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A Note On k-Tridiagonal k-Toeplitz Matrices

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In this note, we give formulas for determinants, permanents, and eigenvalues of k-tridiagonal k-Toeplitz matrices.

Introduction

The *determinant* of an $n \times n$ matrix $A = (a_{ij})$ is defined by

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} ,$$

where S_n represents the symmetric group of degree n. Analogously, the permanent of A is

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} .$$

In literature, there are many methods for computing determinants. But less is known for permanent computation.

In matrix theory, a *permutation matrix* is a square binary matrix that has exactly one entry 1 in each row and each column and 0s elsewhere (Zhang, 1999). Let P be a permutation matrix. Then, P^T is also a permutation matrix. Furthermore, $P^T = P^{-1}$.

A matrix $A = [a_{i,j}] \in M_{n+1}$ of the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_n \\ a_{-1} & a_0 & a_1 & \ddots & & \vdots \\ a_{-2} & a_{-1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_1 & a_2 \\ \vdots & & \ddots & a_{-1} & a_0 & a_1 \\ a_{-n} & \cdots & \cdots & a_{-2} & a_{-1} & a_0 \end{pmatrix}$$

is called a *Toeplitz matrix* (Horn & Johnson, 1985). The general term is $a_{i,j} = a_{j-i}$ for some given sequences $a_{-n}, a_{-n+1}, ..., a_{-1}, a_0, a_1, a_2, ..., a_{n-1}, a_n \in \mathbb{C}$. The entries of A are constant down the diagonals parallel to the main diagonal.

Eigenvalues of a matrix are a fundemental tool in mathematics and have many applications, such as linear equation systems, determinants, ordinary differential equations, partial differential equations and so on. Let A be an $n-$square matrix. The characteristic polynomial of A is

$$\det(\lambda I_n - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

and roots of the polynomial are called *eigenvalues* of A.

Recently, there has been a number of papers on k-tridiagonal matrices and their applications. For example, Asci, Tasci, and El-Mikkawy (2012) gave algorithms for determinants and permanents of k-tridiagonal matrices using LU factorization. Kilic and Tasci (2007) obtained some identities for relationship between some famous number sequences and permanents of some tridiagonal matrices. Yalciner (2011) gave LU factorizations for k-tridiagonal matrices. Then, Yalçıner obtained determinants of k-tridiagonal matrices by using LU factorization. Moreover, Yalciner found eigenvalues of k-tridiagonal matrices by Chebyshev polinomials.

Brualdi and Gibson (1977) showed that

$$\operatorname{per}(P^T AP) = \operatorname{per}(P^{-1}AP) = \operatorname{per}(A), \tag{1}$$

where P is a permutation matrix. Brualdi and Ryser (1991) showed that for a block matrix

$$A = \begin{pmatrix} A_1 & 0 \\ A_3 & A_2 \end{pmatrix},$$

$$\operatorname{per}(A) = \operatorname{per}(A_1)\operatorname{per}(A_2). \tag{2}$$

Sogabe and El-Mikkawy (2011) obtained a fast block diagonalization of k-tridiagonal matrices using permutation matrices. In other words, they considered an n-square k-

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tridiagonal matrix $T_n^{(k)}$,

$$T_n^{(k)} = \begin{pmatrix} a_1 & 0 & \dots & 0 & b_1 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \vdots & & b_2 & \ddots & \vdots \\ \vdots & & \ddots & 0 & & & \ddots & 0 \\ 0 & & & a_{n-k} & & & & b_{n-k} \\ c_{k+1} & & & & \ddots & & & 0 \\ 0 & c_{k+2} & & & & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & & & a_{n-1} & 0 \\ 0 & \dots & 0 & c_n & 0 & \dots & 0 & a_n \end{pmatrix}$$

Sogabe and El-Mikkawy (2011) obtained a permutation matrix P as following:

$$P = \left[P_{\overline{0}}, P_{\overline{1}}, \dots, P_{\overline{k-1}}\right]$$

where

$$\overline{r} = \{i : i \equiv r \,(\mathrm{mod}\, k), i = 1, 2, ..., n\}, \quad r \in \{0, 1, 2, ..., k-1\}$$

and $P_{\overline{r}}$ is $n \times |\overline{r}|$ matrix such that each column is the ith unit vector e_i, where $i \in \overline{r}$ and $|\overline{r}|$ denotes number of elements of $\overline{r}$. So, by matrix multiplication

$$P^T T_n^{(k)} P = T_0 \oplus T_1 \oplus \dots \oplus T_{k-1}$$

where $\oplus$ denotes the direct sum of matrices and T_i's are $\left|\overline{i}\right|$-*square* tridiagonal matrices.

In this paper, we consider k-tridiagonal k-Toeplitz matrices of the form

$$T_{n(k)}^{(k)} = \begin{pmatrix} a_1 & 0 & \cdots & 0 & b_1 & 0 & & & & 0 \\ 0 & a_2 & & & & b_2 & & & & \\ \vdots & & \ddots & & & & \ddots & & & \\ 0 & & & a_k & & & & b_k & & \\ c_1 & & & & a_1 & & & & b_1 & 0 \\ 0 & c_2 & & & & a_2 & & & & \ddots \\ & & \ddots & & & & \ddots & & & \\ & & & c_k & & & & a_k & & \\ & & & & c_1 & & & & a_1 & \\ 0 & & & & 0 & \ddots & 0 & \cdots & 0 & \ddots \end{pmatrix} \qquad (3)$$

and we will obtain eigenvalues, determinants, and permanents of the matrix family.

Main results

Diagonalization of k-tridiagonal k-Toeplitz matrices

Using the similar method of used by Sogabe and El-Mikkawy (2011), one can transform k-tridiagonal k-Toeplitz matrices to the following form

$$P^T T_{n(k)}^{(k)} P = \begin{pmatrix} a_k & b_k & 0 & & & & & & & & & & 0 \\ c_k & a_k & \ddots & & & & & & & & & & \\ 0 & \ddots & \ddots & b_k & & & & & & & & & \\ & & c_k & a_k & 0 & & & & & & & & \\ & & & 0 & a_1 & b_1 & & & & & & & \\ & & & & c_1 & a_1 & \ddots & & & & & & \\ & & & & & \ddots & \ddots & b_1 & & & & & \\ & & & & & & c_1 & a_1 & & & & & \\ & & & & & & & & \ddots & & & & \\ & & & & & & & & & a_{k-1} & b_{k-1} & & \\ & & & & & & & & & c_{k-1} & a_{k-1} & \ddots & 0 \\ & & & & & & & & & & \ddots & \ddots & b_{k-1} \\ 0 & & & & & & & & & & 0 & c_{k-1} & a_{k-1} \end{pmatrix}$$

where T_i's are $\left|\overline{i}\right|$-*square* tridiagonal Toeplitz matrices.

Determinants

Zhang (1999) considered tridiagonal Toeplitz matrices of the following form

$$T_n = \begin{pmatrix} a & b & 0 & \cdots & 0 \\ c & a & b & & \vdots \\ 0 & c & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & b \\ 0 & \cdots & 0 & c & a \end{pmatrix}_{n \times n} . \qquad (4)$$

By Laplace expansion, $\det(T_n) = a \det(T_{n-1}) - bc \det(T_{n-2})$. In other words,

$$det(T_n) = v_n$$

where $v_n = av_{n-1} - bcv_{n-2}$ with initial conditions $v_{-1} = 0, v_0 = 1, v_1 = a$. Then, we have the following theorem.

Theorem 1.

$$\det T_{n(k)}^{(k)} = \prod_{i=0}^{k-1} v_{|\overline{i}|},$$

where $v_n = a_i v_{n-1} - b_i c_i v_{n-2}$ *with initial conditions* $v_{-1} = 0, v_0 = 1, v_1 = a$.

Proof. It is clear that

$$\det T_{n(k)}^{(k)} = \det(T_0) \det(T_1) \dots \det(T_{k-1}).$$

Since $\det(T_i) = v_{|\overline{i}|}$,

$$\det T_{n(k)}^{(k)} = \det(T_0) \det(T_1) \dots \det(T_{k-1}) = \prod_{i=0}^{k-1} v_{|\overline{i}|}.$$

□

Eigenvalues

Kulkarni, Schimdt, and Tsui (1999) considered tridiagonal Toeplitz matrices of the form (4), then obtained the eigenvalues as

$$\lambda_k = a - 2\sqrt{bc}\cos\left(\frac{k\pi}{n+1}\right) \quad k = 1, 2, \ldots, n.$$

Theorem 2. *The eigenvalues of $T^{(k)}_{n(k)}$ are*

$$\lambda_j = a_i - 2\sqrt{b_i c_i}\cos\left(\frac{j\pi}{\left|\bar{i}\right| + 1}\right)$$

where $j = 1, 2, \ldots, \left|\bar{i}\right|$, $i = 0, 1, \ldots, k-1$.

Proof. Since

$$\det(\lambda I_n - P^T T^{(k)}_{n(k)} P) = \det(\lambda I_n - T^{(k)}_{n(k)}) = \prod_{i=0}^{k-1} \det(\lambda I_{\left|\bar{i}\right|} - T_i)$$

the eigenvalues of each T_i matrices are the eigenvalues of $T^{(k)}_{n(k)}$ matrix. So, we get the eigenvalues of $T^{(k)}_{n(k)}$ matrix as below:

$$\lambda_j = a_i - 2\sqrt{b_i c_i}\cos\left(\frac{j\pi}{\left|\bar{i}\right| + 1}\right)$$

here $j = 1, 2, \ldots, \left|\bar{i}\right|$, $i = 0, 1, \ldots, k-1$, which is desired. □

Permanents

El-Mikkawy (2003) considered determinants of tridiagonal matrices of the form

$$A = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_2 & a_2 & b_2 & & \vdots \\ 0 & c_3 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & b_{i-1} \\ 0 & \cdots & 0 & c_i & a_i \end{pmatrix}, \quad i = 2, 3, ..., n$$

here $f_1 = a_1$. Then, he gave the determinants of the matrix family satisfying a three-term recurrence; i.e. $f_i = \det A$:

$$f_i = a_i f_{i-1} - b_{i-1} c_i f_{i-2}$$

with initial conditions $f_0 = 1$ and $f_{-1} = 0$. Using the converter matrix S, given by Kilic and Tasci (2010), and Hadamard multiplication, the permanent of the matrix (4) is

$$\mathrm{per}(T_n) = u_n$$

where $u_n = au_{n-1} + bcu_{n-2}$ with initial conditions $u_0 = 1$ and $u_{-1} = 0$.

Moreover, this result can be also verified by applying a consecutive *"contraction"* method on last column, which is given by Brualdi and Gibson (1977). Then, one can see that rth contraction step is

$$T_n^{\{r\}} = \begin{pmatrix} a & b & 0 & & 0 \\ c & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & c & a & b \\ 0 & & & cu_r & u_{r+1} \end{pmatrix}_{(n-r)\times(n-r)}$$

for $1 \le r \le n-3$. Going on with this process

$$T_n^{\{n-2\}} = \begin{pmatrix} a & b \\ cu_{n-2} & u_{n-1} \end{pmatrix}$$

Since $\mathrm{per}(T_n) = \mathrm{per}(T_n^{\{r\}})$, we have $\mathrm{per}(T_n) = au_{n-1} + bcu_{n-2}$. So $\mathrm{per}(T_n) = u_n$, which is desired.

Theorem 3. *Permanents of k-tridiagonal k-Toeplitz matrices are*

$$\mathrm{per}(T^{(k)}_{n(k)}) = \prod_{i=0}^{k-1} u_{\left|\bar{i}\right|}$$

where $u_n = a_i u_{n-1} + b_i c_i u_{n-1}$ with $u_0 = 1$ and $u_{-1} = 0$.

Proof. By (1) and (2),

$$\begin{aligned} \mathrm{per}(P^T T^{(k)}_{n(k)} P) &= \mathrm{per}(T^{(k)}_{n(k)}) \\ &= \mathrm{per}(T_0 \oplus T_1 \oplus \cdots \oplus T_{k-1}) = \prod_{i=0}^{k-1} per(T_i) \end{aligned}$$

and

$$\mathrm{per}(T_i) = u_{\left|\bar{i}\right|}.$$

So,

$$\mathrm{per}(T^{(k)}_{n(k)}) = \prod_{i=0}^{k-1} u_{\left|\bar{i}\right|}.$$

The proof is completed. □

Illustrative Example

Let us consider a 3-tridiagonal 3-Toeplitz matrix of order 8:

$$T^{(3)}_{8(3)} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 4 & 0 & 0 & 2 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 2 \end{pmatrix}$$

The permutation matrix is $P = [e_3, e_6, e_1, e_4, e_7, e_2, e_5, e_8]$ and

$$P^T = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

then

$$P^T T_{8(3)}^{(3)} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 2 \end{pmatrix}.$$

So,

$$\begin{aligned} P^T T_{8(3)}^{(3)} P &= \begin{pmatrix} -1 & 1 & & & & & & 0 \\ 2 & -1 & & & & & & \\ & & 1 & -1 & 0 & & & \\ & & -2 & 1 & -1 & & & \\ & & 0 & -2 & 1 & & & \\ & & & & & 2 & 3 & 0 \\ & & & & & 4 & 2 & 3 \\ 0 & & & & & 0 & 4 & 2 \end{pmatrix} \\ &= T_0 \oplus T_1 \oplus T_2. \end{aligned}$$

Consequently,

$$\begin{aligned} \det(T_{8(3)}^{(3)}) &= \det(T_0)\det(T_1)\det(T_2) = (-1)(-3)(-40) = -120, \\ \mathrm{per}(T_{8(3)}^{(3)}) &= \mathrm{per}(T_0).\mathrm{per}(T_1).\mathrm{per}(T_2) = 3.5.56 = 840. \end{aligned}$$

Since the eigenvalues of T_i are also the eigenvalues of the matrix $T_{8(3)}^{(3)}$, the eigenvalues of T_0 are

$$\begin{aligned} \lambda_1 &= -1 - 2\sqrt{2}\cos\left(\frac{\pi}{3}\right) = -1 - \sqrt{2} \\ \lambda_2 &= -1 - 2\sqrt{2}\cos\left(\frac{2\pi}{3}\right) = -1 + \sqrt{2}. \end{aligned}$$

The eigenvalues of T_1 are

$$\begin{aligned} \lambda_3 &= 1 - 2\sqrt{2}\cos\left(\frac{\pi}{4}\right) = -1 \\ \lambda_4 &= 1 - 2\sqrt{2}\cos\left(\frac{\pi}{2}\right) = 1 \\ \lambda_5 &= 1 - 2\sqrt{2}\cos\left(\frac{3\pi}{4}\right) = 3. \end{aligned}$$

The eigenvalues of T_2 are

$$\begin{aligned} \lambda_6 &= 2 - 4\sqrt{3}(\cos\frac{\pi}{4}) = 2 - 2\sqrt{6} \\ \lambda_7 &= 2 - 4\sqrt{3}(\cos\frac{\pi}{2}) = 2 \\ \lambda_8 &= 2 - 4\sqrt{3}(\cos\frac{3\pi}{4}) = 2 + 2\sqrt{6}. \end{aligned}$$

Consequently, the eigenvalues of $T_{8(3)}^{(3)}$ are $\lambda_1 = -1 - \sqrt{2}$, $\lambda_2 = -1 + \sqrt{2}$, $\lambda_3 = -1$, $\lambda_4 = 1$, $\lambda_5 = 3$, $\lambda_6 = 2 - 2\sqrt{6}$, $\lambda_7 = 2$, $\lambda_8 = 2 + 2\sqrt{6}$.

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