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On Subsums of Series with Positive Terms

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A necessary and sufficient condition for subsums of a series $\sum_{k=1}^{\infty} a_k$, such that $0 < a_k \to 0$ as $k \to \infty$, cover the interval $[0, s]$, $s = \sum_{k=1}^{\infty} a_k$, is proved.

Introduction

Throughout, $\mathbb{N}$ will denote the set of non-negative integers. For a series $\sum_{k=0}^{\infty} a_k$ in which $a_k \geq 0$ for all $k \in \mathbb{N}$, a subsum of the series is a sum of the form $\sum_{k \in S} a_k$ for some $S \subseteq N$.

We use the convention that $\sum_{k \in \emptyset} a_k = 0$. If $0 < |S| < \infty$, then $\sum_{k \in S} a_k$ is an ordinary finite subsum. If S is infinite, then $\sum_{k \in S} a_k$ is an infinite series in its own right.

We allow ∞ as a series sum. We shall not distinguish between $\sum_{k \in S} a_k$ as a formal sum and the value of the sum. The set of subsum values of a series $\sum_{k=0}^{\infty} a_k$ is

$$\text{Subsum}((a_k)) := \left\{ \sum_{k \in S} a_k \;\middle|\; S \subseteq \mathbb{N} \right\}.$$

If $s = \sum_{k=0}^{\infty} a_k$ then

$$\text{Subsum}((a_k)) \subseteq [0, s] \subseteq [0, \infty].$$

Theorem 1. *Suppose that $a_0, a_1, \ldots$ are positive real numbers, $a_k \to 0$ as $k \to \infty$, and $s = \sum_{k=0}^{\infty} a_k$.*

(1) If $a_k \leq \sum_{j=k+1}^{\infty} a_j \; \forall k \in \mathbb{N}$, then

$$\text{Subsum}((a_k)) = \left\{ \sum_{j \in S} a_j \;\middle|\; S \subseteq \mathbb{N} \right\} = [0, s].$$

(2) If $a_0 \geq a_1 \geq \ldots$, and $a_k > \sum_{j=k+1}^{\infty} a_j$ for some $k \in \mathbb{N}$, then $I = \left(\sum_{j \in \mathbb{N} \setminus \{k\}} a_j, \sum_{0 \leq j \leq k} a_j \right)$ is nonempty, and $\text{Subsum}((a_k)) \cap I = \emptyset$

Proof. Suppose that $a_i \leq \sum_{j=i+1}^{\infty} a_j \; \forall i \in \mathbb{N}$. Clearly $0, s \in \text{Subsum}((a_k))$. Suppose that $x \in (0, s)$. We will define a sequence $\lambda_0, \lambda_1, \ldots$ such that for each $k \in \mathbb{N}$, $\lambda_k \in \{0, a_k\}$ and $\sum_{k=0}^{\infty} \lambda_k = x$. From this it will follow that $x = \sum_{k \in S} a_k$ where $S = \{k \in \mathbb{N} \mid \lambda_k \neq 0\}$.

Having determined λ_j for all $j < k$, set $\lambda_k = a_k$ if $x - \left(\sum_{j<k} \lambda_j + a_k \right) > 0$ and $\lambda_k = 0$, otherwise. It remains to be seen that $x = \sum_{j=0}^{\infty} \lambda_j$.

Clearly the partial sums $\sum_{j=0}^{k} \lambda_j$ are non-decreasing and bounded above by x, so we know our series converges to something less than or equal to x.

If $\lambda_k = 0$ then $x - \left(\sum_{j<k} \lambda_i + a_k \right) \leq 0$, so $0 < x - \sum_{j<k} \lambda_j \leq a_k$. Since $a_k \to 0$, it follows that if $\lambda_k = 0$ for infinitely many values of k, then $\sum_{j=0}^{\infty} \lambda_j = x$.

Otherwise, $\lambda_k = 0$ for only finitely many values of k. Since $\lambda_k \in \{0, a_k\}$ for each $k \in \mathbb{N}$ and $x < s = \sum_{k=0}^{\infty} a_k$, $Z = \{k \in \mathbb{N} \mid \lambda_k = 0\}$ is nonempty.

Let z be the largest element of Z. So $x - \left(\sum_{k<z} \lambda_k + a_z \right) \leq 0$, and

$$\sum_{k=0}^{\infty} \lambda_k \leq x \leq \sum_{k<z} \lambda_k + a_z = \sum_{k \leq z} \lambda_k + a_z \quad (\textit{because } \lambda_z = 0)$$

$$\leq \sum_{k \leq z} \lambda_k + \sum_{k=z+1}^{\infty} a_k$$

$$= \sum_{k=0}^{\infty} \lambda_k$$

Therefore $\sum_{k=0}^{\infty} \lambda_k \leq x \leq \sum_{k=0}^{\infty} \lambda_k$, so $x = \sum_{k=0}^{\infty} \lambda_k$, and because x was chosen arbitrarily from $(0, s)$, (1) is proven.

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Now suppose that $a_0, a_1, \ldots$ is a non-increasing sequence of positive real numbers which converges to 0. Suppose that for some $k \in \mathbb{N}$, $a_k > \sum_{j>k} a_j$. Therefore

$$\sum_{j\in\mathbb{N}\setminus\{k\}} a_j = \sum_{j<k} a_j + \sum_{j>k} a_j < \sum_{j<k} a_j + a_k = \sum_{0\le j\le k} a_j ,$$

so $I = \left(\sum_{j\in\mathbb{N}\setminus\{k\}} a_j, \sum_{0\le j\le k} a_j\right)$ is a nonempty open interval contained inside $\left[0, \sum_{j=0}^{\infty} a_j\right]$, and we will see that
$I \cap \text{Subsum}((a_j)) = \emptyset$.

Suppose that $S \subseteq \mathbb{N}$. If some $r \in \{0, \ldots, k\}$ is missing from S, then because $a_r \ge a_k$,

$$\sum_{j\in S} a_j \le \sum_{j\in\mathbb{N}\setminus\{r\}} a_j \le \sum_{j\in\mathbb{N}\setminus\{k\}} a_j$$

Otherwise, $\{0, \ldots, k\} \subseteq S$, so $\sum_{0\le j\le k} a_j \le \sum_{j\in S} a_j$. In either case, $\sum_{j\in S} a_j \notin I$, so (2) is proven. □

Corollary 1. *If $a_0 \ge a_1 \ge \cdots > 0$, $a_k \to 0$ as $k \to \infty$, and $s = \sum_{k=0}^{\infty} a_k$, then the finite subsums of the series $\sum_{k=0}^{\infty} a_i$ are dense in $[0, s]$ if and only if $a_k \le \sum_{j=k+1}^{\infty} a_j$ for all $k \in \mathbb{N}$.*

Corollary 2. *If $a_0, a_1, \ldots$ are positive real numbers tending to 0 such that $\{a_i\}_{i=0}^{\infty}$ satisfies the hypothesis of (1) of Theorem 1, then so does the non-increasing rearrangement of $\{a_i\}_{i=0}^{\infty}$.*

Let ζ denote the Riemann zeta function. We have the following:

Theorem 2. *Suppose that $p \ge 1$. Then*

$$\left\{1 + \sum_{k\in S} \frac{1}{n^p} \;\middle|\; S \subseteq \{2, 3, \ldots\}, |S| < \infty\right\}$$

is dense in $[1, \zeta(p)]$ if and only if $p \le q$, where q (approx. 2.424) is the unique solution of the equation

$$\frac{1}{2^q} = \sum_{n=3}^{\infty} \frac{1}{n^q}$$

Proof. Since $\frac{1}{k} < \sum_{n>k} \frac{1}{n} = \infty$ for all $k = 2, 3, \ldots$, the claim of Theorem 2 for $p = 1$ follows from Corollary 1 applied to the sequence $\frac{1}{2}, \frac{1}{3}, \ldots$. Suppose $p > 1$. For $k \ge 2$,

$$\left(\sum_{n>k} \frac{1}{n^p}\right) / \frac{1}{k^p} = \sum_{t=0}^{\infty} \left(\frac{k}{k+t+1}\right)^p$$
$$= \sum_{t=0}^{\infty} \left(1 - \frac{t+1}{k+t+1}\right)^p ,$$

which increases as k increases. Therefore, $\frac{1}{k^p} > \sum_{n>k} \frac{1}{n^p}$ for some $k \in \{2, 3, \ldots\}$ if and only if $\frac{1}{2^p} > \sum_{n>2} \frac{1}{n^p}$. Applying Corollary 1 to the sequence $\frac{1}{2^p}, \frac{1}{3^p}, \ldots$, we see that $\left\{1 + \sum_{n\in S} \frac{1}{n^p} \;\middle|\; S \subseteq \{2, 3, \ldots\}, |S| < \infty\right\}$ is dense in $[1, \zeta(p)]$ if and only if $\frac{1}{2^p} \le \sum_{n>2} \frac{1}{n^p}$. Note that $\sum_{n>2} \frac{1}{n^p} / \frac{1}{2^p} = \sum_{n=3}^{\infty} \left(\frac{2}{n}\right)^p$ decreases as p increases. Since the ratio is large for p close to 1 and goes to 0 (continuously) as p approaches infinity, the equation $\frac{1}{2^q} = \sum_{n=3}^{\infty} \frac{1}{n^q}$ has a unique solution, q. Then $\frac{1}{2^p} \le \sum_{n>2} \frac{1}{n^p}$ if and only if $p \le q$, and the theorem is proved. □

Since $\sum_{n=3}^{\infty} \frac{1}{n^2} > \int_3^{\infty} \frac{dx}{x^2} = \frac{1}{3} > \frac{1}{2^2}$ and $\sum_{n=3}^{\infty} \frac{1}{n^3} < \int_2^{\infty} \frac{dx}{x^3} = 1/8 = \frac{1}{2^3}$, $2 < q < 3$. Estimating q by a simple program written in Java, we have q is approximately 2.424.

Remarks and Open Questions

The questions that led to the results of this paper arose from Defant (2015), which gives an answer similar to Theorem 2 to a much more difficult question. Here is the question answered:

If we define σ_{-r} on the positive integers by $\sigma_{-r}(n) = \sum_{d|n,d>0} d^{-r}$, in which $r \ge 1$, is the range of σ_{-r} dense in $[1, \zeta(r)]$? This is a question about a very special set of subsums of $\sum_{k=1}^{\infty} k^{-r}$. The answer (Theorem 2.3 in Defant (2015)): the range of σ_{-r} is dense in $[1, \zeta(r)]$ for $1 \le r \le \kappa$, where κ is the unique solution in $(1, 2)$ of the equation $\frac{2^\kappa}{2^\kappa-1}\frac{3^\kappa+1}{3^\kappa-1} = \zeta(\kappa)$ (Defant estimates $\kappa \approx 1.888$.)

We end with two questions: suppose that $a_0 \ge a_1 \ge \cdots > 0$, $a_k \to 0$ as $k \to \infty$, and $s = \sum_{k=0}^{\infty} a_k$.

1. Is Subsum$((a_k))$ necessarily closed? By Theorem 1, the answer is obviously yes if $a_k \le \sum_{j>k} a_j$ for every $k \in \mathbb{N}$. It follows that it is true if this holds for all but finitely many k.

2. Can there be any maximal open intervals in $[0, s] \setminus$ Subsum$((a_k))$ other than the intervals $\left(\sum_{j\in\mathbb{N}\setminus\{k\}} a_j, \sum_{0\le j\le k} a_j\right)$ for $k \in \mathbb{N}$ such that $a_k > \sum_{j=k+1}^{\infty} a_j$?

References

Defant, C. (2015). On the density of ranges of generalized divisor functions. *Notes on Number Theory and Discrete Mathematics*, *21*, 80 - 87.