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A short proof of an inverse eigenvalue problem for Jacobi matrices

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A short proof for an inverse eigenvalue problem on alternating sign eigenvalues for Jacobi matrices is provided.

Introduction

The main goal of an inverse eigenvalue problem is to reconstruct a matrix with certain specific structure from the knowledge of its eigenvalues. The case of Jacobi matrices has been thoroughly scrutinized by the linear algebra community, producing a rich variety of interesting results.

It is widely known that a Jacobi matrix is uniquely determined by its eigenvalues and those of the largest leading principal submatrix (Elsner and Hershkowitz (2003); Gesztesy and Simon (1997); Hochstadt (1967, 1974); Holtz (2005); Lu and Sun (1999); Shieh (2004), just to name few). This result is frequently attributed to Harry Hochstadt Hochstadt (1974), but was evidently first discovered by Burton Wendroff in 1960 Wendroff (1961). In fact, since early 1930 the general theoretical and practical properties of the zeros of the orthogonal polynomials have been extensively investigated by many authors Favard (1935); Shohat (1937); Szegő (1939). In 1960, Wendroff establishes and gives, according to Jean Favard, a "simple et élégant" proof of the following result of orthogonal polynomials theory, here translated to matrices.

Theorem 1 (Wendroff's Theorem)**.** *Let* $\{\lambda_1, \ldots, \lambda_n\}$ *and* $\{\mu_1, \ldots, \mu_{n-1}\}$ *be two sets of real numbers such that* $\lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n$. *Then there is a unique Jacobi matrix whose eigenvalues are* $\{\lambda_i\}$ *and such that the eigenvalues of its largest leading principal submatrix are* $\{\mu_i\}$.

The converse of Wendroff's theorem is the well-known separation theorem for orthogonal polynomials (Szegő, 1939, Theorem 3.3.2). It seems that many authors working on matrix analysis have not been aware of the relevance Wendroff's result and the intimate connection between Jacobi matrices and orthogonal polynomials.

Theorem 1 has been used in da Fonseca and Petronilho (2004) to prove a result on the spectra of close-to-Schwarz matrices presented in Elsner and Hershkowitz (2003), and the original version on orthogonal polynomials has been thoroughly studied by many (see Beardon and Driver (2005); Brezinski, Driver, and Redivo-Zaglia (2004); Joulak (2005), just to cite few).

The aim of this short note is twofold. On the one hand, it should serve a useful purpose in bringing orthogonal polynomials as a powerful tool to the attention of the linear algebra community. One the other hand, it provides a concise proof of the following interesting result obtained in Holtz (2005).

Theorem 2 (Holtz (2005))**.** *There exists a unique Jacobi matrix*

$$T_n = \begin{pmatrix} 0 & b_1 & & & \\ b_1 & \ddots & \ddots & & \\ & \ddots & \ddots & b_{n-2} & \\ & & b_{n-2} & 0 & b_{n-1} \\ & & & b_{n-1} & a \end{pmatrix}, \tag{1}$$

for $a > 0$, *with eigenvalues* $\{\lambda_i\}$ *if and only if*

$$\lambda_1 > -\lambda_2 > \lambda_3 > \cdots > (-)^{n-1}\lambda_n > 0\,. \tag{2}$$

We first observe that this result is originally stated for anti-bidiagonal matrices, but the underlying graph of such matrices is in fact a path da Fonseca (2006a). Note that, if $a < 0$, then (2) comes

$$-\lambda_1 > \lambda_2 > \cdots > (-)^n\lambda_n > 0\,. \tag{3}$$

We also point out that the reconstruction of the matrix (1) depends on a single set of real eigenvalues (2) (or (3)).

The proof we can find in Holtz (2005) uses the Cauchy-Binet formula for the necessity and some intricate sign manipulations on some inequalities, for the sufficiency. It contains a flaw which can be fixed as it was pointed out in (Tyaglov, 2012, Remark 3.7).

Our approach depends on some elementary calculations motivated by Wendroff's Theorem. In fact, it can be seen as

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a corollary of Theorem 2. A simple illustrative example is provided. In the end, we present an alternative proof for the necessity of our approach.

Other proofs appeared recently as a consequence of more elaborated results Bebiano and da Providência (2011, 2012); Tyaglov (2012).

The proof

We start with the necessity. Setting $P_n(x)$ for the characteristic polynomial of T_n, we have

$$P_n(x) = (x-a)P_{n-1}(x) - b_{n-1}^2 P_{n-2}(x) \qquad (4)$$

and

$$P_k(x) = x\,P_{k-1}(x) - b_{k-1}^2 P_{k-2}(x)\,, \quad \text{for } k = n-1, \ldots, 2, \quad (5)$$

with $P_1(x) = x$ and $P_0(x) = 1$. Observe that, for $k < n$, $P_k(x)$ is even whenever k is even; otherwise is odd. All the eigenvalues of T_n can be obtained from the intersection of

$$f(x) = \frac{P_{n-2}(x)}{P_{n-1}(x)} \quad \text{and} \quad g(x) = \frac{x-a}{b_{n-1}^2}$$

da Fonseca (2006b). Taking into account that f is a strictly decreasing function (in each interval where it is defined) (Szegő, 1939, p. 43) and the opposite parity of $P_{n-2}(x)$ and $P_{n-1}(x)$, if $a > 0$, then we get (2).

We turn now to the sufficiency. First, we observe that, from the parities of $P_{n-2}(x)$ and $P_{n-1}(x)$, and from (4), we have

$$P_{n-1}(x) = \frac{(-)^n P_n(-x) - P_n(x)}{2\,a}\,.$$

Therefore $P_{n-1}(x)$ is either even or odd. We point out that λ is a zero of $P_{n-1}(x)$ if and only if $-\lambda$ it is as well. We may apply now Wendroff's algorithm, starting with

$$a = \lambda_1 + \cdots + \lambda_n$$

which is always positive, from (2). Setting

$$\lambda_{\min} = \begin{cases} \lambda_n & \text{if } n \text{ is even} \\ \lambda_{n-1} & \text{if } n \text{ is odd} \end{cases}$$

we get

$$\lambda_{\min} - a < 0\,.$$

Moreover, the polynomial

$$-b_{n-1}^2 R(x) = P_n(x) - (x-a)P_{n-1}(x)$$

is of degree exactly $n-2$. Making $R(x) = P_{n-2}(x)$, we have

$$b_{n-1}^2 = \frac{(\lambda_{\min} - a)P_{n-1}(\lambda_{\min})}{P_{n-2}(\lambda_{\min})} > 0\,.$$

The above procedure is repeated in order to obtain $P_{n-2}(x), \ldots, P_1(x) = x$, getting successively $b_{n-2}, \ldots, b_1$.

Example

As an example, let us construct a Jacobi matrix of the type (1) with eigenvalues $4, -3, 2, -1$. Then, $a = 2$, $P_4(x) = (x-4)(x+3)(x-2)(x+1)$, and $P_3(x) = x(x^2-7)$. Since

$$P_4(x) - (x-2)P_3(x) = -6x^2 + 24\,,$$

we have $b_3 = \sqrt{6}$ and $P_2(x) = x^2 - 4$. Now, we get

$$P_3(x) - xP_2(x) = -3x$$

and, consequently, we have $b_2 = \sqrt{3}$ and $P_1(x) = x$. Finally,

$$P_2(x) - xP_1(x) = -4$$

and $b_1 = 2$. Therefore, the Jacobi matrix we are looking for is

$$\begin{pmatrix} 0 & 2 & & \\ 2 & 0 & \sqrt{3} & \\ & \sqrt{3} & 0 & \sqrt{6} \\ & & \sqrt{6} & 2 \end{pmatrix}.$$

A final remark

In his crucial paper, Shohat Shohat (1937) in 1937 considered some questions for linear combinations of orthogonal polynomials. This subject has been deeply explored. The next result can be seen as a refinement of the particular case of Weyl Theorem (Horn & Johnson, 2013, Theorem 4.3.1) and it was stated in (Brezinski et al., 2004, Theorem 3) or (Joulak, 2005, Theorem 5).

Lemma 1. *Let J_n be the Jacobi matrix of order n*

$$J_n = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{pmatrix},$$

Let us assume that e_n is the column unit vector $(0, \ldots, 0, 1) \in \mathbb{R}^n$ and α is a nonzero real number. Setting $\lambda_1 < \cdots < \lambda_n$, $\mu_1 < \cdots < \mu_{n-1}$, and $\theta_1 < \cdots < \theta_n$ for the eigenvalues of J_n, J_{n-1}, and $J_n + \alpha\, e_n e_n^T$, respectively, then

1. *$\alpha < 0$ if and only if $\lambda_1 < \theta_1 < \mu_1 < \lambda_2 < \theta_2 < \mu_2 < \cdots < \mu_{n-1} < \lambda_n < \theta_n$*
2. *$\alpha > 0$ if and only if $\theta_1 < \lambda_1 < \mu_1 < \theta_2 < \lambda_2 < \mu_2 < \cdots < \mu_{n-1} < \theta_n < \lambda_n$.*

This result allows us to derive immediately the necessity part of Theorem 2.

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