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Solutions of Equations Reducible to the Form $z^n(1-z)^m = d$

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Many equations can be converted to the form $z^n(1-z)^m = d$ with n, m integer. In this paper, we show how this type of equation can be solved efficiently. The main advantage of our method is that the complex roots are indexed by a parameter k, where k must be even if $d > 0$ and odd if $d < 0$. For each k, the roots can then be found by solving a simple equation. We also give narrow bounds for all roots. Though we have no proof, we found experimentally that the complex roots are ordered according to their modulus by k: If $m > 0$, the modulus decreases with k, and if $m < 0$, it increases with k. This allows one to select specific complex roots, something not possible when using standard root-finding algorithms.

Introduction

In this paper, we consider the solution of the following equation for z:

$$z^n(1-z)^m = d. \tag{1}$$

This equation appears in Syski (1960), and it was used to find the waiting time of the $E_n/E_m/1$ queue, a one server queue with Erlang arrivals and Erlang service times. In Grassmann (2011), we provided a solution method to solve this equation for the case that n and m are natural numbers, and that $d > 0$. Here, we generalize this method to deal with any integers n and m, and any real number d.

The main contribution of this paper is a novel method to find the complex roots of (1), which is important since all except at most 4 roots are complex. The complex roots are indexed by a parameter k, where k must be even if $d > 0$ and odd if $d < 0$. For each k, one only needs to solve a simple equation for a single variable. Also, in the over 40'000 cases we observed, the absolute value (modulus) of the solution decreased with k if $m > 0$, and increased with k is $m < 0$. Getting the roots ordered by their modulus is a definite advantage over the methods of Laguerre Kincaid and Cheney (2002) or Newton Press, Flannery, Teukolsky, and Vetterling (1986). Even if the roots would not be ordered, the fact that they are indexed by k is a definite advantage. Specifically, in Newton's method, there is no way to determine to which root the algorithm converges because the domain of attraction is fractal (Kincaid & Cheney, 2002, page 127). In fact, to avoid finding roots already found, it is suggested to deflate the polynomial Kincaid and Cheney (2002), Press et al. (1986), but this leads to polynomials with all coefficients non-zero. As a consequence, the number of operations to find all roots is $O((n+m)^2)$ as opposed to $O(n+m)$ for our method.

Though the problem of solving equations was very active in the past, there is not much recent literature on this topic aside from textbooks. For a review of the older literature, see Kulkarni (2006) and references therein. In this paper, we use polar coordinates, an approach also used by Pukhta (2011).

To justify the importance of equation (1), note that any equation of degree less than 6 can be converted to a similar form by Tschirnhaus transformations (Tschirnhaus (2003), citation from Adamchik and Jeffrey (2003)). Consider, e.g. a polynomial of degree 5, which can be converted to

$$x^5 + px + q = 0.$$

If we set $z = -xp/q$, then this equation can be written as

$$\frac{z^5}{1-z} = -\frac{p^5}{q^4},$$

and this has the form of (1) with $n = 5$ and $m = -1$, provided p and q are real. Also, all Bring radicals with real coefficients, that is, solutions of the equations $x^n - x + c = 0$ can be brought into this form.

As a second example, consider the discrete-time renewal process. In a renewal process, we consider an item, such as a light bulb, which is subject to failure and has to be replaced. If p_i is the probability that the item has a lifetime of i, and r_n is the probability that the item fails at time n, we have:

$$r_n = r_{n-1}p_1 + r_{n-2}p_2 + \ldots.$$

If $p_i = 0$ for $i > b$, and $p_i = q^i/c$, $1 \le i \le b$, with $c = \sum_{i=1}^{b} q^i$, we obtain

$$r_n = r_{n-1}q/c + r_{n-2}q^2/c + \ldots + r_{n-b}q^b/c. \tag{2}$$

This is a difference equation, and we set set $r_n = x^n$. After substituting this expression for r_n and dividing by x^n, we

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obtain for $qx \neq 1$

$$1 = \frac{1}{c}\sum_{i=1}^{b}(q/x)^i = \frac{1}{c}\sum_{i=0}^{b}(q/x)^i - \frac{1}{c} = \frac{1}{c}\frac{1-(q/x)^{b+1}}{1-q/x} - \frac{1}{c}.$$

If $y = q/x$, this yields

$$(c+1)(1-y) = 1 - y^{b+1}$$

or

$$c - (c+1)y = c\left(1 - \frac{c+1}{c}y\right) = -y^{b+1}.$$

Setting $z = \frac{c+1}{c}y$, this yields

$$z^{b+1}(1-z)^{-1} = -(c+1)^{b+1}c^{-b}.$$

Note that if $q = 1$, one obtains the uniform distribution.

Generally, given an equation of the form

$$(c_1 + c_2x)^n(c_3 + c_4x)^m = d_0,$$

we first set $y = c_1 + c_2x$ to obtain $y^n(d_1 - d_2y)^m = d_0$, where $d_1 = c_3 - c_1c_4/c_2$ and $d_2 = c_4/c_2$. We then set $z = \frac{d_2}{d_1}y$, and after minor calculations, this yields

$$z^n(1-z)^m = \frac{d_2^n}{d_1^{n+m}}d_0.$$

To simplify our task, we make the following assumptions. Except for cases where the solution of (1) is trivial, these assumptions do not lead to any loss of generality.

1. $d \neq 0$. If $d = 0$, the only solutions of (1) are 0 and 1.
2. $n + m \neq 0$. If $n + m = 0$, we merely have to solve the binomial equation $\left(\frac{z}{z-1}\right)^n = d$.
3. $n + m > 0$. If $n + m < 0$, we convert the equation to $z^{-n}(1-z)^{-m} = d^{-1}$, and this has the form of (1).
4. $n > m$. If $n < m$, let $y = 1 - z$, and we have $y^m(1-y)^n = d$, which has the form of (1).

Note that our assumptions imply that $n > 0$. On the other hand, m can be positive or negative.

We now first describe how to find the real roots of (1), and then how to find the complex roots.

The Real Roots

Let $f(z) = z^n(1-z)^m$, and note that $f(z)$ has a stationary point at $z_s = \frac{n}{n+m}$. This follows because $f'(z)$, the derivative of $f(z)$, is equal to

$$f'(z) = z^{n-1}(1-z)^{m-1}(n - z(m+n)). \qquad (3)$$

Clearly, z_s is between 0 and 1 if $m > 0$, but if $m < 0$, $z_s > 1$. In principle, we now have to deal with 4 intervals, which are, if $m > 0$, $(-\infty, 0)$, $(0, z_s]$, $[z_s, 1)$ and $(1, \infty)$. If $d = f(z_s)$, there is a double root of $f(z) = d$ at z_s. We associate one root with the interval ending at z_s, and the second one with the interval starting at z_s. Except for the interval $(0, 1)$, we have to consider 4 possibilities if $m > 0$, depending on whether n and m are even or odd. This yields 8 cases, in addition to the cases arising for the interval $(0, 1)$, meaning there are 10 cases to consider for $m > 0$. Another 10 cases arise if $m < 0$. However, we were able to restrict the number of cases to 4 for $m > 0$, and 4 for $m < 0$, as shown by the following theorem.

Theorem 1. *If $z_s = \frac{n}{n+m}$, $u = \sqrt[n+m]{|d|}$ and $m > 0$, then each of the following intervals contains exactly one root of $f(z) = d$, given the conditions as stated, and there are no roots outside these intervals.*

1. The interval $(-u, 0)$ contains exactly one root if $(-1)^n d > 0$.

2. The intervals $(0, z_s]$ contains exactly one root if $0 < d \leq f(z_s)$.

3. The interval $[z_s, 1)$ contains exactly one root if $0 < d \leq f(z_s)$.

4. The interval $(1, u+1)$ contains exactly one root if $(-1)^m d > 0$.

If $m < 0$, only the intervals listed below can contain roots, and under the conditions stated, each interval contains exactly one root.

1. The interval $(a, 0)$, with $a = \min(-1, -2^{-m/(n+m)}u)$ contains exactly one root if $(-1)^n d > 0$.

2. The interval $(0, 1)$ contains exactly one root if $d > 0$.

3. The interval $(1, z_s]$ contains exactly one root if $d \leq f(z_s) < 0$ or $d \geq f(z_s) > 0$.

4. The interval $[z_s, u)$ contains exactly one root if $(-1)^m d \geq f(z_s)$.

Instead of considering the roots of $f(z) = d$, we consider the roots of $|f(z)| = |d|$. This is possible because $f(z) = d$ and $|f(z)| = |d|$ have the same roots as long as $f(z)$ has the same sign as d. We have

$$\begin{aligned} |f(z)| &= (-1)^n f(z) && z < 0 \\ |f(z)| &= f(z) && 0 < z < 1 \\ |f(z)| &= (-1)^m f(z) && z > 1. \end{aligned}$$

Hence, for $z < 0$, $f(z)$ is positive if n is even, and negative if n is odd. It follows that for $z < 0$, $f(z)$ and d have the same sign if $(-1)^n d > 0$, that is, $|f(z)| = |d|$ implies $f(z) = d$ if $d(-1)^n > 0$. A similar argument shows that for $z > 1$, $|f(z)| = |d|$ implies $f(z) = d$ if $(-1)^m d > 0$. Also note that

$$|f(z)| = |z|^n|z-1|^m.$$

The intervals of Theorem 1 are chosen such that $|f(z)|$ is continuous and monotonic within the interval. Because of the intermediate value theorem, the interval (a_1, a_2) contains exactly one roots if $|f(a_1)| < |d| < |f(a_2)|$ or $|f(a_2)| < |d| < |f(a_1)|$. This frequently involves finding a value z such that $|f(z)| > |d|$. To this end, we introduce functions $|g(z)|$ such that $|g(z)| < |f(z)|$ within the interval in question, and we solve $|g(z)| = |d|$. For this value of z, $|f(z)| > |d|$.

We now have for $m > 0$

1. For the interval $(-u, 0)$, we set $|g(z)| = |z|^{n+m}$, which is less than $|f(z)|$ for $z < 0$. Solving $|g(z)| = |d|$ yields $|z| = \sqrt[n+m]{|d|} = u$. This implies that $|f(-u)| > |d|$. Since $f(0) = 0$, there is a root in $(-u, 0)$ if $(-1)^n d > 0$.

2. For the interval $(0, z_s]$, $f(0) = 0$ and $f(z_s) > 0$. Hence, there is a root if $0 < d \leq f(z_s)$

3. For the interval $[z_s, 1)$, $f(z_s) > 0$ and $f(1) = 0$. Hence, there is a root if $0 < d \leq f(z_s)$

4. For the interval $(1, u+1)$, we set $|g(z)| = |z-1|^{n+m}$, which is greater than $|f(z)|$ for $z > 1$. Solving $|g(z)| = |d|$ yields $z = u + 1$. Since $f(1) = 0$, we obtain the result.

If $m < 0$, we have

1. For the interval $(a, 0)$, we use

$$|f(z)| = \left|\frac{z}{z-1}\right|^{-m} |z|^{n+m}.$$

For $z < -1$, $|z/(z-1)| > 1/2$. We can thus set $|g(z)| = (1/2)^{-m}|z|^{n+m} = |d|$, yielding $|z| = 2^{-m/(n+m)}u$. If this is smaller than 1, the interval in question is $(-1, 0)$, otherwise it is $(-2^{-m/(n+m)}u, 0)$

2. For the interval $(0, 1)$, $|f(0)| = 0$, and $|f(z)| \to \infty$ as $z \to 1$, and there is therefore one root in $(0, 1)$ if $d > 0$.

3. For the interval $(1, z_s]$, $|f(z)| \to \infty$ as $z \to 1$ from above, and the lowest value of $|f(z)|$ is $|f(z_s)|$. Hence, there is a root of $|f(z)| = |d|$ if $|d| \geq |f(z_s)|$.

4. For the interval $[z_s, u)$, we set $|g(z)| = |z|^{n+m}$, yielding the bound u. Note that $u > z_s$: if $z_d > 0$ is a solution of $|f(z_d)| = |d|$, $|g(z_d)| < |f(z_d)|$, and we have to increase z to satisfy $|g(z)| = |d|$.

This completes the proof of Theorem 1.

The Complex Roots

Formulas Determining the Complex Roots

First, we prove

Theorem 2. *Equation (1) has no complex double roots.*

To prove this theorem, note that if z^* is a double root, z^* satisfies both $f(z) = d$ and $f'(z) = 0$. Now

$$f'(z) = nz^{n-1}(1-z)^m - mz^n(1-z)^{m-1} = f(z)\left(\frac{n}{z} - \frac{m}{1-z}\right).$$

Unless $f(z) = 0$, the only solution of $f'(z) = 0$ is z_s. Since $d \neq 0$, no root of $f(z) = d$ can satisfy $f(z) = 0$. It follows that there is only a double root if $f(z_s) = d$, and this is the only double root. However, z_s is real. For a different, but less general proof, see Syski (1960).

To find the complex roots, we write

$$\begin{aligned} z &= a + \mathrm{i}b = r_A(\cos A + \mathrm{i}\sin A), \\ 1 - z &= 1 - a - \mathrm{i}b = r_B(\cos B - \mathrm{i}\sin B). \end{aligned} \tag{4}$$

We only need the roots with $b > 0$, because the roots are conjugate complex. If we use the canonical range $(-\pi, \pi)$, then we can assume $0 < A, B < \pi$ because for A negative, $b < 0$. Also, because of (4), $\sin A$ and $\sin B$ must have the same sign, which implies $B > 0$. We now have

Theorem 3. *For any complex root of (1) with $b > 0$, there is an integer k, $-\max(0, m) < k < n$ such that*

$$\frac{\sin^n B \sin^m A}{\sin^{n+m}(A+B)}(-1)^k = d. \tag{5}$$

with

$$B = \frac{nA - k\pi}{m}, \quad A + B = \frac{(n+m)A - k\pi}{m} \tag{6}$$

Moreover, if d is positive, there is only a root of (5) if k is even, and if d is negative, there is only a root of (5) if k is odd. Finally

$$0 < A + B < \pi. \tag{7}$$

To prove the theorem, we use

$$z^n(1-z)^m = r_A^n r_B^m(\cos(nA - mB) + \mathrm{i}\sin(nA - mB)). \tag{8}$$

Since this must be equal to d, which is real, the imaginary term must vanish, that is

$$nA - mB = k\pi, \ k \text{ integer}. \tag{9}$$

Solving this equation for B proves (6). Also, $\cos(nA - mB) = \cos(k\pi) = (-1)^k$. Hence, to prove (5), we only need to show

$$r_A = \frac{\sin B}{\sin(A+B)}, \quad r_B = \frac{\sin A}{\sin(A+B)}. \tag{10}$$

We have (see (4)):

$$\tan A = \frac{b}{a}, \quad \tan B = \frac{b}{1-a}.$$

We solve these two equations for a and b to obtain

$$1 - a = a\frac{\tan A}{\tan B}, \quad a = \frac{\tan B}{\tan A + \tan B}, \quad b = a\tan A. \tag{11}$$

This leads to the following expressions

$$a = \frac{\cos A \sin B}{\sin(A+B)}, \quad b = \frac{\sin A \sin B}{\sin(A+B)}, \quad 1 - a = \frac{\sin A \cos B}{\sin(A+B)}. \tag{12}$$

Since $b > 0$, $\sin(A+B) > 0$, and with $0 < A, B < \pi$, this implies (7). r_A^2 can now be found as follows

$$r_A^2 = a^2 + b^2 = a^2 + a^2\tan^2 A = \frac{a^2}{\cos^2 A},$$

which immediately leads to r_A as given in (10). To find r_B, note that $b = a\tan A$ and $1 - a = a\frac{\tan A}{\tan B}$ implies $b =$

$(1-a)\tan B$. Once this is established, the proof is similar to the one for r_A.

Since (5) can be written as $r_A^n r_B^m = (-1)^k d$, we conclude that $(-1)^k d > 0$, that is, k must be even if $d > 0$ and odd if $d < 0$.

Next, we show

$$-m < k < n \text{ if } m > 0 \tag{13}$$
$$0 < k < n \text{ if } m < 0. \tag{14}$$

The inequality (13) follows immediately from

$$-m\pi < -mB < nA - mB = k\pi < n\pi.$$

If $m < 0$, we have, since $A + B < \pi$:

$$0 < nA - mB = k\pi = (n+m)A - m(A+B) < (n+m)A - m\pi < n\pi.$$

Hence, $0 < k < n$ as claimed. This completes the proof of Theorem 3.

It is convenient to define

$$\phi_k(A) = \frac{sin^n B \sin^m A}{\sin^{n+m}(A+B)},\ B = \frac{nA - k\pi}{m}.$$

Hence, we have to solve $\phi_k(A) = |d|$, where k must be even if $d > 0$, and odd if $d < 0$, and $-\max(m, 0) < k < n$. In the next two sections, we show that the equation $\phi_k(A) = |d|$ has a unique root A_k, and we provide intervals for each A_k. Given A_k, we can find the corresponding values for a, b and z by (11). We denote these values by a_k, b_k and z_k. For our discussion, it is convenient to treat the case $m > 0$ and $m < 0$ separately.

Intervals for A_k if m is Positive

To facilitate the solution $\phi_k(A) = |d|$ for positive m, we find for each k an interval that A must satisfy in order to be a root. We have

Theorem 4. *If $m > 0$, then*

1. If $-m < k < 0$, $\phi_k(A) = |d|$ has exactly one root in the interval from 0 to $\frac{m+k}{n+m}\pi$.

2. If $k = 0$, $\phi_k(A) = |d|$ has exactly one root in the interval from 0 to $\frac{m+k}{n+m}\pi$, provided $|d| > f(z_s)$.

3. For $0 < k < n$, $\phi_k(A) = |d|$ has exactly one root in the interval from $\frac{k}{n}\pi$ to $\frac{m+k}{n+m}\pi$.

For our discussions, we define $\ell_k = \max(0, \frac{k}{n})$. and $u_k = \frac{n+k}{n+m}$. Within the interval (ℓ_k, u_k), $\phi_k(A)$ is continuous, and unless $k = 0$, $\phi_k(A)$ ranges from 0 to ∞, which implies that there is at least one A where $\phi_k(A) = |d|$. There would be exactly one root if $\phi_k(A)$ increases with A. $\phi_k(A)$ approaches 0 as A approaches ℓ_k because either A or B approaches 0. As A approaches u_k, $A + B$ approach π.

If $k = 0$, $\phi_k(A)$ does not approach zero as $A \to 0$. The reason is that A, B and $A + B$ all converge to zero together. In this case, $\sin A \to A$, $\sin B \to B$ and $\sin(A + B) \to A + B$. Consequently

$$\phi_0(A) \approx \frac{B^n A^m}{(A+B)^{n+m}}.$$

For k=0, (6) implies $B = nA/m$ and $A + B = (n + m)A/m$. Using these values, the equation above becomes

$$\lim_{A\to 0} \phi_0(A) = \frac{n^n m^m}{(n+m)^{n+m}} = f(z_s).$$

Suppose now that $\phi_k(A)$ increases with k. In this case, $\phi_k(A)$ ranges from $f(z_s)$ to ∞, and there would be no root for $|d| < f(z_s)$, and one root otherwise.

Instead of proving that $\phi_k(A)$ increases with A, we prove that if $\phi_k(A) = |d|$ has more than one root in the interval (ℓ_k, u_k), we get more than $n+m$ roots for (1), which is impossible. For $k = 0$, the argument is slightly more complicated, but the fact is that we get two real roots in $(0, 1)$ if $|d| \le f(z_s)$ and a complex root z_0 and its conjugate if $|d| > f(z_s)$. Indeed, when $k = 0$, we combine, for the root count, the two real roots in $(0, 1)$ arising when $|d| \le f(z_s)$ with the complex root and its conjugate arising when $|d| > f(z_s)$. We also use the fact that the number of times k satisfying $0 < k < n$ is even is $\frac{n-2}{2}$ if n is even, and $\frac{n-1}{2}$ if n is odd. A similar result applies for the negative k: there are $\frac{m-2}{2}$ even values of k when m is even, and $\frac{m-1}{2}$ values if m is odd. The case of k odd can be dealt with in a similar fashion. The following table shows that the number of roots for the case $d > 0$ (k even) is indeed $n + m$. In this and later tables, we abbreviate even by "e" and odd by "o".

Number of roots of (1) if $d > 0$ and $m > 0$)

n	m	$k > 0$	$k < 0$	$k = 0$ or $z \in (0, 1)$	$z \notin (0, 1)$	Total
e	e	$n-2$	$m-2$	2	2	$n+m$
e	o	$n-2$	$m-1$	2	1	$n+m$
o	e	$n-1$	$m-2$	2	1	$n+m$
o	o	$n-1$	$m-1$	2	0	$n+m$

A similar table can be found for $d < 0$, in which case k must be odd.

Number of roots of (1) if $d < 0$ and $m > 0$

n	m	$k > 0$	$k < 0$	$z \notin (0, 1)$	Total
e	e	n	m	0	$n+m$
e	o	n	$m-1$	1	$n+m$
o	e	$n-1$	m	1	$n+m$
o	o	$n-1$	$m-1$	2	$n+m$

We solved many problems with our method. Among other problems, we found A_k and $|z_k|$, which is of course r_A for $A = A_k$, for n ranging from 3 to 50 and m from 1 to n. We chose d to be $cf(z_s)$, where c ranged from 1/3 to 4/3, and from -4/3 to -1/3. Hence, we looked at roughly 20,000 cases.

In all these 20,000 cases, $|z_k|$ decreased with k. We tried hard to prove that this is always the case, but without success. We could not find any counterexamples either. Our conclusion is that for most practical applications, one can safely assume that $|z_k|$ decreases with k.

We also note that A_k typically increases with k, but there are exceptions. Obviously, if $|d|$ is high, then A_k approaches its upper bound u_k, and since the upper bounds increase with k, the same is true for A_k when $|d|$ is high. A similar result applies when $k > 0$ and $|d|$ is low. On the other hand, A_0 is sometimes less than A_k, $k < 0$. As the reader may prove, this follows from the fact that as A approaches 0 from above, $\phi_0(A)$ approaches $f(z_s)$.

Intervals for A_k if m is Negative

Let us now consider the case $m < 0$. We have

Theorem 5. *If $m < 0$, then:*
1. If $0 < k < -m$, $\phi_k(A) = |d|$ has exactly one root in the interval from 0 to $\frac{k}{n}\pi$.
2. If $k = -m$, $\phi_k(A) = |d|$ has exactly one root in the interval from 0 to $\frac{k}{n}\pi$, provided $|d| < f(z_s)$.
3. For $-m < k < n$, $\phi_k(A) = |d|$ has exactly one root in the interval from $\frac{m+k}{n+m}\pi$ to $\frac{k}{n}\pi$.

The proof is very similar to the proof of Theorem 4, except that $\phi_k(A) \to \infty$ as A converges to its lower limit from above, and for $k \neq -m$, $\phi_k(A)$ converges to 0 as A converges to its upper limit. Also, for $k = -m$, as A converges to 0, so does B and $A + B$, with the result that $\phi_0(A)$ converges to $f(z_s)$.

As before, we have to count the roots to make sure we have n roots. This is done in the tables below.

Number of roots of (1) if $d > 0$ and $m < 0$

n	m	$k \neq -m$	$k = -m$ or $z > 1$	$0 < z < 1$	$z < 0$	Total
e	e	$n-2-2$	2	1	1	n
e	o	$n-2$	0	1	1	n
o	e	$n-1-2$	2	1	0	n
o	o	$n-1$	0	1	0	n

Number of roots of (1) if $d < 0$ and $m < 0$

n	m	$k \neq -m$	$k = -m$ or $z > 1$	$0 < z < 1$	$z < 0$	Total
e	e	n	0	0	0	n
e	o	$n-2$	2	0	0	n
o	e	$n-1$	0	0	1	n
o	o	$n-1-2$	2	0	1	n

We also conducted numerical experiments for the case $m < 0$. In these experiments, n ranged from 3 to 50, and m from $-n + 1$ to -1, and we used the same values for d as in the case where $m > 0$. In all these cases, $|z_k|$ increased with k. Hence, the results come out in ascending order if the program finds the roots in ascending order of k.

We also note that like in the case $m > 0$, the A_k tend to increase with k, but there are exceptions. If $|d|$ is small, A_k will be close to its bound ℓ_k, and ℓ_k increases with k. A similar result applies when $|d|$ is low, provided $k > |m|$.

Numerical Considerations and Conclusions

In this paper, we showed that all roots of the equation $z^n(1 - z)^m = d$ can be found by solving equations in only one real variable which is an angle in the case of complex roots. Also, in all cases, the roots can be restricted to small intervals, which makes it easy to apply Newton's method, a method which requires derivatives. For real roots, we use the derivative of the logarithm of (1) to obtain

$$\frac{f'(z)}{f(z)} = \frac{n}{z} - \frac{m}{1-z},$$

and this can easily be solved for $f'(z)$. If we restrict our search to the appropriate interval, then $f(z) = d$ if and only if $|f(z)| = |d|$, and we can use

$$x_{n+1} = x_n - \frac{f(z) - |d|}{f(z)\left(\frac{n}{z} - \frac{m}{z}\right)}.$$

For solving (5), we find after some calculation

$$\frac{\phi_k'(A)}{\phi_k(A)} = \frac{1}{m}(m^2 \cot A + n^2 \cot B - (n+m)^2 \cot(A+B)).$$

The width of the interval in which the root lies increases with k. We typically needed less than 10 interactions for each root.

Often only certain roots are needed. For instance, in Syski (1960), only the roots for $k < 0$ were needed. Our method allows to find only these roots, and none other. In other cases, only the complex root with the largest or smallest modulus is needed.

In view of the simplicity of our method, the question arises to which extent it can be generalized. For instance, what happens if d is complex? Of course, if $d = r_d(\cos\alpha + \mathrm{i}\sin\alpha)$, we can use our method if there is an integer ν such that $\nu\alpha = \pi$. In this case, we merely have take the νth power on both sides of (1) and apply our method to this new equation. Another interesting question is the following: which polynomials can be converted to the form given by (1)? A more general question is as follows: in our case, each complex root was associated with an integer k, and given a value for k, the problem then becomes to solve an equation depending only on the angle A. Since the Bring form can be reduced to (1), any equation reducible to the Bring form with real coefficients can be parametrized by the integer k. It is an open question what other equations can be parametrized in this fashion.

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