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Discrete Inequalities of Steffensen and Hayashi Type in Inner Product Spaces with Applications

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In this paper, a unified approach to Steffensen and Hayashi type inequalities in inner product spaces is presented. Single and double Steffensen–Hayashi type inequalities are established. In particular, some refinements and extensions of the classical results from [J. C. Evard and H. Gauchman, Steffensen type inequalities over general measure spaces, Analysis, 17 (1997), 301-322] and [H.-N. Shi and S.-H. Wu, Majorized proof and improvement of the discrete Steffensen's inequality, Taiwanese J. Math., 11 (2007) 1203-1208] are demonstrated. Applications are provided to bounding expectations of discrete random variables.

Keywords: Steffensen inequality, Hayashi inequality, inner product space, separability of vector, discrete random variables

Introduction and motivation

In this expository section we present the well-known Steffensen's inequality for integrable real functions and its extension by Hayashi. We also demonstrate a discrete version of Steffensen's inequality for finite sequences and its refinement.

Theorem A. (Steffensen (1918)) *Let $x(t)$ and $y(t)$ be integrable real functions on $[a,b]$ such that $x(t)$ is a nonincreasing function and $0 \le y(t) \le 1$ for $t \in [a,b]$. Then*

$$\int_{b-\lambda}^{b} x(t)\,dt \le \int_{a}^{b} x(t)y(t)\,dt \le \int_{a}^{a+\lambda} x(t)\,dt, \tag{1}$$

where $\lambda = \int_a^b y(t)\,dt$.

The Hayashi's modification of (1) reads as follows. (See Hayashi (1919), Mitrinović (1969), (Mitrinović, Pečarić, & Fink, 1993, pp. 311-312).)

Theorem B. (Hayashi (1919)) *Let $x(t)$ and $y(t)$ be integrable real functions on $[a,b]$ such that $x(t)$ is a nonincreasing function and $0 \le y(t) \le A$ for $t \in [a,b]$. Then*

$$A\int_{b-\lambda}^{b} x(t)\,dt \le \int_{a}^{b} x(t)y(t)\,dt \le A\int_{a}^{a+\lambda} x(t)\,dt, \tag{2}$$

where $\lambda = \frac{1}{A}\int_a^b y(t)\,dt$.

Here is a discrete version of Steffensen's inequality (see Evard and Gauchman (1997); Liu (2004), (Marshall, Olkin, & Arnold, 2011, p. 640)).

Theorem C. (Evard and Gauchman (1997)) *Let $x_1 \ge x_2 \ge \ldots \ge x_n \ge 0$ and $0 \le y_1, y_2, \ldots, y_n \le 1$. Let $k_1, k_2 \in \{1, 2, \ldots, n\}$ satisfy $k_2 \le \sum_{i=1}^{n} y_i \le k_1$. Then*

$$\sum_{i=n-k_2+1}^{n} x_i \le \sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{k_1} x_i. \tag{3}$$

Shi and Wu gave an improvement of Theorem C (see (Shi & Wu, 2007, Theorem 2), (Marshall et al., 2011, pp. 640-641)).

Theorem D. (Shi and Wu (2007)) *Let $x_1 \ge x_2 \ge \ldots \ge x_n$ and $0 \le y_1, y_2, \ldots, y_n \le 1$. Let $k_1, k_2 \in \{1, 2, \ldots, n\}$ satisfy $k_2 \le \sum_{i=1}^{n} y_i \le k_1$. Then*

$$\sum_{i=n-k_2+1}^{n} x_i + \left(\sum_{i=1}^{n} y_i - k_2\right) x_n \le \sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{k_1} x_i - \left(k_1 - \sum_{i=1}^{n} y_i\right) x_n. \tag{4}$$

For other Steffensen type inequalities, see e.g. Agarwal and Dragomir (1996); Cerone (2001); Jakšetić, Pečarić, and Perušić (2014); Liu (2004); Masjed-Jamei, Qi, and Srivastava (2010); Mercer (2000, 2008); Milovanović and Pečarić (1979); Mitrinović (1969); Pečarić, Perušić, and Smoljak (2013, 2014); Wu and Srivastava (2007).

In this paper, we show a unified approach to Steffensen and Hayashi's inequalities. Our aim is to provide some generalizations of the above results for vectors x and y in an

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inner-product space. Special attention is paid on "discrete" Theorems C and D.

In next section we collect some relevant notation, definitions and preliminary results in the subject. In further section we derive some Steffensen – Hayashi (S–H) type inequalities in an inner product space via the notion of similarly separable vectors (see Niezgoda (2006, 2008, 2012)). Our estimates utilize information on local position of a vector x among given vectors $s_1, s_2, \dots, s_n$. In particular, some of the corresponding coordinates $t_1, t_2, \dots, t_n$ of x can be negative. Such an approach leads to general results on S–H like inequalities. Next, we discuss some ways to simplify hypotheses of our theorems.

In the last section we present applications of S - H type inequalities. In particular, we give some refinements and extensions of the classical results by Evard and Gauchman (see Theorem C) and by Shi and Wu (see Theorem D). We also employ our results to bound expectations of discrete random variables.

Preliminaries

Throughout this paper $(V, \langle\cdot,\cdot\rangle)$ is a real inner product space.

A nonempty set $C \subset V$ is said to be a *convex cone* if **(i)** $a, b \in C$ implies $a + b \in C$, and **(ii)** $a \in C$ and $0 \le \mu \in \mathbb{R}$ imply $\mu a \in C$.

For a nonempty set $S \subset V$, the symbol cone S stands for the convex cone of all nonnegative linear combinations of vectors in S.

If $C \subset V$ is a convex cone, then by $\le_C$ we denote the preorder on V defined as follows: for $a, b \in V$,

$$a \le_C b \quad \text{iff} \quad b - a \in C.$$

Let $e = (e_1, \dots, e_n) \in V^n$. Let I_1 and I_2 be two sets of indices such that $I_1 \cup I_2 = I$, where $I = \{1, 2, \dots, n\}$. For a given vector $y \in V$ and a scalar $\xi \in \mathbb{R}$, a vector $z \in V$ is said to be (ξ, y)-*separable* on I_1 and I_2 with respect to e, if

$$\begin{aligned} \langle z - \xi y, e_i\rangle \ge 0 \ \text{ for } i \in I_1, \ \text{ and} \\ \langle z - \xi y, e_j\rangle \le 0 \ \text{ for } j \in I_2 \end{aligned} \tag{5}$$

(see Niezgoda (2006)).

It is easily seen that z is (ξ, y)-separable on I_1 and I_2 w.r.t. e if and only if

$$\frac{\langle z, e_j\rangle}{\langle y, e_j\rangle} \le \xi \le \frac{\langle z, e_i\rangle}{\langle y, e_i\rangle} \quad \text{for } i \in I_1 \text{ and } j \in I_2,$$

whenever the above denominators are positive.

For example, if

$$\frac{\langle z, e_1\rangle}{\langle y, e_1\rangle} \ge \frac{\langle z, e_2\rangle}{\langle y, e_2\rangle} \ge \dots \ge \frac{\langle z, e_n\rangle}{\langle y, e_n\rangle} \tag{6}$$

(with positive denominators), then for each $m \in \{1, 2, \dots, n\}$, the vector z is (ξ, y)-separable on $I_1 = \{1, 2, \dots, m\}$ and $I_2 = \{m + 1, \dots, n\}$ w.r.t. e, for any number ξ between $\frac{\langle z, e_m\rangle}{\langle y, e_m\rangle}$ and $\frac{\langle z, e_{m+1}\rangle}{\langle y, e_{m+1}\rangle}$.

Let $e = (e_1, \dots, e_n) \in V^n$ and $d = (d_1, \dots, d_n) \in V^n$. For given vectors $y, v \in V$ and scalars $\xi, \mu \in \mathbb{R}$, two vectors $z, x \in V$ are said to be *similarly separable* with respect to $(\xi, y, e; \mu, v, d)$ if there exist index sets I_1 and I_2 with $I_1 \cup I_2 = I$, where $I = \{1, 2, \dots, n\}$, such that

(i) z is (ξ, y)-separable on I_1 and I_2 with respect to e,

(ii) x is (μ, v)-separable on I_1 and I_2 with respect to d.

In the problem of deriving the right-hand side of Hayashi type inequalities, the similar separability of vectors plays an important role, as shown in the following result.

Lemma 1 (see (Niezgoda, 2006, Theorem 3.5))**.** *Let x, y, z, $v \in V$ and $0 < A \in \mathbb{R}$ be such that $\langle z, v\rangle > 0$, and*

$$\langle y, v\rangle = A\ \langle z, v\rangle. \tag{7}$$

Assume that there exist dual bases $e = (e_1, e_2, \dots, e_n)$ and $d = (d_1, d_2, \dots, d_n)$ of a finite-dimensional subspace V_0 in V, and there exist index sets I_1 and I_2 with $I_1 \cup I_2 = I = \{1, 2, \dots, n\}$ such that

(i) *z is $(\frac{1}{A}, y)$-separable on I_1 and I_2 w.r.t. e, that is*

$$\begin{aligned} A\langle z, e_i\rangle \ge \langle y, e_i\rangle \quad \text{for } i \in I_1, \text{ and} \\ A\langle z, e_j\rangle \le \langle y, e_j\rangle \quad \text{for } j \in I_2, \end{aligned} \tag{8}$$

(ii) *x is (ξ, v)-separable on I_1 and I_2 w.r.t. d for some $\xi \in \mathbb{R}$, that is*

$$\begin{aligned} \langle x, d_i\rangle \ge \xi\langle v, d_i\rangle \quad \text{for } i \in I_1, \text{ and} \\ \langle x, d_j\rangle \le \xi\langle v, d_j\rangle \quad \text{for } j \in I_2. \end{aligned} \tag{9}$$

Then the following inequality holds:

$$\langle x, y\rangle \le A\ \langle x, z\rangle. \tag{10}$$

Proof. Apply (Niezgoda, 2006, Theorem 3.5). ■

Observe that the statements **(i)**-**(ii)** in Lemma 1 say that the vectors z, x are similarly separable on I_1 and I_2 w.r.t. $(\frac{1}{A}, y, e; \xi, v, d)$.

In the sequel we will relax the restriction (7) to the form (11) (see also Theorem A and Theorem C with $A = 1$). This will allow to obtain a refinement of (10) involving an error measuring the deviation of the non-equality case (11) from the equality case (7).

Separability and Steffensen-Hayashi type inequalities

In the first part of this section, we aim to provide sufficient conditions for refinements (14)-(15) of one-sided Steffensen-Hayashi type inequality (10) to hold. Here our interest is only on the right-hand side of the double S-H inequalities. The corresponding left-hand side inequalities can be obtained as consequences of the right-hand versions.

Theorem 1. *Let* $z, y, v \in V$ *and* $0 < A \in \mathbb{R}$ *be such that* $\langle z, v\rangle > 0$, *and*

$$\langle y, v\rangle \le A\ \langle z, v\rangle. \tag{11}$$

Let $s_1, s_2, \dots, s_n \in V$ *with* $V_0 = \operatorname{span}\{s_1, s_2, \dots, s_n\}$ *and* $v \in V_0$. *Suppose that there exist a basis* $e = (e_1, e_2, \dots, e_n)$ *of* V_0 *and index sets* I_1 *and* I_2 *with* $I_1 \cup I_2 = I = \{1, 2, \dots, n\}$, $I_1 \cap I_2 = \emptyset$, *such that*

(i)

$$0 \le_{C_1} s_i \text{ and } s_j \le_{C_2} v \text{ for } i \in I_1 \text{ and } j \in I_2, \tag{12}$$

where $C_1 = \operatorname{cone}\{e_i : i \in I_1\}$ *and* $C_2 = \operatorname{cone}\{e_j : j \in I_2\}$,

(ii) z *is* $(\frac{1}{A}, y)$*-separable on* I_1 *and* I_2 *w.r.t.* e, *that is*

$$\begin{aligned} \langle y, e_i\rangle \le A\langle z, e_i\rangle \quad \text{for } i \in I_1, \text{ and} \\ A\langle z, e_j\rangle \le \langle y, e_j\rangle \quad \text{for } j \in I_2. \end{aligned} \tag{13}$$

Then the following inequalities hold:

(A).

$$\begin{aligned} \langle x, y\rangle &\le A\ \langle x, z\rangle - (A\langle z, v\rangle - \langle y, v\rangle) \sum_{j\in I_2} t_j \\ &\le A\ \langle x, z\rangle \end{aligned} \tag{14}$$

for all $x \in D = \operatorname{cone}\{s_1, s_2, \dots, s_n\}$, *where* $x = \sum_{i=1}^{n} t_i s_i$ *with* $t_i \ge 0$, $i = 1, 2, \dots, n$,

(B).

$$\begin{aligned} \langle x, y\rangle &\le A\ \langle x, z\rangle - \sum_{i\in I\setminus I_x^+} t_i \langle s_i, Az - y\rangle \\ &- (A\langle z, v\rangle - \langle y, v\rangle) \sum_{j\in I_2 \cap I_x^+} t_j \\ &\le A\ \langle x, z\rangle - \sum_{i\in I\setminus I_x^+} t_i \langle s_i, Az - y\rangle \end{aligned} \tag{15}$$

for all $x \in V_0 = \operatorname{span}\{s_1, s_2, \dots, s_n\}$, *where* $x = \sum_{i=1}^{n} t_i s_i$ *with* $t_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, *and* I_x^+ *is a subset of* $\{i \in I : t_i \ge 0\}$.

In (15), if $I \setminus I_x^+ = \emptyset$, *then the first and last sums* $\sum_{i\in I\setminus I_x^+} (\dots)$ *are absent. Likewise, if* $I_2 \cap I_x^+ = \emptyset$ *then the middle term* $(\dots) \sum_{i\in I_2\cap I_x^+} (\dots)$ *is absent.*

Proof. **(A).** From (13) we get

$$\langle Az - y, e_i\rangle \ge 0 \quad \text{for } i \in I_1 \tag{16}$$

and

$$\langle Az - y, e_j\rangle \le 0 \quad \text{for } j \in I_2. \tag{17}$$

On account of (12) we have

$$s_i = \sum_{k\in I_1} \alpha_{ik} e_k \quad \text{for } i \in I_1, \tag{18}$$

and

$$v - s_j = \sum_{l\in I_2} \beta_{jl} e_l \quad \text{for } j \in I_2, \tag{19}$$

for some $\alpha_{ik} \ge 0$ $(i, k \in I_1)$ and $\beta_{jl} \ge 0$ $(j, l \in I_2)$, respectively.

It follows from (16) and (18) that

$$\langle Az - y, s_i\rangle \ge 0 \quad \text{for } i \in I_1. \tag{20}$$

Furthermore,

$$\langle Az - y, s_j\rangle \ge 0 \quad \text{for } j \in I_2. \tag{21}$$

In fact, by (19), (17) and (11), for $j \in I_2$ we obtain

$$\begin{aligned} \langle Az - y, s_j\rangle &= \langle Az - y, v - \sum_{l\in I_2} \beta_{jl} e_l\rangle \\ &= \langle Az - y, v\rangle - \sum_{l\in I_2} \beta_{jl} \langle Az - y, e_l\rangle \\ &\ge \langle Az - y, v\rangle \ge 0, \end{aligned} \tag{22}$$

completing the proof of (21).

Now, fix any $x \in D = \operatorname{cone}\{s_1, s_2, \dots s_n\}$. So, we get $x = \sum_{i=1}^{n} t_i s_i$ for some scalars $t_i \ge 0$, $i = 1, 2, \dots, n$. By employing (20) and (22) we find that

$$\langle Az - y, t_i s_i\rangle \ge 0 \quad \text{for } i \in I_1,$$

$$\langle Az - y, t_j s_j\rangle \ge t_j \langle Az - y, v\rangle \quad \text{for } j \in I_2.$$

Therefore we have

$$\begin{aligned} \langle Az - y, x\rangle &= \langle Az - y, \sum_{i=1}^{n} t_i s_i\rangle \\ &= \sum_{i\in I_1} \langle Az - y, t_i s_i\rangle + \sum_{j\in I_2} \langle Az - y, t_j s_j\rangle \\ &\ge \langle Az - y, v\rangle \sum_{j\in I_2} t_j, \end{aligned}$$

which proves the first inequality in (14).

The second inequality in (14) follows readily from (11), because $t_j \ge 0$, $j \in I_2$.

(B). We shall prove (15).

We take any $x \in V_0 = \operatorname{span}\{s_1, s_2, \ldots, s_n\}$. So, we have $x = \sum_{i=1}^{n} t_i s_i$ for some scalars $t_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$.

Now, we introduce

$$\widetilde{x} = \sum_{i \in I_x^+} t_i s_i.$$

It is clear that

$$\widetilde{x} = x - \sum_{i \in I \setminus I_x^+} t_i s_i = \sum_{i \in I} \widetilde{t_i} s_i,$$

where $\widetilde{t_i} = t_i$ for $i \in I_x^+$, and $\widetilde{t_i} = 0$ for $i \in I \setminus I_x^+$. Thus $\widetilde{x} \in D = \operatorname{cone}\{s_1, s_2, \ldots, s_n\}$. By making use of (14) for $\widetilde{x}$ and $\widetilde{t_i}$ in place of x and t_i, respectively, we establish

$$\langle \widetilde{x}, y \rangle \le A \langle \widetilde{x}, z \rangle - (A\langle z, v \rangle - \langle y, v \rangle) \sum_{j \in I_2} \widetilde{t_j} \le A \langle \widetilde{x}, z \rangle.$$

Consequently,

$$\begin{aligned} \langle x - \sum_{i \in I \setminus I_x^+} t_i s_i, y \rangle &\le A \langle x - \sum_{i \in I \setminus I_x^+} t_i s_i, z \rangle \\ &- (A\langle z, v \rangle - \langle y, v \rangle) \sum_{j \in I_2 \cap I_x^+} t_j \\ &\le A \langle x - \sum_{i \in I \setminus I_x^+} t_i s_i, z \rangle, \end{aligned}$$

and further

$$\begin{aligned} \langle x, y \rangle &\le A \langle x, z \rangle - \sum_{i \in I \setminus I_x^+} t_i \langle s_i, Az - y \rangle \\ &- (A\langle z, v \rangle - \langle y, v \rangle) \sum_{j \in I_2 \cap I_x^+} t_j \\ &\le A \langle x, z \rangle - \sum_{i \in I \setminus I_x^+} t_i \langle s_i, Az - y \rangle, \end{aligned}$$

which gives the required result (15). ■

We now discuss some simplifications, specializations and corollaries to Theorem 1.

Remark 1. If $I_1 = \{1, 2, \ldots, m\}$ and $I_2 = \{m+1, m+2, \ldots, n\}$ for some $m \in I = \{1, 2, \ldots, n\}$, and $v = s_n$ with

$$s_i = e_1 + e_2 + \ldots + e_i, \quad i = 1, 2, \ldots, n, \tag{23}$$

then the assumption **(i)** in Theorem 1 is automatically satisfied. Therefore it does not have to be specified. (See Corollaries 1-2 for details.)

Remark 2. In Theorem 1, if $x \in V_0$ is such that $I_x^+ = \{1, 2, \ldots, n\}$ then the statement **(B)** reduces to the statement **(A)**.

Remark 3. In Theorem 1, if in addition the vectors $s_1, s_2, \ldots, s_n$ are linearly independent, then $s = (s_1, s_2, \ldots, s_n)$ is a basis of V_0. In consequence, there exists the *dual basis* $r = (r_1, r_2, \ldots, r_n)$ of V_0 satisfying $\langle s_i, r_j \rangle = \delta_{ij}$ (Kronecker delta), $i, j = 1, 2, \ldots, n$. So, the coefficients t_i in the representation of $x \in V_0$ has the form $t_i = \langle x, r_i \rangle$, $i = 1, 2, \ldots, n$. This formula will be used extensively in the last section.

In the next theorem we present two-sided estimations for $\langle x, y \rangle$.

Theorem 2. *Let $z, y, w, v \in V$ and $0 < A \in \mathbb{R}$ be such that $\langle z, v \rangle > 0$, $\langle w, v \rangle > 0$, and*

$$A \langle w, v \rangle \le \langle y, v \rangle \le A \langle z, v \rangle, \tag{24}$$

Let $s_1, s_2, \ldots, s_n \in V$ with $V_0 = \operatorname{span}\{s_1, s_2, \ldots, s_n\}$ and $v \in V_0$. Suppose that there exist a basis $e = (e_1, e_2, \ldots, e_n)$ of V_0 and index sets I_1 and I_2 with $I_1 \cup I_2 = I$ and $I_1 \cap I_2 = \emptyset$ such that

(i)

$$0 \le_{C_1} s_i \quad \text{and} \quad s_j \le_{C_2} v \quad \text{for } i \in I_1 \text{ and } j \in I_2, \tag{25}$$

where $C_1 = \operatorname{cone}\{e_i : i \in I_1\}$ and $C_2 = \operatorname{cone}\{e_j : j \in I_2\}$,

(ii) *z is $(\frac{1}{A}, y)$-separable on I_1 and I_2 w.r.t. e, that is*

$$\begin{aligned} &\langle y, e_i \rangle \le A\langle z, e_i \rangle \quad \text{for } i \in I_1, \text{ and} \\ &A\langle z, e_j \rangle \le \langle y, e_j \rangle \quad \text{for } j \in I_2, \end{aligned} \tag{26}$$

(iii) *w is $(\frac{1}{A}, y)$-separable on I_2 and I_1 w.r.t. e, that is,*

$$\begin{aligned} &\langle y, e_j \rangle \le A\langle w, e_j \rangle \quad \text{for } j \in I_2, \text{ and} \\ &A\langle w, e_i \rangle \le \langle y, e_i \rangle \quad \text{for } i \in I_1. \end{aligned} \tag{27}$$

Then the following inequalities hold:

(A).

$$\begin{aligned} &A \langle x, w \rangle + (\langle y, v \rangle - A \langle w, v \rangle) \sum_{j \in I_2} t_j \\ \le\ &\langle x, y \rangle \\ \le\ &A \langle x, z \rangle - (A \langle z, v \rangle - \langle y, v \rangle) \sum_{j \in I_2} t_j \end{aligned} \tag{28}$$

for all $x \in D = \operatorname{cone}\{s_1, s_2, \ldots, s_n\}$, where $x = \sum_{i=1}^{n} t_i s_i$ with $t_i \ge 0$, $i = 1, 2, \ldots, n$.

(B).

$$\begin{aligned} \langle x, y \rangle &\le A \langle x, z \rangle - \sum_{i \in I \setminus I_x^+} t_i \langle s_i, Az - y \rangle \\ &- (A \langle z, v \rangle - \langle y, v \rangle) \sum_{j \in I_2 \cap I_x^+} t_j \\ &\le A \langle x, z \rangle - \sum_{i \in I \setminus I_x^+} t_i \langle s_i, Az - y \rangle, \end{aligned} \tag{29}$$

$$\begin{aligned}\langle x,y\rangle &\geq A\,\langle x,w\rangle + \sum_{i\in I\setminus I_x^+} t_i\langle s_i, y-Aw\rangle \\ &+ (\langle y,v\rangle - A\,\langle w,v\rangle)\sum_{j\in I_2\cap I_x^+} t_j \qquad (30)\\ &\geq A\,\langle x,w\rangle + \sum_{i\in I\setminus I_x^+} t_i\langle s_i, y-Aw\rangle\end{aligned}$$

for all $x\in V_0=\operatorname{span}\{s_1,s_2,\ldots,s_n\}$, *where* $x=\sum_{i=1}^n t_is_i$ *with* $t_i\in\mathbb{R}$, $i=1,2,\ldots,n$, *and* I_x^+ *is a subset of* $\{i\in I: t_i\geq 0\}$.

Proof. **(A).** The latter inequality of (28) is a direct consequence of (14) in Theorem 1, part **(A)**.

To prove the former inequality of (28), we define

$$\widetilde{y}=Av-y \quad\text{and}\quad \widetilde{z}=v-w. \qquad (31)$$

Then the first inequality of (24) gives (11) for $\widetilde{y}$ and $\widetilde{z}$ instead of y and z, respectively. Likewise, (27) implies (13) for $\widetilde{y}$ and $\widetilde{z}$ in place of y and z, respectively. So, by making use of inequality (14) in Theorem 1, we infer that

$$\langle x,\widetilde{y}\rangle \leq A\langle x,\widetilde{z}\rangle - (A\langle\widetilde{z},v\rangle - \langle\widetilde{y},v\rangle)\sum_{j\in I_2} t_j \qquad (32)$$

for each $x\in D=\operatorname{cone}\{s_1,s_2,\ldots,s_n\}$ with $x=\sum_{i=1}^n t_is_i$, $t_i\geq 0$. By standard algebra, from (32) via (31) we get the former inequality of (28), as desired.

(B). The first inequality of (29) follows directly from (15) in Theorem 1, part **(B)**. The second inequality of (29) is a consequence of the first and (24).

In order to show the first inequality of (30), it is sufficient to employ (15) in Theorem 1, part **(B)**, applied to $\widetilde{y}$ and $\widetilde{z}$ given by (31). Then direct computation yields the required result. The second inequality of (30) is a consequence of the first and (24). ■

To see how to construct the required vectors s_i, $i=1,2,\ldots,n$, satisfying condition (12) in Theorem 1, with

$$I_1=\{1,2,\ldots,m\} \quad\text{and}\quad I_2=\{m+1,m+2,\ldots,n\}, \qquad (33)$$

the reader is referred to (Niezgoda, 2012, Lemma 2.5). In particular, we have

Corollary 1. *Let* $z,y,v\in V$ *and* $0<A\in\mathbb{R}$ *be such that* $\langle z,v\rangle>0$, *and*

$$\langle y,v\rangle\leq A\,\langle z,v\rangle. \qquad (34)$$

Let $s_1,s_2,\ldots,s_n\in V$ *with* $V_0=\operatorname{span}\{s_1,s_2,\ldots,s_n\}$ *and* $v\in V_0$. *Suppose that there exist a basis* $e=(e_1,e_2,\ldots,e_n)$ *of* V_0 *and index sets* I_1 *and* I_2 *of the form (33) for some* $m\in I=\{1,2,\ldots,n\}$ *such that*

(i)

$$\begin{pmatrix} s_1\\ \vdots\\ s_n\end{pmatrix}=\begin{pmatrix}\mathcal{A} & 0\\ \mathcal{K} & \mathcal{B}\end{pmatrix}\begin{pmatrix} e_1\\ \vdots\\ e_n\end{pmatrix} \qquad (35)$$

where $\mathcal{A}=(\alpha_{ik})$ $(i,k=1,2,\ldots,m)$ *is an* $m\times m$ *matrix with nonnegative entries, and* $\mathcal{B}=(\beta_{jl})$ $(j,l=m+1,m+2,\ldots,n)$ *is an* $(n-m)\times(n-m)$ *matrix, and* $\mathcal{K}$ *is an* $(n-m)\times m$ *matrix with all rows equal to* $(\gamma_1,\gamma_2,\ldots,\gamma_m)$ *for some* $\gamma_1,\gamma_2,\ldots,\gamma_m\in\mathbb{R}$, *and*

$$v=\gamma_1e_1+\gamma_2e_2+\ldots+\gamma_me_m+\delta_{m+1}e_{m+1}+\ldots+\delta_ne_n, \qquad (36)$$

$$\delta_l\geq\beta_{jl}\ \text{for}\ j,l=m+1,m+2,\ldots,n, \qquad (37)$$

(ii) z *is* $(\frac{1}{A},y)$*-separable on* I_1 *and* I_2 *w.r.t.* e, *that is*

$$\begin{aligned}\langle y,e_i\rangle\leq A\langle z,e_i\rangle \quad &\text{for } i\in I_1, \text{ and}\\ A\langle z,e_j\rangle\leq\langle y,e_j\rangle \quad &\text{for } j\in I_2. \qquad (38)\end{aligned}$$

Then the statements **(A)** *and* **(B)** *of Theorem 1 hold true.*

Proof. By (Niezgoda, 2012, Lemma 2.5), conditions (35)-(37) imply that condition (12) of Theorem 1 is fulfilled. It is now sufficient to apply Theorem 1. ■

In order to give an illustration of the above conditions (35)-(37), notice that for vectors $e_1,\ldots,e_n\in V$ and $s_1,\ldots,s_n\in V$ related by $v=s_n$ and

$$s_i=e_1+e_2+\ldots+e_i,\quad i=1,2,\ldots,n, \qquad (39)$$

(see Remark 1), we have

$$\begin{pmatrix} s_1\\ \vdots\\ s_n\end{pmatrix}=\begin{pmatrix} 1&0&0&\ldots&0&0\\ 1&1&0&\ldots&0&0\\ \vdots&\vdots&\ddots&\ldots&\vdots&\vdots\\ 1&1&1&\ldots&1&0\\ 1&1&1&\ldots&1&1\end{pmatrix}\begin{pmatrix} e_1\\ \vdots\\ e_n\end{pmatrix} \qquad (40)$$

with the triangular matrices $\mathcal{A}$ and $\mathcal{B}$ of sizes $m\times m$ and $(n-m)\times(n-m)$, respectively, and the $(n-m)\times m$ matrix $\mathcal{K}$ of all ones.

Corollary 2. *Let* $z,y,v\in V$ *and* $0<A\in\mathbb{R}$ *be such that* $\langle z,v\rangle>0$, *and*

$$\langle y,v\rangle\leq A\,\langle z,v\rangle. \qquad (41)$$

Let $s_1,s_2,\ldots,s_n\in V$ *with* $V_0=\operatorname{span}\{s_1,s_2,\ldots,s_n\}$ *and* $v\in V_0$. *Suppose that there exists a basis* $e=(e_1,e_2,\ldots,e_n)$ *of* V_0 *such that*

(i) *the equation (39) holds true with* $v=s_n$,

(ii)

$$\begin{aligned}\frac{\langle z,e_1\rangle}{\langle y,e_1\rangle} &\geq \frac{\langle z,e_2\rangle}{\langle y,e_2\rangle} \geq \ldots \geq \frac{\langle z,e_m\rangle}{\langle y,e_m\rangle}\\ &\geq \frac{1}{A}\\ &\geq \frac{\langle z,e_{m+1}\rangle}{\langle y,e_{m+1}\rangle} \geq \ldots \geq \frac{\langle z,e_n\rangle}{\langle y,e_n\rangle}\end{aligned}$$

with positive denominators.

Then the statements **(A)** *and* **(B)** *of Theorem 1 hold true.*

Proof. Apply Corollary 1 and Theorem 1 with condition (6). ■

The idea of simplification of Theorem 1 by the usage of (39)-(40) will be applied extensively in the next section.

Applications

Analysis of Theorems C and D

In this subsection we analyze inequalities (3)-(4) of Theorems C and D, respectively, from the point of view of Theorem 1.

To this end we consider $V = V_0 = \mathbb{R}^n$ with standard inner product $\langle\cdot,\cdot\rangle$ and standard orthonormal basis $e = (e_1, \ldots, e_n)$. Let $s_1, \ldots, s_n \in \mathbb{R}^n$ be defined by (23) (see Remark 1). That is,

$$s_i = (\underbrace{1,\ldots,1}_{i\ times},0,\ldots,0) \in \mathbb{R}^n, \quad i \in I = \{1,2,\ldots,n\}.$$

Because $(s_1, \ldots, s_n)$ is a basis of $\mathbb{R}^n$, there is the dual basis $(r_1, \ldots, r_n)$ of $\mathbb{R}^n$ defined by

$$r_i = (\underbrace{0,\ldots,0}_{i-1\ times},1,-1,0,\ldots,0) \in \mathbb{R}^n,\ i = 1,2,\ldots,n-1,$$

$$r_n = (\underbrace{0,\ldots,0}_{n-1\ times},1).$$

Since for any $x \in \mathbb{R}^n$,

$$x = \sum_{i=1}^{n} \langle x, r_i\rangle s_i,$$

we have

$$t_i = \langle x, r_i\rangle \quad \text{for } i = 1,2,\ldots,n$$

(see Remark 3).

We put $v = s_n$, $z = s_{k_1}$ with $k_1 \in I$ and $A = 1$. Next, we set $I_1 = \{1,2,\ldots,k_1\}$ and $I_2 = \{k_1+1,\ldots,n\}$.

(A). We shall show that Evard-Gauchman's double inequality (3) (see Theorem C) is a special case of Theorem 1, part **A**, with $A = 1$.

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$ and $0 \leq y_1, y_2, \ldots, y_n \leq 1$. Let $k_2 \leq \sum\limits_{i=1}^{n} y_i \leq 1$ with integers $0 \leq k_2 \leq k_1 \leq n$. Then

$$t_i = \langle x, r_i\rangle \geq 0 \quad \text{for } i = 1,2,\ldots,n.$$

Therefore we have $I_x^+ = I$ and $I \setminus I_x^+ = \emptyset$.

We take $v = s_n$, $z = s_{k_1}$, and $I_1 = \{1,2,\ldots,k_1\}$ and $I_2 = \{k_1+1,\ldots,n\}$. Then conditions (11), (12) and (13) are fulfilled. It is not hard to verify that the second inequality of (3) in Theorem C follows from the outer inequality of (14) in Theorem 1, part **(A)**.

To show the first inequality in (3) it is sufficient to consider $\widetilde{y} = s_n - y$, $z = s_{n-k_2}$ and $I_1 = \{1,2,\ldots,n-k_2\}$ and $I_2 = \{n-k_2+1,\ldots,n\}$. In this case condition (11), (12) and (13) are satisfied for $\widetilde{y}$ and $\widetilde{z}$ in place of y and z, respectively. Making use of the outer inequality of (14), applied to $v, \widetilde{y}$ and $\widetilde{z}$, leads to the first inequality of (3), as required.

(B). We shall derive Shi-Wu's double inequality (4) (see Theorem D) as a consequence of Theorem 1, part **B**, with $A = 1$.

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_1 \geq x_2 \geq \ldots \geq x_n$ and $0 \leq y_1, y_2, \ldots, y_n \leq 1$. Let $k_2 \leq \sum\limits_{i=1}^{n} y_i \leq k_1$ with integers $0 \leq k_2 \leq k_1 \leq n$. Then

$$\begin{aligned} t_i &= \langle x, r_i\rangle \geq 0 \quad \text{for } i = 1,2,\ldots,n-1, \text{ and}\\ t_n &= \langle x, r_n\rangle = x_n.\end{aligned}$$

Hence $I_x^+ = \{1,2,\ldots,n-1\}$ and $I \setminus I_x^+ = \{n\}$.

It is easily seen that for $v = s_n$, $z = s_{k_1}$, and $I_1 = \{1,2,\ldots,k_1\}$ and $I_2 = \{k_1+1,\ldots,n\}$ conditions (11)-(13) are met. In this situation, the outer inequality of (15) in Theorem 1, part **(B)**, reduces to the second inequality of (4) in Theorem D.

To prove the first inequality of (4) we define $v = s_n$, $\widetilde{y} = s_n - y$, $\widetilde{z} = s_{n-k_2}$, and $I_1 = \{1,2,\ldots,n-k_2\}$ and $I_2 = \{n-k_2+1,\ldots,n\}$. In this case, conditions (11)-(13) hold true for $\widetilde{y}$ and $\widetilde{z}$ instead of y and z, respectively. By using direct computation, it now follows that the outer inequality of (15), applied to $v, \widetilde{y}$ and $\widetilde{z}$, becomes the first inequality of (4), as claimed.

Refinements of inequalities (3) - (4)

By specifying Theorem 1 in a similar manner as in the previous subsection, we can obtain some refinements of the inequalities in Theorems C and D.

Namely, let $x_1, x_2, \ldots, x_n \in \mathbb{R}$, $x_{n+1} = 0$, $0 \leq y_i \leq A$, $i = 1,2,\ldots,n$, and $Ak_2 \leq \sum\limits_{i=1}^{n} y_i \leq Ak_1$ with integers $0 \leq k_2 \leq k_1 \leq n$.

It is not hard to check that for the vectors $v = s_n$ and $z = s_{k_1}$ and the index sets $I_1 = \{1,\ldots,k_1\}$ and $I_2 = \{k_1+1,\ldots,n\}$, Theorem 1, part **(B)**, gives the following.

If $x_1 \geq x_2 \geq \ldots \geq x_m$ for some $2 \leq m \leq n+1$, then $I_x^+ = \{1, 2, \ldots, m-1\}$ and

$$\sum_{i=1}^{n} x_i y_i \leq A\sum_{i=1}^{k_1} x_i - \sum_{i=m}^{n}(x_i - x_{i+1})c_i - \left(Ak_1 - \sum_{i=1}^{n} y_i\right)(x_{k_1+1} - x_m), \quad (42)$$

where

$$c_i = Ak_1 - \sum_{k=1}^{i} y_k \quad \text{for } i \geq m \geq k_1 + 2.$$

In (42), if $m = n+1$, then the middle term $\sum_{i=m}^{n} \ldots$ is absent. Likewise, if $m < k_1 + 2$ then the last term $(\ldots)(\ldots)$ is absent.

Furthermore, inequality (42) can be extended, since the hypothesis $0 \leq y_i \leq A$, $i = 1, \ldots n$, can be weakened to the form

$$y_i \leq A \quad \text{for } i = 1, \ldots, k_1, \text{ and} \qquad 0 \leq y_i \quad \text{for } i = k_1 + 1, \ldots n, \quad (43)$$

(see condition (13)).

By utilizing Theorem 1, part **(B)**, for the vectors $v = s_n$, $z = s_{n-k_2}$, and $\widetilde{y} = As_n - y$ in place of y, with index sets $I_1 = \{1, \ldots, n-k_2\}$ and $I_2 = \{n-k_2+1, \ldots, n\}$, we establish the following inequality

$$\sum_{i=1}^{n} x_i y_i \geq A\sum_{i=n-k_2+1}^{n} x_i + \sum_{i=m}^{n}(x_i - x_{i+1})d_i + \left(\sum_{i=1}^{n} y_i - Ak_2\right)(x_{n-k_2+1} - x_m), \quad (44)$$

where

$$d_i = A(n-k_2) - \sum_{k=1}^{i}(A - y_k) \quad \text{for } i \geq m \geq n - k_2 + 2.$$

In (44), if $m = n+1$, then the middle term $\sum_{i=m}^{n} \ldots$ is absent. Moreover, if $k_2 < n - m + 2$ then the last term $(\ldots)(\ldots)$ is absent.

Also, inequality (44) can be extended, because the hypothesis $0 \leq y_i \leq A$, $i = 1, \ldots n$, can be weakened to the form

$$y_i \leq A \quad \text{for } i = 1, \ldots, n-k_2, \text{ and} \qquad 0 \leq y_i \quad \text{for } i = n-k_2+1, \ldots n \quad (45)$$

(see condition (13)). In particular, if $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$ then $m = n+1$ and $I_x^+ = \{1, 2, \ldots, n\}$ and

$$\sum_{i=1}^{n} x_i y_i \leq A\sum_{i=1}^{k_1} x_i - \left(Ak_1 - \sum_{i=1}^{n} y_i\right)(x_{k_1+1} - x_{n+1}), \quad (46)$$

$$\sum_{i=1}^{n} x_i y_i \geq A\sum_{i=n-k_2+1}^{n} x_i + \left(\sum_{i=1}^{n} y_i - Ak_2\right)(x_{n-k_2+1} - x_{n+1}). \quad (47)$$

These are the mentioned refinements of the inequality (3) due to Evard and Gauchman (see Theorem C).

If $x_1 \geq x_2 \geq \ldots \geq x_n$ then $I_x^+ = \{1, 2, \ldots, n-1\}$ and

$$\sum_{i=1}^{n} x_i y_i \leq A\sum_{i=1}^{k_1} x_i - \sum_{i=n}^{n}(x_i - x_{i+1})c_i - \left(Ak_1 - \sum_{i=1}^{n} y_i\right)(x_{k_1+1} - x_n), \quad (48)$$

$$\sum_{i=1}^{n} x_i y_i \geq A\sum_{i=n-k_2+1}^{n} x_i + \sum_{i=n}^{n}(x_i - x_{i+1})d_i + \left(\sum_{i=1}^{n} y_i - Ak_2\right)(x_{n-k_2+1} - x_n). \quad (49)$$

These are the mentioned refinements of the inequality (4) of Shi and Wu (see Theorem D).

If $x_1 \geq x_2 \geq \ldots \geq x_{n-1}$ then $I_x^+ = \{1, \ldots, n-2\}$ and

$$\sum_{i=1}^{n} x_i y_i \leq A\sum_{i=1}^{k_1} x_i - \sum_{i=n-1}^{n}(x_i - x_{i+1})c_i - \left(Ak_1 - \sum_{i=1}^{n} y_i\right)(x_{k_1+1} - x_{n-1}), \quad (50)$$

$$\sum_{i=1}^{n} x_i y_i \geq A\sum_{i=n-k_2+1}^{n} x_i + \sum_{i=n-1}^{n}(x_i - x_{i+1})d_i + \left(\sum_{i=1}^{n} y_i - Ak_2\right)(x_{n-k_2+1} - x_{n-1}). \quad (51)$$

Bounding expectation of a discrete random variable

Steffensen and Hayashi type inequalities (28)-(30) (see Theorem 1) can be applied to estimating expectations of random variables (cf. Balakrishnan and Rychlik (2006); Gajek and Okolewski (2001)).

Let X be a random variable taking values $x_1, x_2, \ldots, x_n$ with probabilities $p_i = y_i$, $0 \leq p_i \leq 1$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} p_i = 1$, respectively. We also define $x_{n+1} = 0$.

For $V = \mathbb{R}^n$, we consider the vectors

$$e_i = (\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, 1, 0, \ldots, 0) \quad \text{for } i = 1, 2, \ldots, n,$$

$$s_i = (\underbrace{1, \ldots, 1}_{i \text{ times}}, 0, \ldots, 0) \quad \text{for } i = 1, 2, \ldots, n,$$

$$r_i = (\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, 1, -1, 0, \ldots, 0) \quad \text{for } i = 1, 2, \ldots, n-1,$$

$$r_n = (\underbrace{0, \ldots, 0}_{n-1 \text{ times}}, 1).$$

By making use of the results of the previous subsection the expectation

$$EX = \sum_{i=1}^{n} x_i y_i = \langle x, y \rangle$$

of the random variable X can be bounded as follows.

We denote $A = \max\limits_{i=1,\ldots,n} y_i$. Let $0 \leq k_2 \leq k_1 \leq n$ be integers satisfying $k_2 \leq \frac{1}{A} \leq k_1$. For $2 \leq m \leq n+1$ we define

$$c_i = Ak_1 - \sum_{k=1}^{i} y_k \quad \text{for } i \geq m \geq k_1 + 2,$$

$$d_i = A(n-k_2) - \sum_{k=1}^{i} (A - y_k) \quad \text{for } i \geq m \geq n - k_2 + 2.$$

If $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$ then $m = n+1$ and

$$EX \leq A \sum_{i=1}^{k_1} x_i - (Ak_1 - 1)(x_{k_1+1} - x_{n+1}), \tag{52}$$

$$EX \geq A \sum_{i=n-k_2+1}^{n} x_i + (1 - Ak_2)(x_{n-k_2+1} - x_{n+1}). \tag{53}$$

If $x_1 \geq x_2 \geq \ldots \geq x_n$ then $m = n$ and

$$\begin{aligned} EX &\leq A \sum_{i=1}^{k_1} x_i - \sum_{i=n}^{n} (x_i - x_{i+1}) c_i \\ &- (Ak_1 - 1)(x_{k_1+1} - x_n), \end{aligned} \tag{54}$$

$$\begin{aligned} EX &\geq A \sum_{i=n-k_2+1}^{n} x_i + \sum_{i=n}^{n} (x_i - x_{i+1}) d_i \\ &+ (1 - Ak_2)(x_{n-k_2+1} - x_n). \end{aligned} \tag{55}$$

If $x_1 \geq x_2 \geq \ldots \geq x_{n-1}$ then $m = n-1$ and

$$\begin{aligned} EX &\leq A \sum_{i=1}^{k_1} x_i - \sum_{i=n-1}^{n} (x_i - x_{i+1}) c_i \\ &- (Ak_1 - 1)(x_{k_1+1} - x_{n-1}), \end{aligned} \tag{56}$$

$$\begin{aligned} EX \geq A &\sum_{i=n-k_2+1}^{n} x_i + \sum_{i=n-1}^{n} (x_i - x_{i+1}) d_i \\ &+ (1 - Ak_2)(x_{n-k_2+1} - x_{n-1}). \end{aligned} \tag{57}$$

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