.
(2)	Set $k \leftarrow 1$.
(3)	[Finding a sublattice] If $k > n/2$, then return the trivial filtration $0 \subset L$. Perform $M \leftarrow$ SmallestSub($L, k, \operatorname{vol} L^{k/n}$). If $M =$ fail, then go to step (4). Check the slope condition: $\max M \leq \min L/M$. If the condition is not satisfied, then go to step (4). Otherwise, go to step (6).
(4)	[Finding a dual sublattice] If $k \geq n/2$, then return the trivial filtration $0 \subset L$. Perform $N \leftarrow$ SmallestSub($L^*, n-k, \operatorname{vol} L^{-k/n}$). If $N =$ fail, then go to step (5). Check the slope condition: $\max N \leq \min L^*/N$. If the condition is not satisfied, then go to step (5). Otherwise, set $M \leftarrow N^\#$, then go to step (6).
(5)	[Loop] $k \leftarrow k+1$ and go to step (3),
(6)	[Recursive step] Perform CanonicalFiltration(M) and CanonicalFiltration(L/M). Combine two filtrations together to obtain the canonical filtration of L and return it. M is a part of the filtration if and only if $\max M \lneq \min L/M$.

Let

$$\pi_i\colon \sum_{j=1}^{k} \mathbb{R}b_j \to \left(\sum_{j=1}^{i-1} \mathbb{R}b_j\right)^{\perp}$$

be the orthogonal projection for $1 \leq i \leq k$. Then

$$\pi_j(b_i) = \mu_{ij}\widetilde{b_j} + \cdots + \mu_{ii-1}\widetilde{b_{i-1}} + \widetilde{b_i} \quad \text{for} \quad 1 \leq j \leq i.$$

In particular,

$$\pi_{i-1}(b_i) = \mu_{ii-1}\widetilde{b_{i-1}} + \widetilde{b_i}. \tag{3}$$

Definition 5 (Korkine and Zolotareff (1873), Lagarias, Lenstra, and Schnorr (1990))**.** *With above notations, the basis B of M is called Korkine-Zolotareff reduced if the following two conditions are satisfied.*

1. $|\mu_{ij}| \leq 1/2$ *for* $1 \leq j < i \leq k$,
2. $\widetilde{b_i}$ *is a shortest nonzero vector of* $\pi_i(M)$ *for* $1 \leq i \leq k$.

Lemma 1. *With above notations, if the basis is Korkin-Zolotareff reduced, then*

1. $|b_j|^2 \leq (4/3)^{i-1}\left|\widetilde{b_i}\right|^2$ *for* $1 \leq j \leq i \leq k$.
2. $|b_j| \leq (4/3)^{(k+j-2)/4}\left(\frac{\operatorname{vol}(M)}{\operatorname{vol}(M_{j-1})}\right)^{\frac{1}{k-j+1}}$ *for* $1 \leq j \leq k$ *where* $M_{j-1} = \sum_{i=1}^{j-1}\mathbb{Z}b_i$.

Proof. This proof is a variation of the proof in (Lenstra, Lenstra, & Lovász, 1982, Proposition 1.6). The second condition of Definition 5 implies that $|\pi_{i-1}(b_i)|^2 \geq \left|\widetilde{b_{i-1}}\right|^2$ for $1 < i \leq k$, or by equation (3),

$$\left|\widetilde{b_i} + \mu_{ii-1}\widetilde{b_{i-1}}\right|^2 \geq \left|\widetilde{b_{i-1}}\right|^2,$$

which implies

$$\left|\widetilde{b_i}\right|^2 \geq (1-\mu_{ii-1}^2)\left|\widetilde{b_{i-1}}\right|^2 \geq (3/4)\left|\widetilde{b_{i-1}}\right|^2.$$

By induction,

$$\left|\widetilde{b_j}\right|^2 \leq (4/3)^{i-j}\left|\widetilde{b_i}\right|^2 \quad \text{for } 1 \leq j \leq i \leq k.$$

Now we obtain

$$\begin{aligned}|b_i|^2 &= \left|\widetilde{b_i}\right|^2 + \sum_{j=1}^{i-1}\mu_{ij}^2\left|\widetilde{b_j}\right|^2 \\ &\leq \left|\widetilde{b_i}\right|^2 + \sum_{j=1}^{i-1}\frac{1}{4}\left(\frac{4}{3}\right)^{i-j}\left|\widetilde{b_i}\right|^2 \\ &= \left(\frac{4}{3}\right)^{i-1}\left|\widetilde{b_i}\right|^2\end{aligned}$$

for $1 \leq i \leq k$. It follows that

$$|b_j|^2 \leq (4/3)^{j-1}\left|\widetilde{b_j}\right|^2 \leq (4/3)^{i-1}\left|\widetilde{b_i}\right|^2$$

for $1 \leq j \leq i \leq k$. This proves the first inequalities of the lemma. For the second part, we multiply the inequalities of the first part for $j \leq i \leq k$ to get

$$\begin{aligned}|b_j| &\leq \left(\frac{4}{3}\right)^{(k+j-2)/4}\left(\prod_{i=j}^{k}\left|\widetilde{b_i}\right|\right)^{\frac{1}{k-j+1}} \\ &= \left(\frac{4}{3}\right)^{(k+j-2)/4}\left(\frac{\prod_{i=1}^{k}\left|\widetilde{b_i}\right|}{\prod_{i=1}^{j-1}\left|\widetilde{b_i}\right|}\right)^{\frac{1}{k-j+1}} \\ &= \left(\frac{4}{3}\right)^{(k+j-2)/4}\left(\frac{\operatorname{vol} M}{\operatorname{vol} M_{j-1}}\right)^{\frac{1}{k-j+1}}.\end{aligned}$$

□

Proposition 2. *Let M be a lattice of dimension k such that* $\operatorname{vol} M \leq V$*. Then there exists a basis $B = (b_1, \ldots, b_k)$ of M such that*

$$|b_j| \leq \left(\frac{4}{3}\right)^{(k+j-2)/4}\left(\frac{V}{\operatorname{vol} M_{j-1}}\right)^{\frac{1}{k-j+1}}. \tag{4}$$

for each $1 \le j \le n$, where M_j is the sublattice of M generated by $b_1, \ldots, b_j$. In this case, the lengths of all vectors of B are bounded above by the constant

$$\max_{1\le j\le k}\left(\frac{4}{3}\right)^{\frac{k^2-k}{4(k-j+1)}}\left(\frac{V}{\lambda_1(M)^{j-1}}\right)^{\frac{1}{k-j+1}}. \qquad (5)$$

Proof. Since Korkine-Zolotareff basis exists for any lattice, the existence of basis satisfying the first inequality follows from Lemma 1. The constant (5) is obtained as follows.

$$\begin{aligned}\operatorname{vol} M_{j-1} &= \prod_{i=1}^{j-1}\left|\widetilde{b_i}\right| \\ &\ge \prod_{i=1}^{j-1}\left(\frac{3}{4}\right)^{(i-1)/2}|b_i| \qquad \text{(by Lemma 1)} \\ &\ge \prod_{i=1}^{j-1}\left(\left(\frac{3}{4}\right)^{(i-1)/2}\lambda_1(M)\right) \\ &= \left(\frac{3}{4}\right)^{(j^2-3j+2)/4}\lambda_1(M)^{j-1}.\end{aligned}$$

This implies

$$\begin{aligned}|b_j| &\le \left(\frac{4}{3}\right)^{\frac{k+j-2}{4}+\frac{j^2-3j+2}{4(k-j+1)}}\left(\frac{V}{\lambda_1(M)^{j-1}}\right)^{\frac{1}{k-j+1}} \qquad \text{(by Lemma 1)} \\ &= \left(\frac{4}{3}\right)^{\frac{k^2-k}{4(k-j+1)}}\left(\frac{V}{\lambda_1(M)^{j-1}}\right)^{\frac{1}{k-j+1}}.\end{aligned}$$

We get the upper bound by taking the maximum of the right hand side for $1 \le j \le k$. □

Semistability of Perfect Lattices

The classification of all perfect lattices of dimension 8 has been completed recently. Laïhem (1992), Baril (1996), Napias (1996), and Batut and Martinet (1996) found 10916 perfect lattices of dimension 8, and Mathieu Dutour Sikirić, Achill Schürmann, and Frank Vallentin proved that there are no more (Sikirić et al., 2007). These lattices can be viewed in Jacques Martinet's homepage `http://www.math.u-bordeaux1.fr/~martinet/` or the web-page `http://www.math.uni-magdeburg.de/lattice_geometry/` that used to be maintained by Achill Schürmann and Frank Vallentin.

The algoritheorem described in the previous section has been implemented by the author to find the canonical filtrations of perfect lattices. The code in GAP computer algebra system (GAP, 2015) can be found at `http://sourceforge.net/projects/cflat/`. The code applied to perfect lattices revealed that all perfect lattices of dimensions from 2 to 7 are semistable and that there exists exactly one perfect lattice of dimension 8 that is unstable. It is listed as 'Form Nr. 68' in the file by Schürmann and Vallentin (2005a). It is also known as the Coxeter-Barnes lattice $\mathbb{A}_8^2$. The following matrix is the Gram matrix of the lattice with respect to the basis

$$\mathcal{B} = (e_1, e_2, e_3, e_4, e_5, e_6, e_7, f)$$

where $\varepsilon_1, \ldots, \varepsilon_8$ are standard basis vectors, $e_i = \varepsilon_i - \varepsilon_0$, and $f = \frac{1}{2}(e_1 + \cdots + e_8)$ (Martinet, 2003, Section 5.1).

$$A_8^2 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 9/2 \\ 1 & 2 & 1 & 1 & 1 & 1 & 1 & 9/2 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 9/2 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 9/2 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 9/2 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 9/2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 9/2 \\ 9/2 & 9/2 & 9/2 & 9/2 & 9/2 & 9/2 & 9/2 & 18 \end{pmatrix}.$$

This lattice is extreme as well as perfect. Its determinant is $9/4$. It has 71 pairs of shortest vectors of norm 2. The unique pair of shortest vectors in $\left(\mathbb{A}_8^2\right)^*$ are

$$\pm(e_1^* + e_2^* + e_3^* + e_4^* + e_5^* + e_6^* + e_7^* + 4f^*)$$

of norm $8/9$, which implies that it has a unique sublattice of codimension 1 of determinant 2. Since $1.10409 \approx 2^{1/7} < (9/4)^{1/8} \approx 1.10668$, this lattice is unstable. The sublattice of $\mathbb{A}_8^2$ of dimension 7 corresponding to the shortest vectors of the dual lattice is spanned by

$$(e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5, e_5 - e_6, e_6 - e_7, \\ f - e_1 - e_2 - e_3 - e_4)$$

and is the exceptional lattice $\mathbb{E}_7 \subset \mathbb{A}_8^2$.

Theorem 3. *1. All perfect lattices in dimension 2–7 are semistable.*

2. Among 10916 perfect lattices of dimension 8, the Coxeter-Barnes lattice $\mathbb{A}_8^2$ is the unique unstable lattice, and $0 \subset \mathbb{E}_7 \subset \mathbb{A}_8^2$ is its canonical filtration. All other 8-dimensional perfect lattices are semistable.

The classification of perfect lattices in dimension 9 is still in progress. There are more than 500000 perfect lattices currently known (Schürmann & Vallentin, 2005b). Unstable lattices among dimension 9 are scarcer. All known 9-dimensional lattices are semistable except the one with the following Gram matrix listed as 'Form Nr. 8' in the file by

Schürmann and Vallentin (2005b).

$$P_9^8 = \begin{pmatrix} 6 & 3 & 2 & 3 & 3 & 3 & 3 & 0 & 3 \\ 3 & 6 & 2 & 3 & 3 & 3 & 3 & 0 & 3 \\ 2 & 2 & 6 & 2 & 2 & 2 & 3 & 0 & 0 \\ 3 & 3 & 2 & 6 & 3 & 3 & 3 & 0 & 3 \\ 3 & 3 & 2 & 3 & 6 & 3 & 3 & 0 & 3 \\ 3 & 3 & 2 & 3 & 3 & 6 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 & 6 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 3 & 3 & 6 & 3 \\ 3 & 3 & 0 & 3 & 3 & 3 & 3 & 3 & 6 \end{pmatrix}.$$

If $b_1, \ldots, b_9$ are the basis vectors of the lattice L giving the above Gram matrix, then the canonical filtration of L is $0 \subset M \subset L$ where M is the 8-dimensional sublattice of L generated by $b_1, b_2, b_4, b_5, b_6, b_7, b_8$, and b_9. The lattice $M/\sqrt{3}$ is even, unimodular, and integral, thus is the exceptional lattice $\mathbb{E}_8$ (Conway & Sloane, 1999, 4.8.1). Therefore, $M = \sqrt{3}\mathbb{E}_8$. The determinant of M and the determinant of L are 6561 and 21870, respectively. The lattice L is unstable as we can see from the inequality $3 = 6561^{1/8} < 21870^{1/9} \approx 3.035$.

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