

On the Construction of Jacobi Matrices from Mixed Data

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We show that a unique Jacobi matrix can be reconstructed from two types of mixed data: 1. its two eigenpairs, 2. its eigenvalues and the eigenvalues of its submatrix obtained by removing the first two rows and columns and its 1-1 entry. For the second case, the condition of existence is provided. In addition, we summarize the equivalent sets of parameters used to recover Jacobi matrices and show some direct connections.

Introduction

An $n \times n$ matrix J is called a Jacobi matrix if it has the form

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdot & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{n-2} & b_{n-1} & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot & 0 & a_{n-1} & b_n \end{pmatrix}, \quad (1)$$

where all a_i, b_i are real. It is called irreducible if $a_i \neq 0$ for $i = 1, \dots, n - 1$. When J is irreducible, its eigenvalues have multiplicity 1.

Jacobi matrices have a wide range of applications in physics and engineering, and are closely and non-trivially linked with many other mathematical objects, such as orthogonal polynomials, one dimensional Schrödinger operators and the Sturm-Liouville problem. In the past couple of decades, constructing Jacobi matrices from different types of data was studied intensively. Hochstadt (1974) proved that a unique Jacobi matrix J with all off diagonal entries positive can be recovered from the eigenvalues of J and the eigenvalues of the submatrix of J obtained by removing the first row and column. He later (Hochstadt, 1979) showed that J can be uniquely determined by its eigenvalues and by its $n - 1$ entries which are those on the left top corner. There have also been other methods of the reconstruction of Jacobi matrices from different types of data widely developed (see (Biegler-König, 1981; Boley & Golub, 1987; Ghanbari, Parvizpour, & Mirzaei, 2014; Gray & Wilson, 1976)). We construct Jacobi matrices from two new types of data.

This paper is organized as follows. Firstly, we show that a Jacobi matrix can be reconstructed from two eigenpairs. We also provide a condition on the uniqueness of the solution.

Secondly, we show that a unique Jacobi matrix J is determined by the eigenvalues of J , the eigenvalues of the submatrix obtained by removing the first two rows and columns from J and the (1,1) entry of J . At the end, we summarize the equivalent sets of parameters used to recover Jacobi matrices and show some direct connections among these sets.

We let $J_{(k,k)}$ denote the submatrix of J as in (1) obtained by removing the first k rows and columns. Denote the spectrum of the irreducible Jacobi matrix J by $\sigma(J) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ with $\lambda_1 > \lambda_2 > \dots > \lambda_n$.

Main Result

Theorem 1. Let J be an irreducible Jacobi matrix as in (1). Given two real numbers λ, μ and two distinct real vectors $u = (u_1, u_2, \dots, u_n)^T, v = (v_1, v_2, \dots, v_n)^T$, if $\begin{pmatrix} u_k & u_{k+1} \\ v_k & v_{k+1} \end{pmatrix}$ is invertible for all $k = 1, \dots, n - 1$, and

$$a_{n-1} = \frac{u_n v_n (\lambda - \mu)}{v_n u_{n-1} - u_n v_{n-1}},$$

then a unique Jacobi matrix J can be constructed such that (λ, u) and (μ, v) are two of its eigenpairs.

Lemma 1. Let J be a Jacobi matrix as in (1). Let λ be an eigenvalue of J and $u = (u_1, \dots, u_n)^T$ be the corresponding eigenvector. Then

$$a_{k-1} u_{k-1} + b_k u_k + a_k u_{k+1} = \lambda u_k, \quad (2)$$

where $k = 1, 2, \dots, n$, and $p_{n+1} = a_0 = a_n = 0$.

Proof. It is due to $Ju = \lambda u$. □

Lemma 2. Let J be an irreducible Jacobi matrix and $u = (u_1, \dots, u_n)^T$ be an arbitrary eigenvector of J . Then $u_1 \neq 0$.

Proof. Assume that $u_1 = 0$. Let $k = 1$ be in (2). Then

$$b_1 u_1 + a_1 u_2 = \lambda u_1$$

is equivalent to

$$a_1 u_2 = 0.$$

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Since J is irreducible, $a_1 \neq 0$; hence $u_2 = 0$. Let $k = 2$ be in (2). Then

$$a_1 u_1 + b_2 u_2 + a_2 u_3 = \lambda u_2$$

is equivalent to

$$a_2 u_3 = 0$$

due to $u_1 = 0, u_2 = 0$. Then $u_3 = 0$ since $a_i \neq 0$. Similarly, all u_i 's vanish. It is a contradiction. \square

Proof of Theorem 1. Due to Lemma 2, we can assume that $u = (1, u_2, u_3, \dots, u_n)^T$ and $v = (1, v_2, v_3, \dots, v_n)^T$ and $u_2 \neq v_2$ because $\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$ is invertible. Then apply Lemma 1. We have a linear system with $2n - 1$ unknowns and $2n$ equations.

$$\begin{cases} b_1 + a_1 u_2 = \lambda & (3.1) \\ b_1 + a_1 v_2 = \mu & (3.2) \\ a_1 + b_2 u_2 + a_2 u_3 = \lambda u_2 & (3.3) \\ a_1 + b_2 v_2 + a_2 v_3 = \mu v_2 & (3.4) \\ \vdots & \vdots \\ a_{n-2} u_{n-2} + b_{n-1} u_{n-1} + a_{n-1} u_n = \lambda u_{n-1} & (3.(2n-3)) \\ a_{n-2} v_{n-2} + b_{n-1} v_{n-1} + a_{n-1} v_n = \mu v_{n-1} & (3.(2n-2)) \\ a_{n-1} u_{n-1} + b_n u_n = \lambda u_n & (3.(2n-1)) \\ a_{n-1} v_{n-1} + b_n v_n = \mu v_n & (3.2n) \end{cases}$$

The first two equations (3.1) and (3.2) form a sub-system that only contains a_1 and b_1 :

$$\begin{cases} b_1 + a_1 u_2 = \lambda & (3.1) \\ b_1 + a_1 v_2 = \mu & (3.2) \end{cases}$$

Since $\lambda \neq \mu$ and $u_2 \neq v_2$, b_1 and a_1 can be solved uniquely. Then b_2 and a_2 can be uniquely solved from (3.3) and (3.4) if and only if $\begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix}$ is invertible. In general, once we have a_{k-1} , the sub-system

$$\begin{cases} b_k u_k + a_k u_{k+1} = \lambda u_k - a_{k-1} u_{k-1} & (4.1) \\ b_k v_k + a_k v_{k+1} = \mu v_k - a_{k-1} v_{k-1} & (4.2) \end{cases}$$

only contains the unknown a_k . So b_k can be solved uniquely if and only if $\begin{pmatrix} u_k & u_{k+1} \\ v_k & v_{k+1} \end{pmatrix}$ is invertible for $k = 1, 2, \dots, n-1$.

Once we solve a_{n-1} , the last two equations

$$\begin{cases} b_n u_n = \lambda u_n - a_{n-1} u_{n-1} & (4.11) \\ b_n v_n = \mu v_n - a_{n-1} v_{n-1} & (4.21) \end{cases}$$

must be linearly dependent, which is equivalent to

$$\frac{u_n}{v_n} = \frac{\lambda u_n - a_{n-1} u_{n-1}}{\mu v_n - a_{n-1} v_{n-1}},$$

i.e.

$$a_{n-1} = \frac{u_n v_n (\lambda - \mu)}{v_n u_{n-1} - u_n v_{n-1}}. \quad (5)$$

The denominator of (5) can be written as $\det \begin{pmatrix} u_{n-1} & u_n \\ v_{n-1} & v_n \end{pmatrix}$ due to the assumption. \square

Theorem 2. Let $\{\lambda_i\}_{i=1}^n$ be the eigenvalues of J and $\{v_i\}_{i=1}^{n-2}$ be the eigenvalues of $J_{(2,2)}$ where $\lambda_1 > \dots > \lambda_n$ and $v_1 > \dots > v_{n-2}$. If

$$\frac{\prod_{i=1}^n (b_1 - \lambda_i)}{\prod_{j=1}^{n-2} (b_1 - v_j)} < 0,$$

then a unique irreducible Jacobi matrix J with 1-1 entry b_1 and positive off diagonal entries can be constructed such that $\sigma(J) = \{\lambda_i\}_{i=1}^n$, $\sigma(J_{(2,2)}) = \{v_i\}_{i=1}^{n-2}$

Lemma 3. Let

$$f(x) = (x - \lambda_1)(x - \lambda_2) \dots (x - \lambda_n),$$

where $\lambda_1 > \lambda_2 > \dots > \lambda_n$ and

$$g(x) = (x - \mu_1)(x - \mu_2) \dots (x - \mu_{n-1}),$$

where $\mu_1 > \mu_2 > \dots > \mu_{n-1}$. Then

$$\lambda_1 > \mu_1 > \lambda_2 > \dots > \mu_{n-1} > \lambda_n \quad (6)$$

if and only if there exist unique positive numbers c_1, c_2, \dots, c_n with $\sum_{i=1}^n c_i = 1$ such that

$$g(x) = c_1 \frac{f(x)}{x - \lambda_1} + c_2 \frac{f(x)}{x - \lambda_2} + \dots + c_n \frac{f(x)}{x - \lambda_n}. \quad (7)$$

Proof. Necessary condition: By plugging λ_i into (7), we have

$$c_i = \frac{g(\lambda_i)}{\prod_{j=1, j \neq i}^n (\lambda_i - \lambda_j)} = \frac{g(\lambda_i)}{f'(\lambda_i)},$$

for $i = 1, 2, \dots, n$. The positivity of c_i is due to the interlacing relation (6). $\sum_{i=1}^n c_i = 1$ because both of $f(x)$ and $g(x)$ are monic.

Sufficient condition: Suppose that $g(x)$ has a representation as in (7). Because all c_i 's are positive, the signs of $g(\lambda_i)$'s are alternating and hence the roots of $g(x)$ interlace roots of $f(x)$. \square

Proof of Theorem 2. Let $P_J(\lambda)$, $P_{J_{(1,1)}}(\lambda)$ and $P_{J_{(2,2)}}(\lambda)$ be the characteristic polynomial of J , $J_{(1,1)}$ and $J_{(2,2)}$ respectively. Then

$$P_J(\lambda) = |\lambda I - J| = \prod_{i=1}^n (\lambda - \lambda_i) \quad (8)$$

and

$$P_{J_{(2,2)}}(\lambda) = |\lambda I - J_{(2,2)}| = \prod_{i=1}^{n-2} (\lambda - v_i).$$

Meanwhile

$$P_J(\lambda) = \begin{vmatrix} \lambda - b_1 & -a_1 & & & & & \\ -a_1 & \lambda - b_2 & -a_2 & & & & \\ & -a_2 & \lambda - b_3 & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & \\ & & & & & \lambda - b_{n-1} & -a_{n-1} \\ & & & & & -a_{n-1} & \lambda - b_n \end{vmatrix} \quad (9)$$

Expanding (9) by the first row, we have

$$P_J(\lambda) = (\lambda - b_1)P_{J_{(1,1)}}(\lambda) - a_1^2 P_{J_{(2,2)}}(\lambda). \quad (10)$$

Letting $\lambda = b_1$, (10) becomes

$$a_1^2 = -\frac{P_J(b_1)}{P_{J_{(2,2)}}(b_1)} = -\frac{\prod_{i=1}^n (b_1 - \lambda_i)}{\prod_{j=1}^{n-2} (b_1 - \nu_j)}.$$

So a positive a_1 can be uniquely determined if $\frac{\prod_{i=1}^n (b_1 - \lambda_i)}{\prod_{j=1}^{n-2} (b_1 - \nu_j)} < 0$.

According to Lemma 3, we have

$$P_{J_{(1,1)}}(\lambda) = c_1 \frac{P_J(\lambda)}{\lambda - \lambda_1} + c_2 \frac{P_J(\lambda)}{\lambda - \lambda_2} + \dots + c_n \frac{P_J(\lambda)}{\lambda - \lambda_n}. \quad (11)$$

Let λ be any eigenvalue λ_i for $i = 1, 2, \dots, n$ in (10) and (11). We have

$$0 = P_J(\lambda_i) = (\lambda_i - b_1)P_{J_{(1,1)}}(\lambda_i) - a_1^2 P_{J_{(2,2)}}(\lambda_i) \quad (12)$$

and

$$P_{J_{(1,1)}}(\lambda_i) = c_i \prod_{j=1, j \neq i}^n (\lambda_i - \lambda_j) = c_i P'_J(\lambda_i). \quad (13)$$

Combining (12) to (13), we have

$$0 = (\lambda_i - b_1)c_i P'_J(\lambda_i) - a_1^2 \prod_{k=1}^{n-2} (\lambda_i - \nu_k),$$

and hence

$$c_i = \frac{a_1^2 \prod_{k=1}^{n-2} (\lambda_i - \nu_k)}{(\lambda_i - b_1) P'_J(\lambda_i)}$$

for $i = 1, 2, \dots, n$.

According to (11),

$$P_{J_{(1,1)}}(\lambda) = \sum_{i=1}^n \frac{a_1^2 \prod_{k=1}^{n-2} (\lambda_i - \nu_k)}{(\lambda_i - b_1) P'_J(\lambda_i)} \frac{P_J(\lambda)}{\lambda - \lambda_i}.$$

Now we have solved the characteristic polynomial of $J_{(1,1)}$, so Hochstadt's method (1974) can be applied to finish the construction of J from $\sigma(J)$ and $\sigma(J_{(1,1)})$. \square

Theorem 3. Let J be an irreducible $n \times n$ Jacobi matrix as in (1). Let

- $\{\lambda_i\}_{i=1}^n, \{\mu_i\}_{i=1}^{n-1}$ and $\{\nu_i\}_{i=1}^{n-2}$ be eigenvalues of $J, J_{(1,1)}$ and $J_{(2,2)}$ respectively;
- $\{w_i\}_{i=1}^n$ be the set of orthonormal eigenvectors associated with $\{\lambda_i\}_{i=1}^n$.

Then the following sets of parameters are equivalent:

- (1) $\{b_i\}_{i=1}^n \cup \{a_i\}_{i=1}^{n-1}$, where $(a_i > 0)$ for $i = 1, \dots, n$.
- (2) $\{\lambda_i\}_{i=1}^n \cup \{\mu_i\}_{i=1}^{n-1}$ satisfying $\lambda_1 > \mu_1 > \dots > \mu_{n-1} > \lambda_n$.
- (3) $\{\lambda_i\}_{i=1}^n \cup \{w_i^j\}_{i=1}^n$ with $w_i^j > 0$ for $i = 1, \dots, n$.
- (4) $\{\lambda_i\}_{i=1}^n \cup \{a_i\}_{i=1}^{\lfloor n/2-1 \rfloor} \cup \{b_i\}_{i=1}^{\lfloor n/2 \rfloor}$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $a_i > 0$ for $i = 1, \dots, \lfloor n/2 - 1 \rfloor$.
- (5) $\{\lambda_i\}_{i=1}^n \cup \{\nu_i\}_{i=1}^{n-2} \cup b_1$ with $\frac{\prod_{i=1}^n (b_1 - \lambda_i)}{\prod_{j=1}^{n-2} (b_1 - \nu_j)} < 0$.

Proof. It is clear that (2),(3),(4) and (5) are determined by (1).

"(2) \Rightarrow (1)" is due to Hochstadt (1974) and Gray and Wilson (1976).

"(3) \Rightarrow (1)" is due to Gesztesy and Simon (1997).

"(4) \Rightarrow (1)" is due to Hochstadt (1979).

"(5) \Rightarrow (1)" is due to Theorem 2.

In addition, there exist direct connections among these sets.

"(2) \Rightarrow (3)" Let $J = U^* \Lambda U$, where Λ is the diagonal matrix and U is a unitary matrix consisting of orthonormal eigenvectors, namely $U = (w^1, w^2, \dots, w^n)$. For any λ which is not an eigenvalue of J , we have

$$(\lambda I - J)^{-1} = U^*(\lambda I - \Lambda)^{-1} U.$$

On one hand,

$$\begin{aligned} & ((\lambda I - J)^{-1} \delta_1, \delta_1) \\ &= ((\lambda I - \Lambda)^{-1} U \delta_1, U \delta_1) \\ &= \left(\left(\frac{w_1^1}{\lambda - \lambda_1}, \frac{w_1^2}{\lambda - \lambda_2}, \dots, \frac{w_1^n}{\lambda - \lambda_n} \right), (w_1^1, \dots, w_1^n) \right) \\ &= \sum_{i=1}^n \frac{|w_1^i|^2}{\lambda - \lambda_i}, \end{aligned} \quad (14)$$

where $\delta_1 = (1, 0, \dots, 0)^T$. On the other hand,

$$\begin{aligned} ((\lambda I - J)^{-1} \delta_1, \delta_1) &= \frac{\det(\lambda I - J_{(1,1)})}{\det(\lambda I - J)} \\ &= \frac{\prod_{i=1}^{n-1} (\lambda - \mu_i)}{\prod_{i=1}^n (\lambda - \lambda_i)} \\ &= \sum_{i=1}^n \frac{P_{J_{(1,1)}}(\lambda_i)}{P'_J(\lambda_i)} \frac{1}{\lambda - \lambda_i}. \end{aligned} \quad (15)$$

Compare (14) and (15), $|w_1^i|^2 = \frac{P_{J_{(1,1)}}(\lambda_i)}{P'_J(\lambda_i)}$ for $i = 1, 2, \dots, n$.

"(2) \Rightarrow (5)" See the proof of Theorem 2. \square

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