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Behavior of solutions of a higher order difference equation

R. Abo-Zeid
Department of Basic Science
The Higher Institute for Engineering & Technology- Al-Obour
Cairo, Egypt

In this paper, we derive the forbidden set and discuss the global behavior of all solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-k}}{B + C\prod_{i=0}^{k} x_{n-i}}, \quad n = 0, 1, \dots,$$

where A, B, C are positive real numbers and the initial conditions $x_{-k}, \dots, x_{-1}, x_0$ are real numbers. This equation was discussed by some authors. Although we have an explicit formula for the solutions of that equation, the global behavior is worth to be discussed.

Introduction

Difference equations, although their forms look very simple, it is extremely difficult to understand thoroughly the global behaviors of their solutions. One can refer to (Agarwal, 1992; Camouzis & Ladas, 2008; Grove & Ladas, 2005; Kocic & Ladas, 1993; Kulenović & Ladas, 2002). The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations.

The aforementioned equation and some special cases attract many authors.

Elabbasy, El-Metwally, & Elsayed (2007) investigated the global stability, boundedness and the periodicity of the positive solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma\prod_{i=0}^{k} x_{n-i}}, \quad n = 0, 1, \dots, \tag{1}$$

with nonnegative real numbers α, β, γ, positive real initial conditions and positive integer k. They introduced an explicit formula of the solutions of equation (1).

They claimed that the positive equilibrium point is locally asymptotically stable for $k \neq 1$ when $\alpha > \beta$. Also they claimed that the positive equilibrium point is a global attractor. But unfortunately, the positive equilibrium point is not locally asymptotically stable for all values of α, β and all values of k. In fact the associated characteristic equation to the linearized equation associated with equation (1) has the root $\frac{\beta}{\alpha}$ and k other roots with modulus 1. Also, the positive equilibrium point is not a global attractor when $\alpha > \beta$, since every solution converges to a $(k + 1)$-periodic solution when $\alpha > \beta$.

Email: abuzead73@yahoo.com

Stević (2012) described the behavior of (well-defined) solutions of the difference equation $x_n = \frac{x_{n-k}}{b+cx_{n-1}\dots x_{n-k}}$, $n = 0, 1, \dots$.

Cinar (2004b; 2004c) obtained and discussed the positive solutions of the rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0, 1, \dots,$$

and

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}, \quad n = 0, 1, \dots,$$

where a, b are positive real numbers. Cinar (2004a) also discussed the behavior of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}}, \quad n = 0, 1, \dots.$$

Stević (2004) showed that every positive solution of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0, 1, \dots,$$

converges to zero.

Aloqeili (2006) investigated the dynamics of the rational difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}, \quad n = 0, 1, \dots,$$

where a is a positive real number.

Andruch-Sobi & Migda (2006) investigated the asymptotic behavior of solutions of the equation

$$x_{n+1} = \frac{ax_{n-1}}{b + cx_n x_{n-1}}, \quad n = 0, 1, \dots,$$

with positive parameters a and c, negative parameter b and nonnegative initial conditions.

They also used the explicit formula for the solutions of the equation

$$x_{n+1} = \frac{ax_{n-1}}{b + cx_n x_{n-1}}, \quad n = 0, 1, \ldots,$$

with positive parameters and nonnegative initial conditions in investigating their behavior (Andruch-Sobi & Migda, 2009).

Sedaghat (2009) determined the global behavior of all solutions of the rational difference equations

$$x_{n+1} = \frac{ax_{n-1}}{x_n x_{n-1} + b}, \text{ and } x_{n+1} = \frac{ax_n x_{n-1}}{x_n + bx_{n-2}}, \quad n = 0, 1, \ldots,$$

where $a, b > 0$.

Bajo & Liz (2011) studied the global behavior of the second-order nonlinear difference equation

$$x_{n+1} = \frac{x_{n-1}}{a + bx_n x_{n-1}}, \quad n = 0, 1, \ldots,$$

with real parameters a, b and real initial conditions.

Khalaf-Allah (2009) investigated the behavior and periodic nature of the two difference equations

$$x_{n+1} = \frac{x_{n-2}}{\pm 1 + x_n x_{n-1} x_{n-2}}, \quad n = 0, 1, \ldots.$$

Cinar, Karatas, & Yalcinkaya (2007) studied the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{-1 + x_n x_{n-1} x_{n-2} x_{n-3}}, \quad n = 0, 1, \ldots.$$

R. Karatas & Cinar (2007) studied the solutions of the difference equation

$$x_{n+1} = \frac{ax_{n-(2k+2)}}{-a + \prod_{i=0}^{2k+2} x_{n-i}}, \quad n = 0, 1, \ldots,$$

with real initial conditions, positive real number a and positive integer k.

C. Karatas & Yalcinkaya (2011) studied the solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-(2k+1)}}{-A + \prod_{i=0}^{2k+1} x_{n-i}}, \quad n = 0, 1, \ldots,$$

with real initial conditions, positive real number A and positive integer k.

In this paper, we discuss the global behavior of all solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-k}}{B + C\prod_{i=0}^{k} x_{n-i}}, \quad n = 0, 1, \ldots, \tag{2}$$

where A, B, C are positive real numbers and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are real numbers.

Linearized stability and solutions of equation (2)

In this section we introduce an explicit formula for the solutions of the difference equation (2) and study their linearized stabilities.

To consider all solutions of equation (2), we determine the forbidden set, which is the set of all initial points $(x_{-k}, x_{-k+1}, \ldots, x_0)$ of equation (2) such that their corresponding solutions are not well-defined.

It is true that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of equation (2) with initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0$ such that $x_{-k} \ldots x_{-1}x_0 = 0$, then the solution $\{x_n\}_{n=-k}^{\infty}$ is well-defined.

Now suppose that $x_{-i} \neq 0$, for all $i \in \{0, 1, \ldots, k\}$. We multiply both sides of equation (2) by $x_n x_{n-1} \ldots x_{n-k+1}$ and substitute

$$x_n x_{n-1} \ldots x_{n-k} = \frac{1}{v_n},$$

we obtain the first order nonhomogeneous equation

$$v_{n+1} = \frac{B}{A}v_n + \frac{C}{A}, \quad v_0 = \frac{1}{x_0 x_{-1} \ldots x_{-k}}. \tag{3}$$

It is clear that the mapping $h(x) = \frac{B}{A}x + \frac{C}{A}$ is invertible and its inverse is $h^{-1}(x) = \frac{A}{B}x - \frac{C}{B}$.
We try to deduce the forbidden set of equation (2).

For, suppose that we start from an initial point $(x_{-k}, \ldots, x_{-1}, x_0)$ such that $x_{-k} \ldots x_{-1}x_0 = -\frac{B}{C}$. The backward orbits $v_n = \frac{1}{x_n x_{n-1} \ldots x_{n-k}}$ satisfy the difference equation

$$v_n = h^{-1}(v_{n-1}) = \frac{A}{B}v_{n-1} - \frac{C}{B}$$

with

$$v_0 = \frac{1}{x_0 x_{-1} \ldots x_{-k}} = -\frac{C}{B},$$

then we obtain

$$v_n = \frac{1}{x_n x_{n-1} \ldots x_{n-k}} = h^{-n}(v_0) = -\frac{C}{B}\sum_{l=0}^{n}(\frac{A}{B})^l.$$

That is

$$x_n x_{n-1} \ldots x_{n-k} = -\frac{B}{C\sum_{l=0}^{n}(\frac{A}{B})^l}.$$

On the other hand, we can observe that if we start from an initial point $(x_{-k}, \ldots, x_{-1}, x_0)$ such that

$$x_{-k} \ldots x_{-1}x_0 = -\frac{B}{C\sum_{l=0}^{n_0}(\frac{A}{B})^l}$$

for some $n_0 \in \mathbb{N}$, then according to equation (3) we obtain

$$v_{n_0} = \frac{1}{x_{n_0} x_{n_0-1} \ldots x_{n_0-k}} = -\frac{C}{B}.$$

This implies that $B + Cx_{n_0} x_{n_0-1} \ldots x_{n_0-k} = 0$.

Therefore, x_{n_0+1} is undefined.

These observations lead us to conclude the following result.

Proposition 1. *The forbidden set F_1 of equation (2) is*

$$F_1 = \bigcup_{n=0}^{\infty}\{(u_0, u_1, \ldots, u_k) : \prod_{i=0}^{k} u_i = -\frac{B}{C\sum_{l=0}^{n}(\frac{A}{B})^l}\}.$$

It is clear that the forbidden set F_1 is contained entirely in the interiors of the 2^k orthants (a quadrant in 2-dimensional Euclidean space or an octant in 3-dimensional Euclidean space) of $\mathbf{R}^{k+1}$. These orthants are of the form

$$\{(u_0, u_1, \ldots, u_k) : \prod_{i=0}^{k} u_i < 0\}.$$

Now assume that $x_{-i} = 0$, for some but not all $i \in \{0, 1, \ldots, k\}$. Then

$$x_n = \begin{cases} (\frac{A}{B})^{\frac{n-1}{k+1}+1} x_{-k}, & n = 1, k+2, 2k+3, \ldots \\ (\frac{A}{B})^{\frac{n-2}{k+1}+1} x_{-k+1}, & n = 2, k+3, 2k+4, \ldots \\ \cdots \\ \cdots \\ \cdots \\ (\frac{A}{B})^{\frac{n-k}{k+1}+1} x_{-1}, & n = k, 2k+1, 3k+2, \ldots \\ (\frac{A}{B})^{\frac{n-k-1}{k+1}+1} x_0, & n = k+1, 2k+2, 3k+3, \ldots \end{cases} \quad (4)$$

Theorem 1. *Let $x_{-k}, x_{-k+1}, \ldots, x_{-1}$ and x_0 be real numbers such that*

$$\mu = x_{-k}x_{-k+1}\ldots x_{-1}x_0 \neq -\frac{B}{C\sum_{l=0}^{n}(\frac{A}{B})^l}$$

for any $n \in N$. Then the solution $\{x_n\}_{n=-k}^{\infty}$ of equation (2) is

$$x_n = \begin{cases} x_{-k}A^{\frac{n-1}{k+1}+1}\prod_{j=0}^{\frac{n-1}{k+1}} \frac{B^{(k+1)j}+\frac{C}{A}A^{(k+1)j}\mu\sum_{l=0}^{(k+1)j-1}(\frac{B}{A})^l}{B^{(k+1)j+1}+\frac{C}{A}A^{(k+1)j+1}\mu\sum_{l=0}^{(k+1)j}(\frac{B}{A})^l}, & n = 1, k+2, 2k+3, \ldots \\ x_{-k+1}A^{\frac{n-2}{k+1}+1}\prod_{j=0}^{\frac{n-2}{k+1}} \frac{B^{(k+1)j+1}+\frac{C}{A}A^{(k+1)j+1}\mu\sum_{l=0}^{(k+1)j}(\frac{B}{A})^l}{B^{(k+1)j+2}+\frac{C}{A}A^{(k+1)j+2}\mu\sum_{l=0}^{(k+1)j+1}(\frac{B}{A})^l}, & n = 2, k+3, 2k+4, \ldots \\ \cdots \\ \cdots \\ \cdots \\ x_{-1}A^{\frac{n-k}{k+1}+1}\prod_{j=0}^{\frac{n-k}{k+1}} \frac{B^{(k+1)j+k-1}+\frac{C}{A}A^{(k+1)j+k-1}\mu\sum_{l=0}^{(k+1)j+k-2}(\frac{B}{A})^l}{B^{(k+1)j+k}+\frac{C}{A}A^{(k+1)j+k}\mu\sum_{l=0}^{(k+1)j+k-1}(\frac{B}{A})^l}, & n = k, 2k+1, 3k+2, \ldots, \\ x_0A^{\frac{n-k-1}{k+1}+1}\prod_{j=0}^{\frac{n-k-1}{k+1}} \frac{B^{(k+1)j+k}+\frac{C}{A}A^{(k+1)j+k}\mu\sum_{l=0}^{(k+1)j+k-1}(\frac{B}{A})^l}{B^{(k+1)j+k+1}+\frac{C}{A}A^{(k+1)j+k+1}\mu\sum_{l=0}^{(k+1)j+k}(\frac{B}{A})^l}, & n = k+1, 2k+2, 3k+3, \ldots \end{cases} \quad (5)$$

It is convenient to reduce the parameters of (2).

The change of variables $\sqrt[k+1]{\frac{C}{B}}x_n = y_n$ reduces equation (2) to the equation

$$y_{n+1} = \frac{ry_{n-k}}{1+\prod_{i=0}^{k} y_{n-i}}, \quad n = 0, 1, \ldots, \quad (6)$$

where $r = \frac{A}{B}$.
We will deal with equation (6) rather than equation (2).

To start navigating the global behavior of the difference equation (6), we classify the nontrivial solutions of equation (6) into two types of solutions:

- Solutions with initial points $(y_{-k}, y_{-k+1}, \ldots, y_0)$ such that $y_{-i} = 0$, for some but not all $i \in \{0, 1, \ldots, k\}$.
- Solutions with initial points $(y_{-k}, y_{-k+1}, \ldots, y_0)$ such that $y_{-i} \neq 0$, for all $i \in \{0, 1, \ldots, k\}$.

These two types of solutions exhibit a global behavior different from each other.

Suppose that $y_{-i} \neq 0$, for all $i \in \{0, 1, \ldots, k\}$. From equation (6) and using the substitution $t_n = \frac{1}{y_ny_{n-1}\ldots y_{n-k}}$, we can obtain the linear nonhomogeneous difference equation

$$t_{n+1} = \frac{1}{r}t_n + \frac{1}{r}, \quad t_0 = \frac{1}{y_0y_{-1}\ldots y_{-k}}. \quad (7)$$

Proposition 2. *The forbidden set F_2 of equation (6) is*

$$F_2 = \bigcup_{n=0}^{\infty}\{(u_0, u_1, \ldots, u_k) : \prod_{i=0}^{k} u_i = -\frac{1}{\sum_{l=0}^{n} r^l}\}.$$

Theorem 2. *Let $y_{-k}, y_{-k+1}, \ldots, y_{-1}$ and y_0 be real numbers such that*

$$\alpha = y_{-k}y_{-k+1}\ldots y_{-1}y_0 \neq -\frac{1}{\sum_{l=0}^{n} r^l}$$

for any $n \in N$. Then the solution $\{y_n\}_{n=-k}^{\infty}$ of equation (6) is

$$y_n = \begin{cases} y_{-k}r^{\frac{n-1}{k+1}+1}\prod_{j=0}^{\frac{n-1}{k+1}} \frac{1+\alpha\sum_{l=0}^{(k+1)j-1} r^l}{1+\alpha\sum_{l=0}^{(k+1)j} r^l}, & n = 1, k+2, 2k+3, \ldots, \\ y_{-k+1}r^{\frac{n-2}{k+1}+1}\prod_{j=0}^{\frac{n-2}{k+1}} \frac{1+\alpha\sum_{l=0}^{(k+1)j} r^l}{1+\alpha\sum_{l=0}^{(k+1)j+1} r^l}, & n = 2, k+3, 2k+4, \ldots, \\ \cdots \\ \cdots \\ \cdots \\ y_{-1}r^{\frac{n-k}{k+1}+1}\prod_{j=0}^{\frac{n-k}{k+1}} \frac{1+\alpha\sum_{l=0}^{(k+1)j+k-2} r^l}{1+\alpha\sum_{l=0}^{(k+1)j+k-1} r^l}, & n = k, 2k+1, 3k+2, \ldots, \\ y_0r^{\frac{n-k-1}{k+1}+1}\prod_{j=0}^{\frac{n-k-1}{k+1}} \frac{1+\alpha\sum_{l=0}^{(k+1)j+k-1} r^l}{1+\alpha\sum_{l=0}^{(k+1)j+k} r^l}, & n = k+1, 2k+2, 3k+3, \ldots \end{cases} \quad (8)$$

Corollary 1. *Assume that $r > 1$ and let $\{y_n\}_{n=-k}^{\infty}$ be a non-trivial solution of equation (6). If $\alpha = y_{-k}y_{-k+1}\ldots y_{-1}y_0 = 0$ and*

$$\alpha = y_{-k}y_{-k+1}\ldots y_{-1}y_0 \neq -\frac{1}{\sum_{l=0}^{n} r^l}$$

for any $n \in N$, then the solution $\{y_n\}_{n=-k}^{\infty}$ is unbounded.

Corollary 2. *Assume that $r = 1$ and*

$$\alpha = y_{-k}y_{-k+1}\ldots y_{-1}y_0 \neq -\frac{1}{n+1}$$

for any $n \in N$. Then the solution $\{y_n\}_{n=-k}^{\infty}$ of equation (6) is

$$y_n = \begin{cases} y_{-k}\prod_{j=0}^{\frac{n-1}{k+1}} \frac{1+(k+1)j\alpha}{1+((k+1)j+1)\alpha}, & n = 1, k+2, 2k+3, \ldots, \\ y_{-k+1}\prod_{j=0}^{\frac{n-2}{k+1}} \frac{1+((k+1)j+1)\alpha}{1+((k+1)j+2)\alpha}, & n = 2, k+3, 2k+4, \ldots, \\ \cdots \\ \cdots \\ y_{-1}\prod_{j=0}^{\frac{n-k}{k+1}} \frac{1+((k+1)j+k-1)\alpha}{1+((k+1)j+k)\alpha}, & n = k, 2k+1, 3k+2, \ldots, \\ y_0\prod_{j=0}^{\frac{n-k-1}{k+1}} \frac{1+((k+1)j+k)\alpha}{1+((k+1)j+k+1)\alpha}, & n = k+1, 2k+2, 3k+3, \ldots \end{cases} \quad (9)$$

We end this section with the discussion of the local stability of the equilibrium points of equation (6).

We give some preliminaries which will be needed in the remainder of this section.

Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{10}$$

where $f : R^{k+1} \to R$.

Definition 1. *(Kocic & Ladas, 1993) An equilibrium point for equation (10) is a point $\bar{x} \in R$ such that $\bar{x} = f(\bar{x}, \bar{x}, \ldots, \bar{x})$.*

Definition 2. *(Kocic & Ladas, 1993)*

1. *An equilibrium point $\bar{x}$ for equation (10) is called* locally stable *if for every $\epsilon > 0$, there exists a $\delta > 0$ such that every solution $\{x_n\}$ with initial conditions*

$$x_{-k}, x_{-k+1}, \ldots, x_0 \in]\bar{x} - \delta, \bar{x} + \delta[$$

is such that

$$x_n \in]\bar{x} - \epsilon, \bar{x} + \epsilon[$$

for all $n \in \mathbb{N}$. Otherwise $\bar{x}$ is said to be unstable.

2. *The equilibrium point $\bar{x}$ of equation (10) is called* locally asymptotically stable *if it is locally stable and there exists $\gamma > 0$ such that for any initial conditions*

$$x_{-k}, x_{-k+1}, \ldots, x_0 \in]\bar{x} - \gamma, \bar{x} + \gamma[,$$

the corresponding solution $\{x_n\}$ tends to $\bar{x}$.

3. *The equilibrium point $\bar{x}$ for equation (10) is called a* global attractor *if every solution $\{x_n\}$ converges to $\bar{x}$ as $n \to \infty$.*

4. *The equilibrium point $\bar{x}$ for equation (10) is called* globally asymptotically stable *if it is locally asymptotically stable and global attractor.*

Suppose that f is continuously differentiable in some open neighborhood of $\bar{x}$.

Let

$$a_i = \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x}), \quad \text{for} \quad i = 0, 1, \ldots, k$$

denote the partial derivatives of $f(x_n, x_{n-1}, \ldots, x_{n-k})$ with respect to x_{n-i} evaluated at the equilibrium point $\bar{x}$ of equation (10). Then the equation

$$z_{n+1} = \sum_{i=0}^{k} a_i z_{n-i} \quad , n = 0, 1, \ldots, \tag{11}$$

is called the linearized equation associated with equation (10) about the equilibrium point $\bar{x}$, and the equation

$$\lambda^{k+1} - \sum_{i=0}^{k} a_i \lambda^{k-i} = 0 \tag{12}$$

is called the characteristic equation associated with equation (11) about the equilibrium point $\bar{x}$.

Theorem 3. *(Kocic & Ladas, 1993) Assume that f is a C^1 function and let $\bar{x}$ be an equilibrium point of equation (10). Then the following statements are true:*

1. *If all roots of equation (12) lie in the open disk $|\lambda| < 1$, then $\bar{x}$ is locally asymptotically stable.*
2. *If at least one root of equation (12) has absolute value greater than one, then $\bar{x}$ is unstable.*

It is clear that the equilibrium point $\bar{y} = 0$ is always an equilibrium point of equation (6) and the nonzero equilibrium points depend on whether k is even or odd.

When k is odd, we have the nonzero equilibrium points $\bar{y} = \pm \sqrt[k+1]{r-1}$ if $r > 1$.

When k is even, we have the nonzero equilibrium point $\bar{y} = \sqrt[k+1]{r-1}$, $r \neq 1$.

Lemma 1. *Let $P(x)$ be the polynomial*

$$x^k + x^{k-1} + \ldots + x + 1.$$

Then the zeros of $P(x)$ are of modulus one.

The following theorem describes the local behavior of the equilibrium points.

Theorem 4. *The following statements are true.*

1. *The equilibrium point $\bar{y} = 0$ is locally asymptotically stable if $r < 1$ and unstable if $r > 1$.*
2. *If k is even, then $\bar{y} = \sqrt[k+1]{r-1}$ is unstable if $r < 1$ and nonhyperbolic if $r > 1$.*
3. *If k is odd, then the equilibrium points $\bar{y} = \pm \sqrt[k+1]{r-1}$ are nonhyperbolic points.*

Proof. The linearized equation associated with equation (6) about an equilibrium point $\bar{y}$ is

$$z_{n+1} + \frac{r\bar{y}^{k+1}}{(1+\bar{y}^{k+1})^2} \sum_{i=0}^{k-1} z_{n-i} - \frac{r}{(1+\bar{y}^{k+1})^2} z_{n-k} = 0, \quad n = 0, 1, \ldots. \tag{13}$$

Its characteristic equation associated with this equation is

$$\lambda^{k+1} + \frac{r\bar{y}^{k+1}}{(1+\bar{y}^{k+1})^2} \sum_{i=0}^{k-1} \lambda^{k-i} - \frac{r}{(1+\bar{y}^{k+1})^2} = 0. \tag{14}$$

Therefore, (1) follows directly.

Equation (13) about a nonzero equilibrium point $\bar{y}$ is

$$z_{n+1} + \frac{r-1}{r} \sum_{i=0}^{k-1} z_{n-i} - \frac{1}{r} z_{n-k} = 0, \quad n = 0, 1, \ldots. \tag{15}$$

Also equation (14) becomes

$$\lambda^{k+1} + \frac{r-1}{r} \sum_{i=0}^{k-1} \lambda^{k-i} - \frac{1}{r} = 0. \tag{16}$$

Let

$$f(\lambda) = \lambda^{k+1} + \frac{r-1}{r}\sum_{i=0}^{k-1}\lambda^{k-i} - \frac{1}{r}.$$

We can see that

$$f(\lambda) = (\lambda - \frac{1}{r})\sum_{l=0}^{k}\lambda^{l} = (\lambda - \frac{1}{r})P(\lambda).$$

Then the roots of equation (16) are the zeros of $f(\lambda)$. Using lemma (1), we see that, the roots of equation (16) are $\frac{1}{r}$ and k other roots with modulus 1. Therefore, (2) and (3) follow directly. □

Global behavior of equation (6)

If we set $n = (k+1)m + i$, $i = 1, 2, \ldots, k+1$ in (8), then the solution of equation (6) can be written as

$$y_{(k+1)m+i} = y_{-(k+1)+i} r^{m+1} \prod_{j=0}^{m} \frac{1 + \alpha \sum_{l=0}^{(k+1)j+i-2} r^l}{1 + \alpha \sum_{l=0}^{(k+1)j+i-1} r^l}, \qquad (17)$$

$$i = 1, 2, \ldots, k+1 \text{ and } m = 0, 1, \ldots$$

This formula is the same as that included in Elabbasy et al. (2007).

But as

$$\frac{1 + \alpha \sum_{l=0}^{(k+1)j+i-2} r^l}{1 + \alpha \sum_{l=0}^{(k+1)j+i-1} r^l} = \frac{\alpha + \theta r^{-(k+1)j-i+1}}{r(\alpha + \theta r^{-(k+1)j-i})},$$

where $\theta = r - 1 - \alpha$, we can write

$$\begin{aligned} y_{(k+1)m+i} &= y_{-(k+1)+i} r^{m+1} \prod_{j=0}^{m} \frac{1 + \alpha \sum_{l=0}^{(k+1)j+i-2} r^l}{1 + \alpha \sum_{l=0}^{(k+1)j+i-1} r^l} \\ &= y_{-(k+1)+i} r^{m+1} \prod_{j=0}^{m} \frac{\alpha + \theta r^{-(k+1)j-i+1}}{r(\alpha + \theta r^{-(k+1)j-i})} \\ &= y_{-(k+1)+i} \prod_{j=0}^{m} \beta_i(j), \end{aligned}$$

$$i = 1, 2, \ldots, k+1 \text{and } m = 0, 1, \ldots,$$

where

$$\beta_i(j) = \frac{\alpha + \theta r^{-(k+1)j-i+1}}{\alpha + \theta r^{-(k+1)j-i}}, \quad i = 1, 2, \ldots, k+1.$$

Theorem 5. *Assume that $\{y_n\}_{n=-k}^{\infty}$ is a solution of equation (6) such that $\alpha \neq \frac{-1}{\sum_{i=0}^{n} r^i}$ for any $n \in N$. If $\alpha = r - 1$, then $\{y_n\}_{n=-k}^{\infty}$ is a periodic solution with period $k+1$.*

Proof. If $\alpha = r - 1$, then $\theta = 0$. Therefore,

$$y_{(k+1)m+i} = y_{-(k+1)+i} \prod_{j=0}^{m} \frac{\alpha + \theta r^{-(k+1)j-i+1}}{\alpha + \theta r^{-(k+1)j-i}} = y_{-(k+1)+i},$$

$$i = 1, 2, \ldots, k+1.$$

□

Proposition 3. *Assume that $r > 1$ and let $\alpha \neq \frac{-1}{\sum_{i=0}^{n} r^i}$ for any $n \in N$. Then there exists $j_0 \in \mathbb{N}$ such that $\beta_i(j) > 0$ for all $j \geq j_0$.*

Proof. We have three situations:

1. If $0 < r - 1 < \alpha$, then $0 < \theta + \alpha < \alpha$. Hence for each $j \in \mathbb{N}$,

$$\begin{aligned} \alpha + \theta r^{-(k+1)j-i+1} &= r^{-(k+1)j-i+1}(\alpha r^{(k+1)j+i-1} + \theta) \\ &> r^{-(k+1)j-i+1}(\alpha + \theta) > 0. \end{aligned}$$

   It follows that $\beta_i(j) > 0$ for all $j \geq 0$.

2. If $0 < \alpha < r - 1$, then $0 < \alpha < \theta + \alpha$. Hence for each $j \in \mathbb{N}$,

$$\begin{aligned} \alpha + \theta r^{-(k+1)j-i+1} &= r^{-(k+1)j-i+1}(\alpha r^{(k+1)j+i-1} + \theta) \\ &> r^{-(k+1)j-i+1}(\alpha + \theta) > 0. \end{aligned}$$

   It follows that $\beta_i(j) > 0$ for all $j \geq 0$.

3. If $\alpha < 0 < r - 1$, then $\alpha + \theta > 0$.

   But

$$\lim_{j\to\infty} \beta_i(j) = \lim_{j\to\infty} \frac{\alpha + \theta r^{-(k+1)j-i+1}}{\alpha + \theta r^{-(k+1)j-i}} = 1.$$

   This implies that there exists $j_0 \in \mathbb{N}$ such that $\beta_i(j) > 0$ for all $j \geq j_0$.

   In all cases there exists $j_0 \in \mathbb{N}$ such that $\beta_i(j) > 0$ for all $j \geq j_0$.

□

Theorem 6. *Assume that $\{y_n\}_{n=-k}^{\infty}$ is a solution of equation (6) such that $\alpha \neq r - 1$ and $\alpha \neq \frac{-1}{\sum_{i=0}^{n} r^i}$ for any $n \in N$. Then the following statements are true.*

1. *If $r < 1$, then $\{y_n\}_{n=-k}^{\infty}$ converges to $\bar{y} = 0$.*
2. *If $r > 1$ and $\alpha \neq 0$, then $\{y_n\}_{n=-k}^{\infty}$ is bounded.*

Proof. Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of equation (6) such that $\alpha \neq \frac{-1}{\sum_{i=0}^{n} r^i}$ for any $n \in \mathbb{N}$.

1. Suppose that $r < 1$. It is clear that, as the equilibrium point $\frac{1}{r-1}$ of equation (7) is repelling, every non-constant solution of equation (7) approaches ∞ or $-\infty$ according to the value of $t_0 = \frac{1}{\alpha}$.
   We shall consider the following situations:

   (a) If $\alpha > 0$, then according to equation (7), $\prod_{i=0}^{k} y_{n-i} > 0$ for each $n \in \mathbb{N}$. Therefore,

$$| y_{n+1} | = \frac{| r y_{n-k} |}{| 1 + \prod_{i=0}^{k} y_{n-i} |} < r \, | y_{n-k} |,$$

$$n = 0, 1, \ldots.$$

(b) If $r-1<\alpha<0$, then according to equation (7), $\prod_{i=0}^{k} y_{n-i} > r-1$ for each $n \in \mathbb{N}$. That is $1+\prod_{i=0}^{k} y_{n-i} > r$ for each $n \in \mathbb{N}$. Therefore,

$$| y_{n+1} | = \frac{| r y_{n-k} |}{| 1+\prod_{i=0}^{k} y_{n-i} |} < | y_{n-k} |, \quad n = 0, 1, \ldots.$$

(c) If $-1<\alpha<r-1$, then according to equation (7), there exists $n_0 \in \mathbb{N}$ such that $\prod_{i=0}^{k} y_{n-i} = \frac{1}{t_n} > 0$ for each $n \geq n_0$. Therefore,

$$| y_{n+1} | = \frac{| r y_{n-k} |}{| 1+\prod_{i=0}^{k} y_{n-i} |} < r \, | y_{n-k} |, \quad n \geq n_0.$$

(d) If $\alpha < -1$, then according to equation (7), $\prod_{i=0}^{k} y_{n-i} = \frac{1}{t_n} > 0$ for each $n > 0$. Therefore,

$$| y_{n+1} | = \frac{| r y_{n-k} |}{| 1+\prod_{i=0}^{k} y_{n-i} |} < r \, | y_{n-k} |, \quad n = 0, 1, \ldots.$$

In all cases, $y_n \to 0$ as $n \to \infty$.

2. Suppose that $r > 1$. Using Proposition 3, there exists $j_0 \in \mathbb{N}$ such that $\beta_i(j) > 0$ for all $j \geq j_0$. Hence for each $i \in \{1, 2, \ldots, k+1\}$, we have for large m

$$\begin{aligned} y_{(k+1)m+i} &= y_{-(k+1)+i} \prod_{j=0}^{m} \beta_i(j) \\ &= y_{-(k+1)+i} \prod_{j=0}^{j_0-1} \beta_i(j) \prod_{j=j_0}^{m} \beta_i(j) \\ &= y_{-(k+1)+i} \prod_{j=0}^{j_0-1} \beta_i(j) \exp\left(\ln \prod_{j=j_0}^{m} \beta_i(j) \right) \\ &= y_{-(k+1)+i} \prod_{j=0}^{j_0-1} \beta_i(j) \exp\left(\sum_{j=j_0}^{m} \ln \beta_i(j) \right). \end{aligned}$$

It is sufficient to test the convergence of the series $\sum_{j=j_0}^{\infty} | \ln \beta_i(j) |$.
Since $\lim_{j\to\infty} \frac{\ln \beta_i(j+1)}{\ln \beta_i(j)} = \frac{0}{0}$, using L'Hospital's rule we obtain $\lim_{j\to\infty} \frac{\ln \beta_i(j+1)}{\ln \beta_i(j)} = \frac{1}{r^{k+1}} < 1$.
It follows from D' Alemberts' test that the series $\sum_{j=j_0}^{\infty} | \ln \beta_i(j) |$ is convergent.
This ensures that the solution $\{y_n\}_{n=-k}^{\infty}$ is bounded.

□

We can observe in case $r > 1$ that, the behavior of the solution $\{y_n\}_{n=-k}^{\infty}$ is totally different according to whether $\alpha = 0$ or $\alpha \neq 0$. This is obvious in Corollary 1 and Theorem 6.

Theorem 7. *Assume that $r > 1$ and let $\{y_n\}_{n=-k}^{\infty}$ be a solution of equation (6) such that $\alpha \neq r-1$ and $\alpha \neq \frac{-1}{\sum_{i=0}^{n} r^i}$ for any $n \in N$. Then $\{y_n\}_{n=-k}^{\infty}$ converges to a $(k+1)$-periodic solution $\{\rho_0, \rho_1, \ldots, \rho_k\}$ of equation (6) with $\rho_0\rho_1 \ldots \rho_k = r-1$.*

Proof. By Theorem 6, there exist $k+1$ real numbers $\rho_i \in \mathbb{R}$ such that

$$\lim_{j\to\infty} y_{(k+1)m+i} = \rho_i, \quad i \in \{0, 1, \ldots, k\}.$$

If we set $n = (k+1)m + i - 1$, $i = 0, 1, \ldots, k$ in equation (6), we get

$$y_{(k+1)m+i} = \frac{r y_{(k+1)(m-1)+i}}{1+\prod_{l=0}^{k} y_{(k+1)(m-1)+i-l+k}}, \quad i = 0, 1, \ldots, k \text{ and } m = 0, 1, \ldots$$

By taking the limit as $m \to \infty$, we obtain

$$\rho_i = \frac{r\rho_i}{1+\prod_{l=0}^{k} \rho_{i-l+k}}, \quad i = 0, 1, \ldots, k.$$

But from equation (7) we have that

$$\prod_{l=0}^{k} y_{n-l} = y_n y_{n-1} \ldots y_{n-k} = \frac{1}{t_n} \to r-1$$

as $n \to \infty$.

This implies that

$$\prod_{i=0}^{k} y_{(k+1)m+i} \to \rho_0\rho_1 \ldots \rho_k = r-1$$

as $m \to \infty$.

Therefore, $\{y_n\}_{n=-k}^{\infty}$ converges to the $(k+1)$-periodic solution

$$\{\ldots, \rho_0, \rho_1, \ldots, \rho_{k-1}, \frac{r-1}{\rho_0\rho_1 \ldots \rho_{k-1}}, \rho_0, \rho_1, \ldots, \rho_{k-1}, \frac{r-1}{\rho_0\rho_1 \ldots \rho_{k-1}}, \ldots\}$$

□

Case $r = 1$

We end this work with the discussion of the case $r = 1$. If we set $r = 1$ in equation (17), then we get

$$y_{(k+1)m+i} = y_{-(k+1)+i} \prod_{j=0}^{m} \gamma_i(j), \quad i = 1, 2, \ldots, k+1 \text{ and } m = 0, 1, \ldots, \tag{18}$$

where

$$\gamma_i(j) = \frac{1+\alpha((k+1)j+i-1)}{1+\alpha((k+1)j+i)}, \quad i = 1, 2, \ldots, k+1.$$

Proposition 4. *Assume that $r = 1$ and let $\alpha \neq \frac{-1}{n+1}$ for any $n \in N$. Then there exists $j_0 \in \mathbb{N}$ such that $\gamma_i(j) > 0$ for all $j \geq j_0$.*

Proof. When $\alpha > 0$, the result is obvious as $\gamma_i(j) > 0$ for each $j \in \mathbb{N}$.

When $\alpha < 0$, It is sufficient to see that,

$$\lim_{j\to\infty} \gamma_i(j) = \lim_{j\to\infty} \frac{1 + \alpha((k+1)j + i - 1)}{1 + \alpha((k+1)j + i)} = 1.$$

This implies that, there exists $j_0 \in \mathbb{N}$ such that $\gamma_i(j) > 0$ for all $j \geq j_0$. □

Theorem 8. *Assume that $r = 1$. Then any solution $\{y_n\}_{n=-k}^{\infty}$ of equation (6) with $\alpha \neq 0$ and $\alpha \neq \frac{-1}{n+1}$ for any $n \in N$ converges to zero.*

Proof. Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of equation (6) such that $\alpha \neq \frac{-1}{n+1}$ for any $n \in N$. Using Proposition 4, there exists $j_0 \in \mathbb{N}$ such that $\gamma_i(j) > 0$ for all $j \geq j_0$. Hence for each $i \in \{1, 2, \ldots, k+1\}$, we have for large m

$$\begin{aligned} y_{(k+1)m+i} &= y_{-(k+1)+i} \prod_{j=0}^{m} \gamma_i(j) \\ &= y_{-(k+1)+i} \prod_{j=0}^{j_0-1} \gamma_i(j) \prod_{j=j_0}^{m} \gamma_i(j) \\ &= y_{-(k+1)+i} \prod_{j=0}^{j_0-1} \gamma_i(j) \exp\left(\ln \prod_{j=j_0}^{m} \gamma_i(j)\right) \\ &= y_{-(k+1)+i} \prod_{j=0}^{j_0-1} \gamma_i(j) \exp\left(\sum_{j=j_0}^{m} \ln \gamma_i(j)\right) \\ &= y_{-(k+1)+i} \prod_{j=0}^{j_0-1} \gamma_i(j) \exp\left((-\sum_{j=j_0}^{m} \ln \frac{1}{\gamma_i(j)}\right). \end{aligned}$$

We shall show that

$$\sum_{j=j_0}^{\infty} \ln \frac{1}{\gamma_i(j)} = \sum_{j=j_0}^{\infty} \ln \frac{1 + \alpha((k+1)j + i)}{1 + \alpha((k+1)j + i - 1)} = \infty,$$

by considering the series $\sum_{j=j_0}^{\infty} \frac{\alpha}{1+\alpha((k+1)j+i)}$. But as

$$\lim_{j\to\infty} \frac{\ln 1 + \alpha((k+1)j + i)/1 + \alpha((k+1)j + i - 1)}{\alpha/1 + \alpha((k+1)j + i)} = 1,$$

using the limit comparison test, we get $\sum_{j=j_0}^{\infty} \ln \frac{1}{\gamma_i(j)} = \infty$. Therefore,

$$y_{(k+1)m+i} = y_{-(k+1)+i} \prod_{j=0}^{j_0-1} \gamma_i(j) \exp\,(-\sum_{j=j_0}^{m} \ln \frac{1}{\gamma_i(j)})$$

converges to zero as $m \to \infty$. □

Example 1. *Figure A1 shows that if $r = 1.1$, $y_{-2} = 1.2$, $y_{-1} = 2$ and $y_0 = -1.3$ ($\alpha = -3.12$), that is ($\alpha \neq r - 1$), then the solution $\{y_n\}_{n=-2}^{\infty}$ of equation (6) converges to the period-3 solution $\{\rho_0, \rho_1, \rho_2\}$, where $\rho_0 = 0.5710$, $\rho_1 = -0.3385$ and $\rho_2 = 0.5175$ (up to 4 decimals), with $\rho_0\rho_1\rho_2 = r - 1 = 0.1$.*

Example 2. *Figure A2 shows that if $r = 0.7$, $y_{-2} = 1.2$, $y_{-1} = 2$ and $y_0 = -1.3$ ($\alpha = -3.12$), that is ($\alpha \neq r - 1$), then the solution $\{y_n\}_{n=-2}^{\infty}$ of equation (6) converges to zero.*

Example 3. *Figure A3 shows that if $r = 2$, $y_{-2} = 1$, $y_{-1} = 0.25$ and $y_0 = 4$ ($\alpha = 1$), that is ($\alpha = r - 1$), then the solution $\{y_n\}_{n=-2}^{\infty}$ of equation (6) is of period 3.*

Example 4. *Figure A4 shows that if $r = 1$, $y_{-2} = 1.5$, $y_{-1} = -1.2$ and $y_0 = 1.3$, then the solution $\{y_n\}_{n=-2}^{\infty}$ of equation (6) converges to zero.*

The following example shows the existence of unbounded solutions.

Example 5. *Figure A5 shows that, if $r = 2.1$, $y_{-2} = 0$, $y_{-1} = 0.2$ and $y_0 = 1.5$, then the solution $\{y_n\}_{n=-2}^{\infty}$ of equation (6) is unbounded.*

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Appendix
Figures

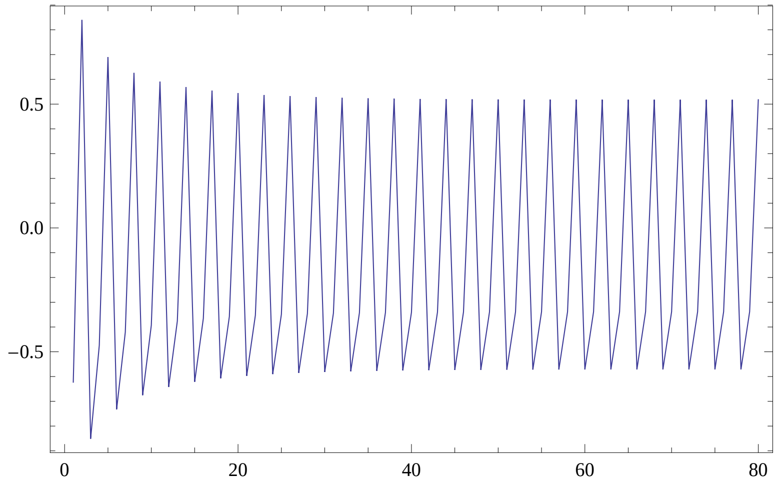

Figure A1. $y_{n+1} = \frac{1.1y_{n-2}}{1+y_n y_{n-1} y_{n-2}}$

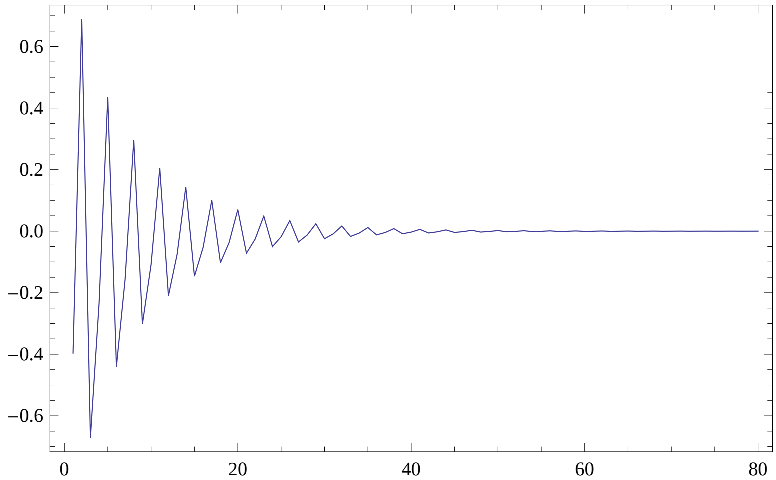

Figure A2. $y_{n+1} = \frac{0.7y_{n-2}}{1+y_n y_{n-1} y_{n-2}}$

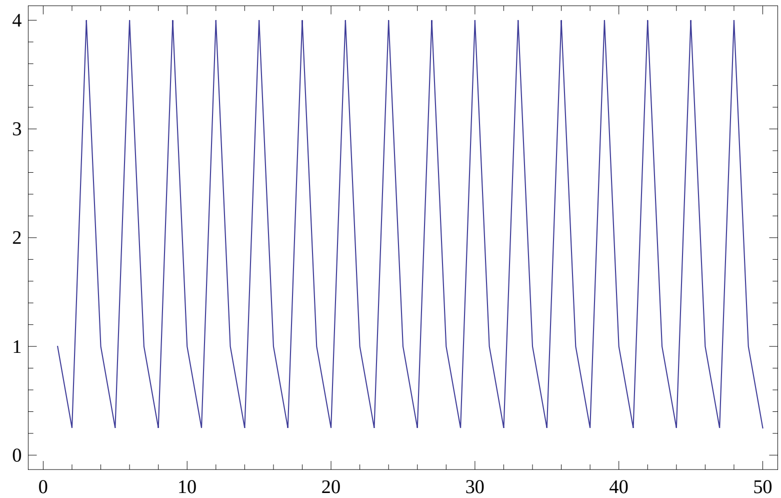

Figure A3. $y_{n+1} = \frac{2y_{n-2}}{1+y_n y_{n-1} y_{n-2}}$

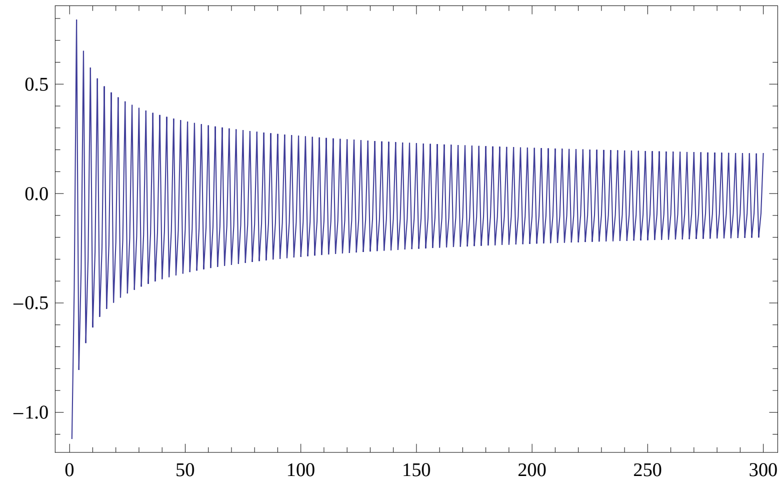

Figure A4. $y_{n+1} = \frac{y_{n-2}}{1+y_n y_{n-1} y_{n-2}}$

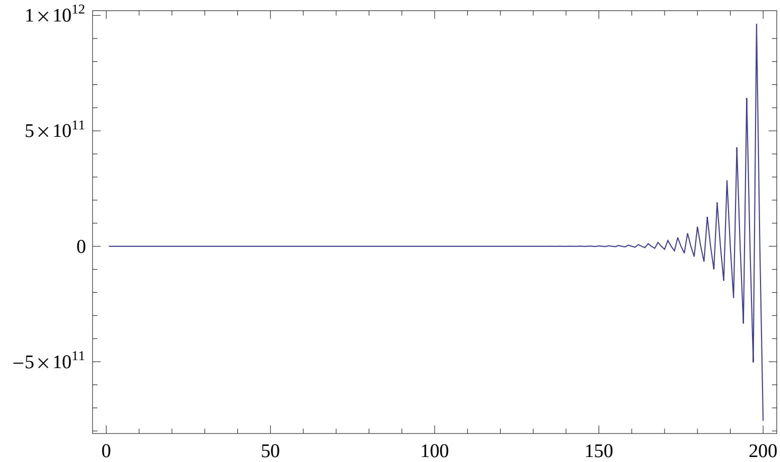

Figure A5. $y_{n+1} = \frac{2.1y_{n-2}}{1+y_n y_{n-1} y_{n-2}}$