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Annuli Containing All the Zeros of a Polynomial

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In this paper, we provide three results concerning the annular regions containing the zeros of a complex polynomial. These results show certain improvements in comparison to some of the earlier results. One specific example of a polynomial for each of new results provided in this paper, has been generated using MATLAB to substantiate the improvement in our results over the existing results. A comparative analysis on the computational results is also done towards the end. Such kind of results on the location of zeros of a polynomial have wide applications in many areas, such as Signal Processing, Communication Theory etc.

Introduction

If $p(z)$ is a complex polynomial of degree n, then by the well known Fundamental Theorem of Algebra, $p(z)$ has exactly n zeros. Since the polynomials in general are unsolvable, it would obviously be of interest to obtain the smallest possible region containing all the zeros of a polynomial. The results related to the location of zeros of a polynomial have significant applications in many areas such as Mathematical Physics, Signal Processing, Communication Theory, Control Theory, Coding Theory, Cryptography, Mathematical Biology, and Computer Engineering. Therefore there is a scope and demand for improving the existing results to better ones.

There are good number of methods, for example, Ehrlich-Aberth's type (see, Aberth (1973), Ehrlich (1967) and Anourein (1977)) for the simultaneous determination of the zeros of algebraic polynomials, and there are studies to accelerate convergence and increase computational efficiency of these methods (see, Milovanovic and Petkovic (1986), Petković (2008)). These methods may possibly have greater efficiency if combined with the results of this paper that provide annulus containing all the zeros of a polynomial.

The first and significant result in this direction is believed to be due to Gauss, who proved that a polynomial $p(z) = a_0+a_1z+a_2z^2+\cdots+a_nz^n$, with all a_k real, has all its zeros on or inside the circle $|z| = R$, where $R = \max\limits_{1\le k\le n}(n\sqrt{2}|a_k|)^{\frac{1}{k}}$. Cauchy (1829) improved this result by proving that, if $p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_{n-1}z^{n-1} + z^n$ is a complex polynomial of degree n, then all the zeros of $p(z)$ lie in the disc $\{z : |z| < \eta\} \subset \{z : |z| < 1 + A\}$, where $A = \max\limits_{1\le k\le n-1}|a_k|$ and η is the unique positive root of the real-coefficient equation

$$z^n - |a_{n-1}|z^{n-1} - |a_{n-2}|z^{n-2} - \cdots - |a_1|z - |a_0| = 0.$$

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In addition to the above result, if we apply the above result of Cauchy to the polynomial $q(z) = z^np(1/z)$, we get the following well-known theorem.

Theorem 1 (Cauchy). *All the zeros of the polynomial $p(z) = a_0 + a_1z + \cdot + a_nz^n$, $a_n \neq 0$, lie in the annulus $r_1 \le |z| \le r_2$, where r_1 is the unique positive root of the equation*

$$|a_n|z^n + |a_{n-1}|z^{n-1} + \cdots + |a_1|z - |a_0| = 0, \tag{1}$$

and r_2 is the unique positive root of the equation

$$|a_0| + |a_1|z + \cdots + |a_{n-1}|z^{n-1} - |a_n|z^n = 0. \tag{2}$$

One cannot be satisfied with the above Theorem of Cauchy because in order to find the annulus containing all the zeros of a polynomial, one needs to compute the zeros of two other polynomials which is equally laborious. The results providing annuli with radii explicitly in terms of coefficients have been given in many papers and monographs (see, Joyal, Labelle, and Rahman (1967), Aberth (1973), Datt and Govil (1978), Milovanovic, Mitrinovic, and Rassias (1994), Sun and Hsieh (1996), Diaz-Barrero (2002b), Jain (2006), Affane-Aji, Agarwal, and Govil (2009) and Affane-Aji, Biaz, and Govil (2010)). Let us begin by stating the following theorem due to Díaz-Barrero (2002a), which gives an annulus containing all the zeros of a polynomial.

Theorem 2. *If $p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_{n-1}z^{n-1} + a_nz^n$ is a non-constant complex polynomial of degree n, with $a_k \neq 0$, $1 \le k \le n$, then all the zeros of $p(z)$ lie in the annulus $C = \{z : r_1 \le |z| \le r_2\}$, where*

$$r_1 = \frac{3}{2}\min_{1\le k\le n}\left\{\frac{2^nF_kC(n,k)}{F_{4n}}\left|\frac{a_0}{a_k}\right|\right\}^{\frac{1}{k}}$$

and

$$r_2 = \frac{2}{3}\max_{1\le k\le n}\left\{\frac{F_{4n}}{2^nF_kC(n,k)}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{\frac{1}{k}}.$$

Here $C(n,k) = \frac{n!}{k!(n-k)!}$, *and* F_k *is the* k^{th} *Fibonacci number, defined by* $F_0 = 0,\ F_1 = 1$ *and* $F_k = F_{k-1} + F_{k-2},\ \ k \geq 2.$

Kim (2005) also proved a similar result as the following.

Theorem 3. *Let* $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n$ *be a non-constant complex polynomial of degree* n, *with* $a_k \neq 0,\ \ 1 \leq k \leq n$. *Then all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \leq |z| \leq r_2\}$, *where*

$$r_1 = \min_{1\leq k\leq n} \left\{ \frac{C(n,k)}{2^n - 1} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}}$$

and

$$r_2 = \max_{1\leq k\leq n} \left\{ \frac{2^n - 1}{C(n,k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}.$$

Here $C(n,k)$ *are the binomial coefficients.*

Díaz-Barrero and Egozcue (2004) proved another result on the annulus containing all the zeros of a polynomial which is stated below.

Theorem 4. *If* $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n$ *is a non-constant complex polynomial of degree* n, *with* $a_k \neq 0,\ \ 1 \leq k \leq n$, *then for* $j \geq 2$, *all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \leq |z| \leq r_2\}$, *where*

$$r_1 = \min_{1\leq k\leq n} \left\{ \frac{C(n,k) A_k B_j^k (bB_{j-1})^{n-k}}{A_{jn}} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}},$$

and

$$r_2 = \max_{1\leq k\leq n} \left\{ \frac{A_{jn}}{C(n,k) A_k B_j^k (bB_{j-1})^{n-k}} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}.$$

Here $C(n,k)$ *are the binomial coefficients,*

$$B_n = \sum_{k=0}^{n-1} r^k s^{n-1-k},$$

and for $j \geq 2$

$$\sum_{k=0}^{n} C(n,k)(bB_{j-1})^{n-k} B_j^k A_k = A_{jn}$$

where $A_n = cr^n + ds^n$, c, d *are real constants and* r, s *are the roots of the equation* $x^2 - ax - b = 0$, *in which* a, b *are strictly positive real numbers.*

Dalal and Govil (2013) (see also, Dalal and Govil (2014) and Dalal and Govil (2017)) gave a new shape to all these kind of results by unifying them into a single result and including all the above Theorems 2, 3 and 4 as special cases.

Theorem 5. *Let* $A_k > 0$ *for* $1 \leq k \leq n$ *and be such that* $\sum_{k=1}^{n} A_k = 1$. *If* $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n$ *is a non-constant complex polynomial of degree* n, *with* $a_k \neq 0$ *for* $1 \leq k \leq n$. *Then all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \leq |z| \leq r_2\}$, *where*

$$r_1 = \min_{1\leq k\leq n} \left\{ A_k \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}}$$

and

$$r_2 = \max_{1\leq k\leq n} \left\{ \frac{1}{A_k} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}.$$

Dalal and Govil (2013) also gave the following result as an application of Theorem 5.

Theorem 6. *Let* $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n$ *be a non-constant complex polynomial of degree* n, *with* $a_k \neq 0,\ 1 \leq k \leq n$. *Then all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \leq |z| \leq r_2\}$, *where*

$$r_1 = \min_{1\leq k\leq n} \left\{ \frac{C_{k-1} C_{n-k}}{C_n} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}}$$

and

$$r_2 = \max_{1\leq k\leq n} \left\{ \frac{C_n}{C_{k-1} C_{n-k}} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}.$$

Here $C_k = \dfrac{C(2k,k)}{k+1}$ *is the* k^{th} *Catalan number in which* $C(2k,k)$ *are the binomial coefficients.*

Dalal and Govil (2013) have mentioned in their paper, Theorem 5 not only includes Theorems 2, 3, and 4, as special cases, but also capable of generating many new results by making appropriate choice of the numbers A_k. In view of this, Bidkham, Zireh, and Mezerji (2017, 2013) and, Rather and Mattoo (2013) obtained results on annulus containing all the zeros of a polynomial involving Fibonacci numbers and generalized Fibonacci numbers respectively. Recently Govil and Kumar (2015), and Nwaeze (2016) also proved similar results using such special array numbers.

In this paper, we use Theorem 5 to obtain few new results, which provide annuli containing all the zeros of a polynomial. Further, we demonstrate that for some polynomials, our theorems sharpen some of the known results in this direction, and this has been done in the last section where we develop MATLAB code to generate examples of polynomials for which our results give better bounds than obtainable from the known results, such as Theorems 2, 3 and 6.

Our first result, stated below gives annulus in terms of Bell numbers. For more details about Bell numbers one can refer Bona (2015).

Theorem 7. *Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n$ be a non-constant complex polynomial of degree n, with $a_k \neq 0$, $1 \leq k \leq n$. Then all the zeros of $p(z)$ lie in the annulus $C = \{z : r_1 \leq |z| \leq r_2\}$, where*

$$r_1 = \min_{1\leq k\leq n} \left\{ \frac{C(n,k)B(k)}{B(n+1)-1} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}},$$

and

$$r_2 = \max_{1\leq k\leq n} \left\{ \frac{B(n+1)-1}{C(n,k)B(k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}.$$

Here $B(n)$, called n^{th} Bell number is the total number of partitions of a set containing n elements and $C(n,k)$ are binomial coefficients.

In the next result, we will make use of Bell numbers with Sterling numbers of the second kind (one can refer Conway and Guy (1996) for Sterling numbers) to evaluate the radii of two circles involved in the annular region containing all the zeros of a polynomial.

Theorem 8. *Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n$ be a non-constant complex polynomial of degree n, with $a_k \neq 0$, $1 \leq k \leq n$. Then all the zeros of $p(z)$ lie in the annulus $C = \{z : r_1 \leq |z| \leq r_2\}$, with*

$$r_1 = \min_{1\leq k\leq n} \left\{ \frac{S(n,k)}{B(n)} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}},$$

and

$$r_2 = \max_{1\leq k\leq n} \left\{ \frac{B(n)}{S(n,k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}},$$

where $B(n)$ are Bell numbers and $S(n,k)$ are the Sterling numbers of second kind where $n \geq k$.

Finally, we present the following result which involves the special combination of binomial coefficients.

Theorem 9. *Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n$ be a non-constant complex polynomial of degree n, with $a_k \neq 0$, $1 \leq k \leq n$. Then all the zeros of $p(z)$ lie in the annulus $C = \{z : r_1 \leq |z| \leq r_2\}$, where*

$$r_1 = \min_{1\leq k\leq n} \left\{ \frac{2kC(2n,n+k)}{nC(2n,n)} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}}$$

and

$$r_2 = \max_{1\leq k\leq n} \left\{ \frac{nC(2n,n)}{2kC(2n,n+k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}.$$

Here $C(2n,n)$ and $C(2n,n+k)$ are binomial coefficients.

For some set of polynomials, Theorems 7, 8, 9 yield sharper bounds than obtainable from some of the already known results given over here and this has been shown in the last section.

Lemmas

We will need the following lemmas to prove our results. Our first lemma connects Bell numbers with Binomial coefficients [Bona (2015) Theorem 2.18, p 65].

Lemma 1. *If $B(n)$ are Bell numbers for the given positive integer n, then*

$$B(n+1) = \sum_{k=0}^{n} C(n,k)B(k)$$

where $B(n)$ is the n^{th} Bell number with $B(0) = 1$ and $C(n,k)$ are binomial coefficients.

Proof. For the sake of completeness, we provide a new different proof over here. Let us count the number of partitions of a set of $n+1$ elements, depending on the size of the block containing the $n+1^{st}$ element. If the block has size m for $1 \leq m \leq n+1$ then we have $C(n, m-1)$ choices for the $m-1$ other elements of the block. But then the remaining $n+1-m$ elements can be partitioned in $B(n+1-m)$ ways. We have therefore,

$$\begin{aligned} B(n+1) &= \sum_{m=1}^{n+1} C(n,m-1)B(n+1-m) \\ &= \sum_{m=1}^{n+1} C(n,n+1-m)B(n+1-m) \\ &= \sum_{k=0}^{n} C(n,k)B(k). \end{aligned}$$

□

Next lemma provides an identity involving Bell numbers and Sterling numbers of second kind, which we will use to prove Theorem 8. We omit its proof since it follows directly from their definitions and also can be found in Conway and Guy (1996).

Lemma 2. *If $B(n)$ is the n^{th} Bell number and $S(n,k)$ are Sterling numbers of second kind, then*

$$B(n) = \sum_{k=1}^{n} S(n,k),$$

for $n \geq 1$.

Our next result is an identity involving binomial coefficients.

Lemma 3. *With the standard binomial coefficient notations,*

$$\sum_{k=1}^{n} 2kC(2n,n+k) = nC(2n,n).$$

Proof. Let

$$\begin{aligned} S_n &= \sum_{k=1}^{n} kC(2n, n+k) \\ &= (2n)! \sum_{k=1}^{n} \frac{k}{(n+k)!(n-k)!}. \end{aligned}$$

We have

$$\begin{aligned} 2S_n &= (2n)! \sum_{k=1}^{n} \frac{(n+k)-(n-k)}{(n+k)!(n-k)!} \\ &= (2n)! \sum_{k=1}^{n} \left(\frac{1}{(n+k-1)!(n-k)!} \right. \\ &\quad \left. - \frac{1}{(n+k)!(n-k-1)!} \right) \\ &= (2n) \sum_{k=1}^{n} (C(2n-1, n+k-1) \\ &\quad - C(2n-1, n+k)). \end{aligned}$$

But the last sum is a telescopic sum and therefore

$$2S_n = nC(2n, n).$$

Hence the proof. □

Proofs of the theorems

Proof of Theorem 7. If $B(n)$ are Bell numbers, then by Lemma 1,

$$\sum_{k=1}^{n} \frac{C(n,k)B(k)}{B(n+1)-1} = 1.$$

Thus, if we take

$$A_k = \frac{C(n,k)B(k)}{B(n+1)-1},$$

then each A_k is positive and $\sum_{k=1}^{n} A_k = 1$, and hence applying Theorem 5 for this set of A_k, $(1 \le k \le n)$, we get the required annulus C given by (3), that contains all the zeros of the polynomial $p(z)$. □

Proof of Theorem 8. Note that if $B(n)$ is the n^{th} Bell number, and $S(n,k)$ Sterling numbers of second kind then from Lemma 2, for all $n \ge 1$ we have

$$\sum_{k=1}^{n} \frac{S(n,k)}{B(n)} = 1,$$

If we take $A_k = \frac{S(n,k)}{B(n)}$, then $A_k > 0$ and $\sum_{k=1}^{n} A_k = 1$, and hence by applying Theorem 5 for this set of values of A_k, $(1 \le k \le n)$, we get (4), and hence Theorem 8 is proved. □

Proof of Theorem 9. By Lemma 3, we have

$$\sum_{k=1}^{n} \frac{2kC(2n, n+k)}{nC(2n, n)} = 1.$$

Now, if we take

$$A_k = \frac{2kC(2n, n+k)}{nC(2n, n)}$$

then $A_k > 0$ and $\sum_{k=1}^{n} A_k = 1$, and hence applying Theorem 5 for this set of values of A_k, we get the required annulus given by (5), and the proof of Theorem 9 is thus complete. □

Computational Results and Analysis

In this section, we demonstrate the improvement of our results over the earlier results by presenting examples of polynomials, for which the annuli obtained by our results are significantly smaller than the annuli obtainable from the other stated results in the first section. We developed MATLAB code for generating these examples. By inputting various values to the coefficients of the polynomial in the MATLAB program, we chose and presented the one which showed considerable betterment in terms of output values. We will also substantiate the improvement of our results over the existing results using the examples discussed in the papers due to Dalal and Govil (2013), Díaz-Barrero (2002a) and Kim (2005).

Let us first consider the example appeared in the paper due to Dalal and Govil (2013).

Example 1. Let $p(z) = z^4 + 0.01z^3 + 0.1z^2 + 0.2z + 0.4$.

Result	r_1	r_2	Area of annulus
Theorem 2	0.1945	1.1290	3.8855
Theorem 3	0.4041	1.5651	7.1823
Theorem 6	0.6148	1.1187	2.7442
Theorem 7	0.1569	1.0799	3.5864
Theorem 8	0.1333	1.5651	7.6394
Actual Bound	0.7413	0.8532	0.5605

Table 1

For the polynomial $p(z)$ given in Example 1, Table 1 shows that, Theorem 7 yields the best possible value for r_2 in the annular region containing all the zeros of the polynomial $p(z)$. In this case, the radius of outer circle of the annular ring containing all the zeros of $p(z)$ is 1.0799 which is less than that given by Theorem 6 by considerable margin of 0.0388. Combining Theorems 6 and 7 we can reduce the area of the annular region containing all the zeros of $p(z)$. One can also observe that the area of the annular region containing all the zeros of $p(z)$ obtained by Theorem 7 is nearly 50% of that obtained by Theorem 3 and 92% of that of obtained by Theorem 2.

Let us consider another example appeared in the paper due to Dalal and Govil (2013).

Example 2. Let $p(z) = z^5 + 0.006z^4 + 0.01z^3 + 0.2z^2 + 0.3z + 1$.

Result	r_1	r_2	Area of annulus
Theorem 2	0.1183	1.4097	6.1995
Theorem 3	0.5032	1.9873	11.6124
Theorem 6	0.7715	1.2806	3.2818
Theorem 7	0.0825	1.3118	5.3848
Actual Bound	0.9530	1.0612	0.6847

Table 2

One can observe from the Table 2 that, our Theorem 7 is giving significantly better bound than obtainable from already known Theorems 2 and 3. In fact the area of the annulus containing all the zeros of the polynomial $p(z)$ obtained by Theorem 7 is about 5.3848, which is about 86.85% of the area of the annulus obtained by Theorem 2, about 46.37% of the area of the annulus obtained by Theorem 3.

Let us consider the example used in the papers due to Kim (2005), and Díaz-Barrero (2002a).

Example 3. Let $p(z) = z^3 + 0.1z^2 + 0.3z + 0.7$.

Result	r_1	r_2	Area of annulus
Theorem 2	0.5833	1.2313	3.6938
Theorem 3	0.4642	1.6985	8.3863
Theorem 7	0.5000	1.2515	4.1349
Theorem 8	0.4667	1.5183	6.5579
Actual Bound	0.8058	0.9320	0.6893

Table 3

For the polynomial $p(z)$ given in Example 3, it is clear from the Table 3 that Theorems 7 and 8 give the considerably better annular region containing all the zeros of the polynomial $p(z)$ comparing to Theorem 3 given by Kim (2005). In this case, the area of the annulus containing all the zeros of the polynomial $p(z)$ obtained by Theorem 7 is about 49.30% of the area of the annulus obtained by Theorem 3, and the area of the annulus obtained by Theorem 8 is about 78.19% of the area of the annulus obtained by Theorem 3.

Finally, let us take the following example.

Example 4. Let $p(z) = z^4 + 0.1z^3 + 1.1z^2 + 0.1z + 0.001$.

Result	r_1	r_2	Area of annulus
Theorem 2	0.0010	2.2420	15.7909
Theorem 3	0.0027	1.6583	8.6394
Theorem 6	0.0036	2.7749	24.1902
Theorem 7	0.0008	2.1622	14.6869
Theorem 8	0.0007	1.5353	7.4052
Theorem 9	0.0040	1.6583	8.6393
Actual Bound	0.0114	1.0480	3.4499

Table 4

For the polynomial $p(z)$ given in Example 4, it is clear from the Table 4 that Theorems 8, 9 perform better than Theorems 2, 3, 6. Theorem 8 gives the best possible annular region containing all the zeros of the polynomial $p(z)$. In this case, the area of the annular region containing all the zeros of $p(z)$ is 7.4052. Also, this area is about 46.689% of the area obtained from Theorem 2, and about 30.61% of the area obtained by Theorem 6. Theorem 9 also shows considerable improvement comparing to Theorem 2 and 6.

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