Alabama Journal of Mathematics
38 (2014)
ISSN 2373-0404

Powers of Complex Tridiagonal Matrices

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In this paper, we derive the general expression of the sth (s in $\mathbb{N}$ if n is odd; s in $\mathbb{Z}$ if n is even) power of some tridiagonal matrices.

Introduction

Rimas (2005a, 2005b, 2006a, 2006b) studied positive integer powers of some tridiagonal matrices. Öteleş and Akbulak (2013, 2014) obtained integer powers and complex factorization formula of a complex tridiagonal matrix for the generalized Fibonacci-Pell numbers and positive integer powers of some complex tridiagonal matrices.

Lemma 1. *(Cahill, D'Erico, & Spence, 2003)*
Let $\{H(n), n = 1, 2, \ldots\}$ be a sequence of tridiagonal matrices of the form

$$H(n) = \begin{bmatrix} h_{1,1} & h_{1,2} & & & \\ h_{2,1} & h_{2,2} & h_{2,3} & & \\ & h_{3,2} & h_{3,3} & \ddots & \\ & & \ddots & \ddots & h_{n-1,n} \\ & & & h_{n,n-1} & h_{n,n} \end{bmatrix}.$$

Then the successive determinants of $H(n)$ are given by the recursive formula

$$\begin{aligned} |H(1)| &= h_{1,1} \\ |H(2)| &= h_{1,1}h_{2,2} - h_{1,2}h_{2,1} \\ |H(n)| &= h_{n,n}\,|H(n-1)| - h_{n-1,n}h_{n,n-1}\,|H(n-2)|. \end{aligned}$$

Let $\{H^{\dagger}(n), n = 1, 2, \ldots\}$ be a sequence of tridiagonal matrices of the form

$$H^{\dagger}(n) = \begin{bmatrix} h_{1,1} & -h_{1,2} & & & \\ -h_{2,1} & h_{2,2} & -h_{2,3} & & \\ & -h_{3,2} & h_{3,3} & \ddots & \\ & & \ddots & \ddots & -h_{n-1,n} \\ & & & -h_{n,n-1} & h_{n,n} \end{bmatrix}.$$

Then

$$\det(H(n)) = \det(H^{\dagger}(n)). \tag{1}$$

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Let

$$A := \begin{bmatrix} a & 2b & & & & 0 \\ -b & a & b & & & \\ & -b & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & & -b & a & 2b \\ 0 & & & & -b & a \end{bmatrix} \tag{2}$$

be tridiagonal matrix with $a, 0 \neq b \in \mathbb{C}$. In this paper, we acquire eigenvalues and eigenvectors of an $n-$square complex tridiagonal matrix in (2).

We derive expression of the rth power ($r \in \mathbb{N}$) of a matrix using the well-known expression $G^r = SJ^rS^{-1}$ (Horn & Johnson, 2012), where J is the Jordan's form of the matrix G and S is the transforming matrix of G. We need the eigenvalues and eigenvectors of the matrix A to calculate transforming matrix.

Let Q be the following n- square tridiagonal matrix

$$Q := \begin{bmatrix} 0 & 2 & & & & \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 2 \\ & & & & -1 & 0 \end{bmatrix}.$$

From Rimas (2006a), the eigenvalues of Q are

$$\mu_k = -2i\cos\left(\frac{(k-1)\pi}{n-1}\right),\ k = 1, 2, \ldots, n.$$

The Chebyshev polynomials of the first kind $T_n(x)$ and second kind $U_n(x)$ are defined by Mason and Handscomb (2003) as

$$T_n(x) = \cos(n \arccos x), \quad -1 \leq x \leq 1 \tag{3}$$

and

$$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}, \quad -1 \leq x \leq 1. \tag{4}$$

All roots of the polynomial $U_n(x)$ are included in the interval $[-1, 1]$ and can be found using the relation

$$x_{nk} = \cos\left(\frac{k\pi}{n+1}\right), \quad k = 1, 2, \ldots, n.$$

Eigenvalues and Eigenvectors of A

Theorem 1. *Let A be as in (2). Then the eigenvalues and eigenvectors of the matrix A are*

$$\lambda_k = a - 2bi\cos\left(\frac{(k-1)\pi}{n-1}\right); k = 1, 2, \ldots, n \tag{5}$$

and for $j = 1, 2, \ldots, n$

$$x_{kj} = \begin{cases} e(k-1)\, T_{k-1}\left(\frac{i\delta_j}{2}\right) & k = 1, 2, \ldots, n-1 \\ \frac{e(k-1)}{2} T_{k-1}\left(\frac{i\delta_j}{2}\right) & k = n \end{cases} \tag{6}$$

here $\delta_j = \frac{\lambda_j - a}{b}, e(k) = e^{-i\frac{k\pi}{2}}$ *and* $T_k(x)$ *is Chebyshev polynomial of the first kind.*

Proof. Let

$$C := \begin{bmatrix} \frac{a}{b} & 2 & & & & 0 \\ -1 & \frac{a}{b} & 1 & & & \\ & -1 & \ddots & \ddots & & \\ & & \ddots & \frac{a}{b} & 1 & \\ & & & -1 & \frac{a}{b} & 2 \\ 0 & & & & -1 & \frac{a}{b} \end{bmatrix}.$$

The characteristic polynomial of C is

$$D_n(\varepsilon) = (\varepsilon^2 + 4)\Delta_{n-2}(\varepsilon) \tag{7}$$

where $\varepsilon = \lambda - \frac{a}{b}$ and

$$\Delta_n(\varepsilon) = \varepsilon\Delta_{n-1}(\varepsilon) + \Delta_{n-2}(\varepsilon) \tag{8}$$

with initial conditions $\Delta_0(\varepsilon) = 1, \Delta_1(\varepsilon) = \varepsilon, \Delta_2(\varepsilon) = \varepsilon^2 + 1$.

The solution of difference equation (8) is

$$\Delta_n(\varepsilon) = (-i)^n\, U_n\left(\frac{i\varepsilon}{2}\right)$$

(Rimas, 2006a). So the equation in (7) can be written as

$$D_n(\varepsilon) = (\varepsilon^2 + 4)(-i)^n\, U_{n-2}\left(\frac{i\varepsilon}{2}\right).$$

The eigenvalues of the matrix C are

$$\varepsilon_k = -2i\cos\left(\frac{(k-1)\pi}{n-1}\right), \quad k = 1, 2, \ldots, n$$

and so we get the eigenvalues of the matrix A:

$$\lambda_k = a - 2bi\cos\left(\frac{(k-1)\pi}{n-1}\right), \text{ for } k = 1, 2, \ldots, n.$$

The eigenvectors of A is the solution of the following linear homogeneous equations system

$$(\lambda_k I_n - A)x = 0$$

where λ_k is the k–th eigenvalue of the matrix A ($k = 1, 2, \ldots, n$), i.e.,

$$\left.\begin{aligned} (\lambda_k - a)x_1 - 2bx_2 &= 0 \\ bx_1 + (\lambda_k - a)x_2 - bx_3 &= 0 \\ bx_2 + (\lambda_k - a)x_3 - bx_4 &= 0 \\ &\vdots \\ bx_{n-2} + (\lambda_k - a)x_{n-1} - 2bx_n &= 0 \\ bx_{n-1} + (\lambda_k - a)x_n &= 0. \end{aligned}\right\} \tag{9}$$

Divide each terms of all the equations in system (9) by $b \neq 0$, and set $\delta_j = \frac{\lambda_j - a}{b}$ $(j = 1, 2, \ldots, n)$. Since rank of system is $n-1$, choosing $x_1 = 1$ and solving the set of the system x_1, we find the jth element of kth eigenvectors of the matrix A for $1 \leq j, k \leq n$. Then the solution of the system (9) is

$$x_{kj} = \begin{cases} e(k-1)\, T_{k-1}\left(\frac{i\delta_j}{2}\right) & k = 1, 2, \ldots, n-1 \\ \frac{e(k-1)}{2} T_{k-1}\left(\frac{i\delta_j}{2}\right) & k = n \end{cases}$$

for $j = 1, 2, \ldots, n$ where $e(k) = e^{-i\frac{k\pi}{2}}$ and $T_k(x)$ is Chebyshev polynomial of the first kind. □

The Integer Powers of the Matrix A

Let $J = P^{-1}AP$, where

$$J = \text{diag}(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n)$$

is the Jordan decomposition of A, λ_k ($k = 1, 2, \ldots, n$; $n \in \mathbb{N}$) are the eigenvalues of A and P is transforming matrix. Since all the eigenvalues of A are simple, columns of the transforming matrix P are the eigenvectors of the matrix A (Horn & Johnson, 2012). From (6), we can write the matrix P as

$$P = [x_{kj}] = \begin{cases} e(k-1)\, T_{k-1}\left(\frac{i\delta_j}{2}\right) & k = 1, 2, \ldots, n-1 \\ \frac{e(k-1)}{2} T_{k-1}\left(\frac{i\delta_j}{2}\right) & k = n \end{cases} \tag{10}$$

for $j = 1, 2, \ldots, n$ where $\delta_j = \frac{\lambda_j - a}{b}$ and $e(k) = e^{-i\frac{k\pi}{2}}$. Considering (10), we write down the matrix P

$$\begin{bmatrix} e(0)\, T_0\left(\frac{i\delta_1}{2}\right) & \cdots & e(0)\, T_0\left(\frac{i\delta_n}{2}\right) \\ \vdots & \ddots & \vdots \\ e(n-2)\, T_{n-2}\left(\frac{i\delta_1}{2}\right) & \cdots & e(n-2)\, T_{n-2}\left(\frac{i\delta_n}{2}\right) \\ \frac{e(n-1)}{2} T_{n-1}\left(\frac{i\delta_1}{2}\right) & \cdots & \frac{e(n-1)}{2} T_{n-1}\left(\frac{i\delta_n}{2}\right) \end{bmatrix}.$$

Denoting jth column of the matrix P^{-1} by υ_j and implementing the essential transformations, we obtain

$$\upsilon_j = \varsigma_j \begin{bmatrix} f_1\overline{e}\,(j-1)\,T_{j-1}\left(\frac{i\delta_1}{2}\right) \\ f_2\overline{e}\,(j-1)\,T_{j-1}\left(\frac{i\delta_2}{2}\right) \\ f_3\overline{e}\,(j-1)\,T_{j-1}\left(\frac{i\delta_3}{2}\right) \\ \vdots \\ f_{n-1}\overline{e}\,(j-1)\,T_{j-1}\left(\frac{i\delta_{n-1}}{2}\right) \\ f_n\overline{e}\,(j-1)\,T_{j-1}\left(\frac{i\delta_n}{2}\right) \end{bmatrix}, \; j = 1,\ldots,n$$

where

$$\varsigma_j = \begin{cases} 1, & j = 1 \\ 2, & 1 < j \le n \end{cases},$$

$$\beta_k = \begin{cases} 1, & k = 1,n \\ 2, & 1 < k < n \end{cases},$$

and $f_k = \frac{\beta_k}{2n-2}$. Hence we have P^{-1} equals

$$\frac{1}{2n-2}\begin{bmatrix} \overline{e}\,(0)\,T_0\left(\frac{i\delta_1}{2}\right) & \cdots & 2\overline{e}\,(n-1)\,T_{n-1}\left(\frac{i\delta_1}{2}\right) \\ \vdots & \ddots & \vdots \\ 2\overline{e}\,(0)\,T_0\left(\frac{i\delta_{n-1}}{2}\right) & \cdots & 4\overline{e}\,(n-1)\,T_{n-1}\left(\frac{i\delta_{n-1}}{2}\right) \\ \overline{e}\,(0)\,T_0\left(\frac{i\delta_n}{2}\right) & \cdots & 2\overline{e}\,(n-1)\,T_{n-1}\left(\frac{i\delta_n}{2}\right) \end{bmatrix}.$$

Let

$$(A)^s = PJ^sP^{-1} = V(s) = (v_{mj}(s))$$

where

$$s = \begin{cases} s \in \mathbb{N}, & n \text{ odd} \\ s \in \mathbb{Z}, & n \text{ even} \end{cases}$$

So

$$v_{mj}(s) = \varsigma_j \sum_{k=1}^{n} \lambda_k^s f_k e(m-1)\overline{e}\,(j-1)\,T_{m-1}\left(\frac{i\delta_k}{2}\right)T_{j-1}\left(\frac{i\delta_k}{2}\right)$$

where $m = 1,2,\ldots,n-1$; $j = 1,2,\ldots,n$ and

$$v_{mj}(s) = \frac{\varsigma_j}{2} \sum_{k=1}^{n} \lambda_k^s f_k e(m-1)\overline{e}\,(j-1)\,T_{m-1}\left(\frac{i\delta_k}{2}\right)T_{j-1}\left(\frac{i\delta_k}{2}\right)$$

$m = n$ and $j = 1,2,\ldots,n$.

Numerical examples

Example 1. *Let $n = 4$, $s = 5$, $a = 2$, and $b = 3$. We obtain*

$$\begin{aligned} J &= \operatorname{diag}(\lambda_1,\lambda_2,\lambda_3,\lambda_4) \\ &= \operatorname{diag}(a-2bi, a-bi, a+bi, a+2bi) \\ &= \operatorname{diag}(2-6i, 2-3i, 2+3i, 2+6i). \end{aligned}$$

Then

$$(A)^5 = v_{mj}(5) = \begin{bmatrix} 3452 & -654 & -6660 & -540 \\ 327 & 6782 & -57 & -6660 \\ -3330 & 57 & 6782 & -654 \\ 135 & -3330 & 327 & 3452 \end{bmatrix}.$$

Example 2. *Let $n = 5$, $s = 3$, $a = 1-i$, and $b = 3+2i$. Then*

$$\begin{aligned} J &= \operatorname{diag}(\lambda_1,\lambda_2,\lambda_3,\lambda_4) \\ &= \operatorname{diag}(a-2bi, a-\sqrt{2}bi, a, a+\sqrt{2}bi, a+2bi) \\ &= \operatorname{diag}(5-7i, 1-i+(2-3i)\sqrt{2}, 1-i, 1-i-(2-3i)\sqrt{2}, -3+5i). \end{aligned}$$

So $(A)^3 = v_{mj}(3)$ equals

$$\begin{bmatrix} -104-44i & 78-312i & 102+42i & -18+92i & 0 \\ -39+156i & -155-65i & 48-202i & 51+21i & -18+92i \\ 51+21i & -48+202i & -104-44i & 48-202i & 102+42i \\ 9-46i & 51+21i & -48+202i & -155-65i & 78-312i \\ 0 & 9-46i & 51+21i & -39+156i & -104-44i \end{bmatrix}.$$

Example 3. *Let $n = 4$, $s = -3$, $a = 2$, and $b = 1$. Then*

$$\begin{aligned} J &= \operatorname{diag}(\lambda_1,\lambda_2,\lambda_3,\lambda_4) \\ &= \operatorname{diag}(a-2bi, a-bi, a+bi, a+2bi) \\ &= \operatorname{diag}(2-2i, 2-i, 2+i, 2+2i). \end{aligned}$$

Thus

$$(A)^{-3} = v_{mj}(-3) = \frac{1}{8000}\begin{bmatrix} 2 & -636 & 252 & -772 \\ 318 & -124 & 68 & 252 \\ 126 & -68 & -124 & -636 \\ 193 & 126 & 318 & 2 \end{bmatrix}.$$

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