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Some infinite product identities involving balancing and Lucas-balancing numbers

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In this article we establish some product identities for balancing and Lucas-balancing numbers, using the telescoping summation formula and inverse hyperbolic tangent function.

Introduction

The balancing sequence, $(B_n)_{n\geq 1}$ and the Lucas-balancing numbers, $(C_n)_{n\geq 1}$ satisfy the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$ and $C_{n+1} = 6C_n - C_{n-1}$ with the initial values $B_0 = 0, B_1 = 1, C_0 = 1, C_1 = 3$ [see Behera and Panda (1999)]. Both of the sequences have the characteristic equation $x^2 - 6x + 1 = 0$. Hence, for the value $\alpha = 3 + \sqrt{8}$ the n^{th} term of these sequences can be written as

$$B_n = \frac{\alpha^n - \alpha^{-n}}{2\sqrt{8}} \text{ and } C_n = \frac{\alpha^n + \alpha^{-n}}{2}. \tag{1}$$

Melham and Shannon in Melham and Shannon (1995) investigated many inverse trigonometric and hyperbolic summation formulas involving generalized Fibonacci numbers. For instance,

$$\prod_{k=1}^{\infty} \frac{F_{2k+2}+1}{F_{2k+2}-1} = 3 \text{ and } \sum_{n=1}^{\infty} \tanh^{-1}\Big(\frac{1}{F_{2n+2}}\Big) = \frac{\ln 3}{2}.$$

Frontczak in Frontczak (2016) investigated several inverse hyperbolic summation and product identities for Fibonacci and Lucas numbers. The present paper deals with finding product identities for balancing and Lucas-balancing numbers.

Preliminaries

The following is the generalized telescoping summation formula [Basu and Apostol (2000),Equation (2.1)]

$$\sum_{k=1}^{N}[f(k)-f(k+m)] = \sum_{k=1}^{m} f(k) - \sum_{k=1}^{m} f(k+N), \quad \text{for } N \geq m \geq 1 \tag{2}$$

and similarly the alternating telescoping summation formula is

$$\begin{aligned}\sum_{k=1}^{N}&(-1)^{k-1}[f(k)+(-1)^{m-1}f(k+m)] \\ &= \sum_{k=1}^{m}(-1)^{k-1}f(k) + (-1)^{N-1}\sum_{k=1}^{m}(-1)^{k-1}f(k+N).\end{aligned} \tag{3}$$

If $f(N) \to 0$ as $N \to \infty$, then from (2) and (3), we obtain

$$\sum_{k=1}^{\infty}[f(k)-f(k+m)] = \sum_{k=1}^{m} f(k) \tag{4}$$

and

$$\sum_{k=1}^{\infty}(-1)^{k-1}[f(k)+(-1)^{m-1}f(k+m)] = \sum_{k=1}^{m}(-1)^{k-1}f(k). \tag{5}$$

The following product identity is useful:

$$\prod_{k=1}^{m} f(k) = \prod_{k=1}^{\lceil m/2 \rceil} f(2k-1) \prod_{k=1}^{\lfloor m/2 \rfloor} f(2k), \tag{6}$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the floor and ceiling function of x respectively. Moreover,

$$\prod_{k=1}^{2q} f(k) = \prod_{k=1}^{q} f(2k-1)f(2k)$$

and

$$\prod_{k=1}^{2q-1} f(k) = \prod_{k=1}^{q} f(2k-1) \prod_{k=1}^{q-1} f(2k)$$

with the trivial product identity

$$\prod_{k=1}^{0} f(k) = 1.$$

Using (1) it is easy to get the following identities involving balancing and Lucas-balancing numbers [see Panda (2009)].

$$\frac{\alpha^n+1}{\alpha^n-1} = \frac{\sqrt{8}B_n}{C_n-1} = \frac{C_n+1}{\sqrt{8}B_n} \tag{7}$$

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$$B_{2n} = 2B_nC_n \tag{8}$$

$$C_{2n} + 1 = 2C_n^2 \tag{9}$$

$$C_{2n} - 1 = 16B_n^2 \tag{10}$$

The following inverse hyperbolic function identities are very important for proving our main results.

$$\tanh^{-1}x + \tanh^{-1}y = \tanh^{-1}\Big(\frac{x+y}{1+xy}\Big),\ xy < 1 \tag{11}$$

$$\tanh^{-1}x - \tanh^{-1}y = \tanh^{-1}\Big(\frac{x-y}{1-xy}\Big),\ xy > -1 \tag{12}$$

$$\tanh^{-1}\Big(\frac{x}{y}\Big) = \frac{1}{2}\ln\Big(\frac{y+x}{y-x}\Big),\ |x| < |y| \tag{13}$$

The following two lemmas are required for proving some of our main results.

Lemma 1. *For every natural number m and n, the following identities hold.*

(a) $B_nB_{n+m+1} - B_{n+1}B_{n+m} = -B_m$

(b) $C_nC_{n+m+1} - C_{n+1}C_{n+m} = 8B_m$

Proof. We prove (a) only. Using (1) and the fact $\alpha\beta = 1$, we have

$$\begin{aligned}
&B_nB_{n+m+1} - B_{n+1}B_{n+m}\\
&= \frac{\alpha^n-\beta^n}{\alpha-\beta}\cdot\frac{\alpha^{n+m+1}-\beta^{n+m+1}}{\alpha-\beta} - \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\cdot\frac{\alpha^{n+m}-\beta^{n+m}}{\alpha-\beta}\\
&= \frac{1}{(\alpha-\beta)^2}[\alpha^m(\alpha-\frac{1}{\alpha}) - \beta^m(\beta-\frac{1}{\beta})]\\
&= -\frac{\alpha^m-\beta^m}{\alpha-\beta} = -B_m
\end{aligned}$$

The proof of (b) is similar. □

Lemma 2. *For every natural number m and n, the following identities hold.*

(a) $B_{n+m}B_{n+m+1} = 6\sum_{i=1}^{m}(-1)^{i+m}B_{n+i}^2 + (-1)^mB_nB_{n+1}$

(b) $C_{n+m}C_{n+m+1} = 6\sum_{i=1}^{m}(-1)^{i+m}C_{n+i}^2 + (-1)^mC_nC_{n+1}$

Proof. We prove (a) only. The proof of (b) is similar. Our proof is based on mathematical induction on m. Since

$$B_{n+1}B_{n+2} = B_{n+1}(6B_{n+1} - B_n) = 6B_{n+1}^2 - B_nB_{n+1},$$

the identity holds for $m = 1$. Assume that the identity holds for every natural number $m \le k$. That is,

$$B_{n+k}B_{n+k+1} = 6\sum_{i=1}^{k}(-1)^{i+k}B_{n+i}^2 + (-1)^kB_nB_{n+1}.$$

It is sufficient to show that the identity holds for $m = k+1$.

$$\begin{aligned}
B_{n+k+1}B_{n+k+2} &= 6B_{n+k+1}^2 - B_{n+k}B_{n+k+1}\\
&= 6B_{n+k+1}^2 - [6\sum_{i=1}^{k}(-1)^{i+k}B_{n+i}^2 + (-1)^kB_nB_{n+1}]\\
&= 6[B_{n+k+1}^2 - \sum_{i=1}^{k}(-1)^{i+k}B_{n+i}^2] + (-1)^{k+1}B_nB_{n+1}\\
&= 6\sum_{i=1}^{k+1}(-1)^{i+k+1}B_{n+i}^2 + (-1)^{k+1}B_nB_{n+1}.
\end{aligned}$$

□

The following is the main reslut of FrontczakFrontczak (2016):

Lemma 3. *Let $g(x)$ and $h(x)$ be real functions of one variable and let $h(x)$ be composite with $h(x) = h(g(x)) < 1$.*

a) Define $H(x)$ by

$$H(x) = \frac{h(g(x)) - h(g(x+1))}{1 - h(g(x))h(g(x+1))}$$

Then we have

$$\sum_{n=1}^{k}\tanh^{-1}H(n) = \tanh^{-1}h(g(1)) - \tanh^{-1}h(g(k+1))$$

and

$$\sum_{n=1}^{\infty}\tanh^{-1}H(n) = \tanh^{-1}h(g(1)) - \lim_{k+1\to\infty}\tanh^{-1}h(g(k+1))$$

b) Define $H^*(x)$ by

$$H^*(x) = \frac{h(g(x)) + h(g(x+1))}{1 + h(g(x))h(g(x+1))}$$

Then we have

$$\sum_{n=1}^{k}(-1)^{n+1}\tanh^{-1}H^*(n) = \tanh^{-1}h(g(1)) + (-1)^{k+1}\tanh^{-1}h(g(k+1))$$

and

$$\sum_{n=1}^{\infty}(-1)^{n+1}\tanh^{-1}H^*(n) = \tanh^{-1}h(g(1)) + \lim_{k+1\to\infty}(-1)^{k+1}\tanh^{-1}h(g(k+1))$$

Infinite product identities

Theorem 1. *For $q, n \in \mathbb{Z}^+$ the following infinite product identities hold.*

$$\prod_{k=1}^{\infty} \frac{B_{(2n-1)(2k+2q-1)} + B_{(2n-1)(2q-1)}}{B_{(2n-1)(2k+2q-1)} - B_{(2n-1)(2q-1)}} = \frac{1}{(\sqrt{8})^{2q-1}} \prod_{k=1}^{q} \frac{C_{(2n-1)(2k-1)}}{B_{(2n-1)(2k-1)}} \prod_{k=1}^{q-1} \frac{C_{(2n-1)2k}}{B_{(2n-1)2k}} \tag{14}$$

$$\prod_{k=1}^{\infty} \frac{B_{2n(2k+2q-1)} + B_{2n(2q-1)}}{B_{2n(2k+2q-1)} - B_{2n(2q-1)}} = \frac{1}{(\sqrt{8})^{2q-1}} \prod_{k=1}^{2q-1} \frac{C_{2nk}}{B_{2nk}} \tag{15}$$

$$\prod_{k=1}^{\infty} \frac{B_{4n(k+q)} + B_{4nq}}{B_{4n(k+q)} - B_{4nq}} = \frac{1}{8^q} \prod_{k=1}^{2q} \frac{C_{2nk}}{B_{2nk}} \tag{16}$$

and

$$\prod_{k=1}^{\infty} \frac{B_{(2n-1)(2k+2q)} + B_{(2n-1)2q}}{B_{(2n-1)(2k+2q)} - B_{(2n-1)2q}} = \frac{1}{8^q} \prod_{k=1}^{q} \frac{C_{(2n-1)(2k-1)}C_{(2n-1)2k}}{B_{(2n-1)(2k-1)}B_{(2n-1)2k}} \tag{17}$$

Proof. Taking $f(k) = \tanh^{-1}(\alpha^{-2pk})$ in (4) and using (12) and(1), we obtain

$$\sum_{k=1}^{\infty} \tanh^{-1}\Big[\frac{B_{pm}}{B_{p(2k+m)}}\Big] = \sum_{k=1}^{m} \tanh^{-1}\Big[\frac{1}{\alpha^{2pk}}\Big]$$

Converting the infinite sum identity to infinite product identity and employing the identities (7)-(10), we have

$$\prod_{k=1}^{\infty} \frac{B_{p(2k+m)} + B_{pm}}{B_{p(2k+m)} - B_{pm}} = \prod_{k=1}^{m} \frac{C_{2pk} + 1}{\sqrt{8}B_{2pk}} = \frac{1}{(\sqrt{8})^m} \prod_{k=1}^{m} \frac{C_{pk}}{B_{pk}} \tag{18}$$

Now, setting $p = 2n - 1$ and $m = 2q - 1$ in (18) and using (6), we have

$$\prod_{k=1}^{\infty} \frac{B_{(2n-1)(2k+2q-1)} + B_{(2n-1)(2q-1)}}{B_{(2n-1)(2k+2q-1)} - B_{(2n-1)(2q-1)}} = \frac{1}{(\sqrt{8})^{2q-1}} \prod_{k=1}^{2q-1} \frac{C_{(2n-1)k}}{B_{(2n-1)k}} = \frac{1}{(\sqrt{8})^{2q-1}} \prod_{k=1}^{q} \frac{C_{(2n-1)(2k-1)}}{B_{(2n-1)(2k-1)}} \prod_{k=1}^{q-1} \frac{C_{(2n-1)2k}}{B_{(2n-1)2k}}$$

which proves (14).

Setting $p = 2n$ and $m = 2q - 1$ in (18), it is easy to get (15). Similarly, Setting $p = 2n$ and $m = 2q$ in (18) proves (16). Further, $p = 2n-1$ and $m = 2q$ in (18) proves (17). □

Theorem 2. *For $q, n \in \mathbb{Z}^+$ the following infinite product identities hold.*

$$\prod_{k=1}^{\infty} \frac{B_{4n(2k+q-1)} + B_{4nq}}{B_{4n(2k+q-1)} - B_{4nq}} = \frac{1}{(\sqrt{8})^q} \prod_{k=1}^{q} \frac{C_{2n(2k-1)}}{B_{2n(2k-1)}} \tag{19}$$

$$\prod_{k=1}^{\infty} \frac{B_{(4n-2)(2k+q-1)} + B_{(2n-1)2q}}{B_{(4n-2)(2k+q-1)} - B_{(2n-1)2q}} = \frac{1}{(\sqrt{8})^q} \prod_{k=1}^{q} \frac{C_{(2n-1)(2k-1)}}{B_{(2n-1)(2k-1)}} \tag{20}$$

$$\prod_{k=1}^{\infty} \frac{B_{(2n-1)(2k+2q-1)} + B_{(2n-1)2q}}{B_{(2n-1)(2k+2q-1)} - B_{(2n-1)2q}} = \frac{1}{8^q} \prod_{k=1}^{2q} \frac{C_{(2n-1)(2k-1)} + 1}{B_{(2n-1)(2k-1)}} \tag{21}$$

and

$$\prod_{k=1}^{\infty} \frac{B_{2(2n-1)(k+q-1)} + B_{(2n-1)(2q-1)}}{B_{2(2n-1)(k+q-1)} - B_{(2n-1)(2q-1)}} = \frac{1}{(\sqrt{8})^{2q-1}} \prod_{k=1}^{2q-1} \frac{C_{(2n-1)(2k-1)} + 1}{B_{(2n-1)(2k-1)}} \tag{22}$$

Proof. Taking $f(k) = \tanh^{-1}(\alpha^{-p(2k-1)})$ in (4) and using (12) and (1), we obtain

$$\sum_{k=1}^{\infty} \tanh^{-1}\Big[\frac{B_{pm}}{B_{p(2k+m-1)}}\Big] = \sum_{k=1}^{m} \tanh^{-1}\Big[\frac{1}{\alpha^{p(2k-1)}}\Big]$$

Converting the infinite sum identity to infinite product identity, we have

$$\prod_{k=1}^{\infty} \frac{B_{p(2k+m-1)} + B_{pm}}{B_{p(2k+m-1)} - B_{pm}} = \prod_{k=1}^{m} \frac{\alpha^{p(2k-1)} + 1}{\alpha^{p(2k-1)} - 1} \tag{23}$$

Using (6)-(10) in (23) for appropriate choice of p and m, the proof of (19)-(22) follows. □

Theorem 3. *For $q, n \in \mathbb{Z}^+$ the following infinite product identities hold.*

$$\prod_{k=1}^{\infty} \frac{B_{4n(k+q)} + (-1)^{k-1}B_{4nq}}{B_{4n(k+q)} + (-1)^k B_{4nq}} = \prod_{k=1}^{q} \frac{C_{2n(2k-1)}B_{4nk}}{B_{2n(2k-1)}C_{4nk}} \tag{24}$$

$$\prod_{k=1}^{\infty} \frac{B_{(2n-1)(2k+2q)} + (-1)^{k-1}B_{(2n-1)2q}}{B_{(2n-1)(2k+2q-1)} + (-1)^k B_{(2n-1)2q}} = \prod_{k=1}^{q} \frac{C_{(2n-1)(2k-1)}B_{(2n-1)2k}}{B_{(2n-1)(2k-1)}C_{(2n-1)2k}} \tag{25}$$

$$\prod_{k=1}^{\infty} \frac{C_{2n(2k+2q-1)} + (-1)^{k-1}C_{2n(2q-1)}}{C_{2n(2k+2q-1)} + (-1)^k C_{2n(2q-1)}} = \frac{1}{\sqrt{8}} \prod_{k=1}^{q} \frac{C_{2n(2k-1)}}{B_{2n(2k-1)}} \prod_{k=1}^{q-1} \frac{B_{4nk}}{C_{4nk}} \tag{26}$$

and

$$\prod_{k=1}^{\infty}\frac{C_{(2n-1)(2k+2q-1)}+(-1)^{k-1}C_{(2n-1)(2q-1)}}{C_{(2n-1)(2k+2q-1)}+(-1)^{k}C_{(2n-1)(2q-1)}} = \frac{1}{\sqrt{8}}\prod_{k=1}^{q}\frac{C_{(2n-1)(2k-1)}}{B_{(2n-1)(2k-1)}}\prod_{k=1}^{q-1}\frac{B_{(2n-1)2k}}{C_{(2n-1)2k}} \tag{27}$$

Proof. Taking $f(k)=\tanh^{-1}(\alpha^{-2pk})$ in (5), setting $m=2q$ and using (12) and (1) we obtain

$$\sum_{k=1}^{\infty}(-1)^{k-1}\tanh^{-1}\Big[\frac{B_{2pq}}{B_{p(2k+2q)}}\Big]=\sum_{k=1}^{2q}(-1)^{k-1}\tanh^{-1}\Big[\frac{1}{\alpha^{2pk}}\Big]$$

and hence

$$\begin{aligned}\prod_{k=1}^{\infty}\frac{B_{p(2k+2q)}+(-1)^{k-1}B_{2pq}}{B_{p(2k+2q)}+(-1)^{k}B_{2pq}} &= \prod_{k=1}^{2q}\frac{\alpha^{2pk}+(-1)^{k-1}}{\alpha^{2pk}+(-1)^{k}}\\ &= \prod_{k=1}^{q}\frac{\alpha^{2p(2k-1)}+1}{\alpha^{2p(2k-1)}-1}\prod_{k=1}^{q}\frac{\alpha^{4pk}-1}{\alpha^{4pk}+1}\\ &= \prod_{k=1}^{q}\frac{\sqrt{8}B_{2p(2k-1)}}{C_{2p(2k-1)}-1}\frac{\sqrt{8}B_{4pk}}{C_{4pk}+1}\\ &= \prod_{k=1}^{q}\frac{C_{p(2k-1)}}{B_{p(2k-1)}}\frac{B_{2pk}}{C_{2pk}}\end{aligned}$$

Setting $p=2n$ and $p=2n-1$ above, proof of (24) and (25) follows respectively.
Taking $f(k)=\tanh^{-1}(\alpha^{-2pk})$ in (5), setting $m=2q-1$ and using (11) and (1) we obtain

$$\sum_{k=1}^{\infty}(-1)^{k-1}\tanh^{-1}\Big[\frac{C_{p(2q-1)}}{C_{p(2k+2q-1)}}\Big]=\sum_{k=1}^{2q-1}(-1)^{k-1}\tanh^{-1}\Big[\frac{1}{\alpha^{2pk}}\Big]$$

and hence

$$\begin{aligned}&\prod_{k=1}^{\infty}\frac{C_{p(2k+2q-1)}+(-1)^{k-1}C_{p(2q-1)}}{C_{p(2k+2q-1)}+(-1)^{k}C_{p(2q-1)}}\\ &= \prod_{k=1}^{2q-1}\frac{\alpha^{2pk}+(-1)^{k-1}}{\alpha^{2pk}+(-1)^{k}}\\ &= \prod_{k=1}^{q}\frac{\alpha^{2p(2k-1)}+1}{\alpha^{2p(2k-1)}-1}\prod_{k=1}^{q-1}\frac{\alpha^{4pk}-1}{\alpha^{4pk}+1}\\ &= \prod_{k=1}^{q}\frac{\sqrt{8}B_{2p(2k-1)}}{C_{2p(2k-1)}-1}\prod_{k=1}^{q-1}\frac{\sqrt{8}B_{4pk}}{C_{4pk}+1}\\ &= \frac{1}{\sqrt{8}}\prod_{k=1}^{q}\frac{C_{p(2k-1)}}{B_{p(2k-1)}}\prod_{k=1}^{q-1}\frac{B_{2pk}}{C_{2pk}}\end{aligned}$$

Setting $p=2n$ and $p=2n-1$ above, proof of (26) and (27) follows respectively. □

Theorem 4. *For $q,n\in\mathbb{Z}^{+}$ the following infinite product identities hold.*

$$\prod_{k=1}^{\infty}\frac{B_{4n(2k+2q-1)}+(-1)^{k-1}B_{8nq}}{B_{4n(2k+2q-1)}+(-1)^{k}B_{8nq}}=\prod_{k=1}^{q}\frac{C_{2n(4k-3)}B_{2n(4k-1)}}{B_{2n(4k-3)}C_{2n(4k-1)}} \tag{28}$$

$$\prod_{k=1}^{\infty}\frac{B_{(4n-2)(2k+2q-1)}+(-1)^{k-1}B_{(2n-1)4q}}{B_{(4n-2)(2k+2q-1)}+(-1)^{k}B_{(2n-1)4q}} = \prod_{k=1}^{q}\frac{C_{(2n-1)(4k-3)}B_{(2n-1)(4k-1)}}{B_{(2n-1)(4k-3)}C_{(2n-1)(4k-1)}} \tag{29}$$

$$\prod_{k=1}^{\infty}\frac{B_{(2n-1)(2k+2q-1)}+(-1)^{k-1}B_{(2n-1)2q}}{B_{(2n-1)(2k+2q-1)}+(-1)^{k}B_{(2n-1)2q}} = \prod_{k=1}^{q}\frac{(C_{(2n-1)(4k-3)}+1)B_{(2n-1)(4k-1)}}{(C_{(2n-1)(4k-1)}+1)B_{(2n-1)(4k-3)}} \tag{30}$$

$$\prod_{k=1}^{\infty}\frac{C_{8n(k+q-1)}+(-1)^{k-1}C_{4n(2q-1)}}{C_{8n(k+q-1)}+(-1)^{k}C_{4n(2q-1)}} = \frac{1}{\sqrt{8}}\prod_{k=1}^{q}\frac{C_{2n(4k-3)}}{B_{2n(4k-3)}}\prod_{k=1}^{q-1}\frac{B_{2n(4k-1)}}{C_{2n(4k-1)}} \tag{31}$$

$$\prod_{k=1}^{\infty}\frac{C_{(8n-4)(k+q-1)}+(-1)^{k-1}C_{(4n-2)(2q-1)}}{C_{(8n-4)(k+q-1)}+(-1)^{k}C_{(4n-2)(2q-1)}} = \frac{1}{\sqrt{8}}\prod_{k=1}^{q}\frac{C_{(2n-1)(4k-3)}}{B_{(2n-1)(4k-3)}}\prod_{k=1}^{q-1}\frac{B_{(2n-1)(4k-1)}}{C_{(2n-1)(4k-1)}} \tag{32}$$

and

$$\prod_{k=1}^{\infty}\frac{C_{(4n-2)(k+q-1)}+(-1)^{k-1}C_{(2n-1)(2q-1)}}{C_{(4n-2)(k+q-1)}+(-1)^{k}C_{(2n-1)(2q-1)}} = \frac{1}{\sqrt{8}}\prod_{k=1}^{q}\frac{(C_{(2n-1)(4k-3)}+1)}{B_{(2n-1)(4k-3)}}\prod_{k=1}^{q-1}\frac{B_{(2n-1)(4k-1)}}{(C_{(2n-1)(4k-1)}+1)} \tag{33}$$

Proof. Taking $f(k)=\tanh^{-1}(\alpha^{-p(2k-1)})$ in (5), setting $m=2q$ and using (12) and (1), we obtain

$$\sum_{k=1}^{\infty}(-1)^{k-1}\tanh^{-1}\Big[\frac{B_{2pq}}{B_{p(2k+2q-1)}}\Big]=\sum_{k=1}^{2q}(-1)^{k-1}\tanh^{-1}\Big[\frac{1}{\alpha^{p(2k-1)}}\Big] \tag{34}$$

Proof of the identities (28)-(30) follows from (34).
Taking $f(k) = \tanh^{-1}(\alpha^{-p(2k-1)})$ in (5), setting $m = 2q-1$ and using (11) and (1) we obtain

$$\sum_{k=1}^{\infty}(-1)^{k-1}\tanh^{-1}\Big[\frac{C_{p(2q-1)}}{C_{2p(k+q-1)}}\Big] = \sum_{k=1}^{2q-1}(-1)^{k-1}\tanh^{-1}\Big[\frac{1}{\alpha^{p(2k-1)}}\Big] \tag{35}$$

Proof of (31)-(33) follows from (35). □

Theorem 5. *If p, j, t and m are natural numbers, then the following identities are true for balancing numbers.*

$$\prod_{n=1}^{\infty}\frac{t^2B^m_{pn+j}B^m_{p(n+1)+j}-1+t(B^m_{p(n+1)+j}-B^m_{pn+j})}{t^2B^m_{pn+j}B^m_{p(n+1)+j}-1-t(B^m_{p(n+1)+j}-B^m_{pn+j})} = \frac{tB^m_{p+j}+1}{tB^m_{p+j}-1},$$

and

$$\prod_{n=1}^{\infty}\frac{t^2B^m_{pn+j}B^m_{p(n+1)+j}+1+(-1)^{n+1}t(B^m_{p(n+1)+j}+B^m_{pn+j})}{t^2B^m_{pn+j}B^m_{p(n+1)+j}+1-(-1)^{n+1}t(B^m_{p(n+1)+j}+B^m_{pn+j})} = \frac{tB^m_{p+j}+1}{tB^m_{p+j}-1}$$

Replacing B_n by C_n, two such identities can be obtained for the Lucas-balancing numbers.

Proof. Taking $g(n) = B_{pn+j}$ and $h(x) = \frac{1}{tx^m}$ in Lemma 3, we have

$$H(n) = \frac{t(B^m_{p(n+1)+j}-B^m_{pn+j})}{t^2B^m_{pn+j}B^m_{p(n+1)+j}-1}$$

and

$$H^*(n) = \frac{t(B^m_{p(n+1)+j}+B^m_{pn+j})}{t^2B^m_{pn+j}B^m_{p(n+1)+j}+1}.$$

Hence,

$$\sum_{n=1}^{k}\tanh^{-1}H(n) = \frac{1}{2}\ln\Big(\frac{tB^m_{p+j}+1}{tB^m_{p+j}-1}\cdot\frac{tB^m_{p(k+1)+j}+1}{tB^m_{p(k+1)+j}-1}\Big),$$

$$\sum_{n=1}^{k}(-1)^{n+1}\tanh^{-1}H^*(n) = \frac{1}{2}\ln\Big(\frac{tB^m_{p+j}+1}{tB^m_{p+j}-1}\cdot\frac{tB^m_{p(k+1)+j}+(-1)^{k+1}}{tB^m_{p(k+1)+j}-(-1)^{k+1}}\Big),$$

and

$$\sum_{n=1}^{\infty}\tanh^{-1}H(n) = \sum_{n=1}^{\infty}(-1)^{n+1}\tanh^{-1}H^*(n) = \frac{1}{2}\ln\Big(\frac{tB^m_{p+j}+1}{tB^m_{p+j}-1}\Big).$$

Converting the infinite sum identity to infinite product identity, the result follows immediately. □

Corollary 1. *Putting $p = 2, t = m = 1$ in Theorem 5, we have*

$$\frac{B_{j+2}+1}{B_{j+2}-1} = \begin{cases} \prod_{n=1}^{\infty}\frac{B^2_{2n+j+1}-2+2C_{2n+j+1}}{B^2_{2n+j+1}-2-2C_{2n+j+1}} \\ \prod_{n=1}^{\infty}\frac{B_{2n+j+1}+6(-1)^{n+1}}{B_{2n+j+1}-6(-1)^{n+1}} \\ \prod_{n=1}^{\infty}\frac{C^2_{2n+j+1}-17+16C_{2n+j+1}}{C^2_{2n+j+1}-17-16C_{2n+j+1}} \\ \prod_{n=1}^{\infty}\frac{C^2_{2n+j+1}-1+48(-1)^{n+1}B_{2n+j+1}}{C^2_{2n+j+1}-1-48(-1)^{n+1}B_{2n+j+1}} \end{cases}$$

and

$$\frac{C_{j+2}+1}{C_{j+2}-1} = \begin{cases} \prod_{n=1}^{\infty}\frac{C^2_{2n+j+1}+7+16B_{2n+j+1}}{C^2_{2n+j+1}+7-16B_{2n+j+1}} \\ \prod_{n=1}^{\infty}\frac{C^2_{2n+j+1}+9+6(-1)^{n+1}C_{2n+j+1}}{C^2_{2n+j+1}+9-6(-1)^{n+1}C_{2n+j+1}} \\ \prod_{n=1}^{\infty}\frac{B^2_{2n+j+1}+1+2B_{2n+j+1}}{B^2_{2n+j+1}+1-2B_{2n+j+1}} \\ \prod_{n=1}^{\infty}\frac{4B^2_{2n+j+1}+5+3(-1)^{n+1}C_{2n+j+1}}{4B^2_{2n+j+1}+5-3(-1)^{n+1}C_{2n+j+1}}. \end{cases}$$

Theorem 6. *If $\alpha = 3+\sqrt{8}$ and $m, j \in \mathbb{N}$, then*

$$\prod_{n=1}^{\infty}\frac{6m^2\sum_{i=1}^{j}(-1)^{i+j}B^2_{n+i}+((-1)^jm^2-1)B_nB_{n+1}-mB_j}{6m^2\sum_{i=1}^{j}(-1)^{i+j}B^2_{n+i}+((-1)^jm^2-1)B_nB_{n+1}+mB_j} = \frac{mB_{j+1}+1}{mB_{j+1}-1}\cdot\frac{m\alpha^j+1}{m\alpha^j-1} \tag{36}$$

and

$$\prod_{n=1}^{\infty}\frac{6m^2\sum_{i=1}^{j}(-1)^{i+j}C^2_{n+i}+((-1)^jm^2-1)C_nC_{n+1}+8mB_j}{6m^2\sum_{i=1}^{j}(-1)^{i+j}C^2_{n+i}+((-1)^jm^2-1)C_nC_{n+1}-8mB_j} = \frac{mC_{j+1}+1}{mC_{j+1}-1}\cdot\frac{m\alpha^j+1}{m\alpha^j-1} \tag{37}$$

Proof. Let $g(n) = \frac{B_n}{B_{n+j}}$ and $h(x) = \frac{x}{m}$ and observe that, $\lim_{n\to\infty}g(n+1) = \alpha^{-j}$. From Lemma 3, we have

$$H(n) = \frac{m(B_nB_{n+j+1}-B_{n+1}B_{n+j})}{m^2B_{n+j}B_{n+j+1}-B_nB_{n+1}}.$$

Applying Lemma 1 to the numerator and lemma 2 to the denominator of $H(n)$, it is easy to have

$$H(n) = \frac{-mB_j}{6m^2 \sum_{i=1}^{j}(-1)^{i+j}B_{n+i}^2 + ((-1)^j m^2 - 1)B_n B_{n+1}}.$$

Hence,

$$\sum_{n=1}^{k} \tanh^{-1} H(n) = \frac{1}{2}\ln\Big(\frac{mB_{j+1}+1}{mB_{j+1}-1} \cdot \frac{mB_{j+k+1} - B_{k+1}}{mB_{j+k+1} + B_{k+1}}\Big)$$

and

$$\sum_{n=1}^{\infty} \tanh^{-1} H(n) = \frac{1}{2}\ln\Big(\frac{mB_{j+1}+1}{mB_{j+1}-1} \cdot \frac{m\alpha^j - 1}{m\alpha^j + 1}\Big).$$

Converting the infinite sum identity to infinite product identity, (36) can be obtained directly. Taking $g(n) = \frac{C_n}{C_{n+j}}$ and $h(x) = \frac{x}{m}$ in Lemma 3, applying Lemma 1 and Lemma 2 we have

$$H(n) = \frac{8mB_j}{6m^2 \sum_{i=1}^{j}(-1)^{i+j}C_{n+i}^2 + ((-1)^j m^2 - 1)C_n C_{n+1}}$$

and (37) can be obtained similarly. □

Theorem 7. *The following product identity holds for every natural number m and j.*

$$\prod_{n=1}^{\infty} \frac{C_{mn+j}C_{m(n+1)+j} - B_{mn+j}B_{m(n+1)+j} - B_m}{C_{mn+j}C_{m(n+1)+j} - B_{mn+j}B_{m(n+1)+j} + B_m} = \frac{C_{m+j} + B_{m+j}}{C_{m+j} - B_{m+j}} \cdot \frac{\sqrt{8}-1}{\sqrt{8}+1}$$

Proof. Let $g(n) = \frac{B_{mn+j}}{C_{mn+j}}$, $h(x) = x$ and observe that $\lim_{n\to\infty} g(n+1) = \frac{1}{\sqrt{8}}$. From Lemma 3, we have

$$H(n) = \frac{B_{mn+j}C_{m(n+1)+j} - C_{mn+j}B_{m(n+1)+j}}{C_{mn+j}C_{m(n+1)+j} - B_{mn+j}B_{m(n+1)+j}}$$

$$= \frac{-B_m}{C_{mn+j}C_{m(n+1)+j} - B_{mn+j}B_{m(n+1)+j}}$$

and the proof is similar to that of the above theorem. □

References

Basu, A., & Apostol, T. (2000). A new method for investigating euler sums. *The Ramanujan Journal*, *4*(4), 397 - 419.

Behera, A., & Panda, G. (1999). On the square roots of triangular numbers. *The Fibonacci Quarterly*, *73*(2), 98 - 105.

Frontczak, R. (2016). Inverse hyperbolic summations and product identities for fibonacci and lucas numbers. *Applied Mathematical Sciences*, *10*(13), 613 - 623.

Melham, R., & Shannon, A. (1995). Inverse trigonometric and hyperbolic summation formulas involving generalized fibonacci numbers. *The Fibonacci Quarterly*, *33*(1), 32 - 40.

Panda, G. (2009). Some fascinating properties of balancing numbers. *In Proc. of Eleventh Internat. Conference on Fibonacci Numbers and Their Applications, Cong. Numerantium*, *194*, 185 - 189.