Alabama Journal of Mathematics
41 (2017) ISSN 2373-0404

Representations for the Drazin inverse of block matrices

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In this paper two explicit representations for the Drazin inverse of a 2×2 block complex matrix M are presented. Moreover, we also present several other representations for the Drazin inverse of M under some conditions and generalize some results in literature.

Introduction

Let $A \in C^{n\times n}$. Then there is a unique matrix $A_d \in C^{n\times n}$ such that

$$\text{(i)}\, A_d A A_d = A_d, \qquad \text{(ii)}\, AA_d = A_d A, \qquad \text{(iii)}\, A^{n+1}A_d = A^n,$$

for some nonnegative integer n. The smallest positive exponent n for which (iii) holds is called the *Drazin index* of A and it is denoted by Ind(A). The matrix A_d is called the *Drazin inverse* of A (See, for example, (Ben-Israel & Greville, 1980, Ch. 4),Drazin (1958), (Piziak & Odell, 1999, Ch. 5), or (Campbell & Meyer, 1979, Ch. 7) for details). The study on representations for the Drazin inverse of block matrices stems essentially from finding the general expressions for the solutions to singular systems of differential equations Campbell (1982); Campbell and Meyer (1979); Campbell, Meyer, and Rose (1976).In 1983, Campbell(Campbell et al. (1976)) established an explicit representation for the Drazin inverse of a 2×2 block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1}$$

in terms of the blocks of the partition, where the blocks $A \in C^{n\times n}, B \in C^{n\times m}, C \in C^{m\times n}$ and $D \in C^{m\times m}$. In 2009, Chunyuan Deng and Yimin Wei (Deng and Wei (2009)) finding an explicit representation for the Drazin inverse of a 2×2 block matrix $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$, where A and BC are generalized Drazin invertible, if $A^\pi AB = 0$, $BC(I - A^\pi) = 0$. Afterwards, several authors have investigated this problem under some limited conditions on the blocks of M, which is mainly as follows:

- $B = 0$ (or $C = 0$). See Meyer and Rose (1977) or (Campbell & Meyer, 1979, Ch. 7).
- $AB = 0$, $D = 0$. See Deng and Wei (2009) .
- $BC = 0$, $DC = 0$ (or $BD = 0$), and D is nilpotent. See Hartwig, Hall, and Katz (1985).
- $BCA = 0$, $BD = 0$, and $DC = 0$ (or BC is nilpotent). See Castro-González, Dopazo, and Martí nez Serrano (2009).
- $BCA = 0$, $BCB = 0$, $DCA = 0$, and $DCB = 0$. See Yang and Liu (2011).
- $BC = 0$ and $DC = 0$. See Cvetković-Ilić (2008).
- $BCA = 0$, $BCB = 0$, $ABD = 0$, and $CBD = 0$. See Elliott and Zsidó (1984).
- $BC = 0$ and $BD = 0$. See Dopazo and Martí nez Serrano (2010).

In this paper, we present respectively the representations for the Drazin inverse of M under the conditions that $AB = 0$, $DC = 0$ and $AB = 0$, $BD = 0$. And we also give several representations for the Drazin inverse of M under some weaker conditions.

Some lemmas and notations

First, we will state some auxiliary lemmas.

Lemma 1. (Meyer & Rose, 1977, Theorems 2.1 and 3.2) or (Campbell & Meyer, 1979, Theorems 7.7.1 and 7.7.2) *Let L and U be of forms*

$$L = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \text{ and } U = \begin{pmatrix} B & C \\ 0 & A \end{pmatrix},$$

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respectively. If $s = \mathrm{Ind}(A)$ *and* $t = \mathrm{Ind}(B)$*, then*

$$L_d = \begin{pmatrix} A_d & 0 \\ S & B_d \end{pmatrix}, \qquad U_d = \begin{pmatrix} B_d & S \\ 0 & A_d \end{pmatrix},$$

where

$$S = \left[\sum_{i=0}^{s-1} B_d^{i+2} C A^i\right] A^\pi + B^\pi \left[\sum_{i=0}^{t-1} B^i C A_d^{i+2}\right] - B_d C A_d. \quad (2)$$

In addition, $\max\{s,t\} \le \mathrm{Ind}(L), \mathrm{Ind}(U) \le s+t$.

Lemma 2. *Let* $A \in C^{m\times n}$*,* $B \in C^{n\times m}$*. Then* $A(BA)_d^i = (AB)_d^i A$ *for every integer* $i \ge 1$*, and* $B(AB)^\pi = (BA)^\pi B$*. Moreover,* $\mathrm{Ind}(BA) - 1 \le \mathrm{Ind}(AB) \le \mathrm{Ind}(BA) + 1$.

Proof. As in the proof of (Campbell & Meyer, 1979, Theorem 7.8.4(iii)), we can obtain $(AB)_d = A(BA)_d^2 B$. The results follow. □

Also, we need the ceiling function $\lceil x \rceil$, the smallest integer greater than or equal to x. In what follows, $A^0 = I$ and $A^\pi \overset{\text{def}}{=} I - AA_d$ for any square matrix A, and the sum $\Sigma_i^j = 0$ if $i > j$. The following ceiling function ($\lceil k/2 \rceil$) are not repetitive for any positive integer k .

The representation in the following lemma is slightly changed for convenience.

Lemma 3. (Hartwig, Wang, & Wei, 2001, Theorem 2.1) *Let* $P, Q \in C^{n\times n}$*. If* $PQ = 0$ *, then*

$$\begin{aligned}(P+Q)_d &= \sum_{i=0}^{2\lceil \frac{k}{2}\rceil -1} Q^\pi Q^i P_d^{i+1} + \sum_{i=0}^{2\lceil \frac{k}{2}\rceil -1} Q_d^{i+1} P^i P^\pi \\ &= \sum_{i=0}^{\lceil \frac{k}{2}\rceil -1} Q^\pi Q^{2i}(I + QP_d) P_d^{2i+1} \\ &\quad + \sum_{i=0}^{\lceil \frac{k}{2}\rceil -1} Q_d^{2i+1}(I + Q_d P) P^{2i} P^\pi,\end{aligned}$$

where $\max\{Ind(P), Ind(Q)\} \le k \le Ind(P) + Ind(Q)$.

Remark 1. *Since* $(P+Q) = [I, Q]\begin{bmatrix} P \\ I \end{bmatrix}$ *and* $\begin{bmatrix} P \\ I \end{bmatrix}[I, Q] = \begin{bmatrix} P & 0 \\ I & Q \end{bmatrix}$ *where* $PQ = 0$,

$$\mathrm{Ind}(\begin{bmatrix} P & 0 \\ I & Q \end{bmatrix}) - 1 \le \mathrm{Ind}(P+Q) \le \mathrm{Ind}(\begin{bmatrix} P & 0 \\ I & Q \end{bmatrix}) + 1$$

by Lemma 2, and then, by Lemma 1,

$$\max\{\mathrm{Ind}(P), \mathrm{Ind}(Q)\} - 1 \le \mathrm{Ind}(P+Q)$$

$$\mathrm{Ind}(P+Q) \le \mathrm{Ind}(P) + \mathrm{Ind}(Q) + 1$$

if $PQ = 0$.

Lemma 4. (Catral, Olesky, & van den Driessche, 2009, Theorem 2.1)*Let* M *be a matrix of the form* (1) *with* $A = 0$ *and* $D = 0$*. Then*

$$M_d = \begin{pmatrix} 0 & (BC)_d B \\ C(BC)_d & 0 \end{pmatrix}.$$

Furthermore, if $\mathrm{Ind}(BC) = p$*, then* $\mathrm{Ind}(M) \le 2p + 1$.

Lemma 5. *Let* $A \in C^{n\times n}$*. Then* $(AA^\pi)_d = 0, (A^2 A_d)_d = A_d, (A^2A_d)^\pi = A^\pi$*, and* $\mathrm{Ind}(AA^\pi) = \mathrm{Ind}(A)$ *and* $\mathrm{Ind}(A^2A_d) = 1$.

Proof. The Jordan canonical form of A permits us to write $A = S(C \oplus N)S^{-1}$, where S and C are nonsingular, and N is nilpotent with index $\mathrm{Ind}(A)$. Thus $A_d = S(C^{-1} \oplus 0)S^{-1}$. Now, it is evident that $A^2A_d = S(C \oplus 0)S^{-1}$ and $AA^\pi = S(0 \oplus N)S^{-1}$, which lead to the affirmations of this lemma. □

Some results on the Drazin inverse of 2×2 block matrices

In this section we shall derivate several representations of the Drazin inverse of a 2×2 block matrix of the form (1) under diverse conditions. The following result, our main theorem, is a generalization of (Deng & Wei, 2009, Theorem 3.1).

Theorem 1. *Let* M *be a matrix of the form* (1)*. If* $AB = 0$ *and* $DC = 0$*, then*

$$M_d = \begin{pmatrix} XA & BY \\ CX & YD \end{pmatrix}, \quad (3)$$

where

$$X = (BC)^\pi \sum_{i=0}^{p-1} (BC)^i A_d^{2i+2} + \sum_{i=0}^{\lceil \frac{s}{2}\rceil -1} (BC)_d^{i+1} A^{2i} A^\pi, \quad (4)$$

$$Y = (CB)^\pi \sum_{i=0}^{q-1} (CB)^i D_d^{2i+2} + \sum_{i=0}^{\lceil \frac{t}{2}\rceil -1} (CB)_d^{i+1} D^{2i} D^\pi, \quad (5)$$

and $s = \mathrm{Ind}(A), t = \mathrm{Ind}(D), p = \mathrm{Ind}(BC)$ *and* $q = \mathrm{Ind}(CB)$.

Proof. Let $M = P + Q$, where

$$P = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}. \quad (6)$$

The conditions $AB = 0$ and $DC = 0$ imply $PQ = 0$. Thus, by Lemma 3,

$$\begin{aligned} M_d = (P+Q)_d &= \sum_{i=0}^{\lceil \frac{l}{2}\rceil -1} Q^\pi Q^{2i}(I + QP_d)P_d^{2i+1} \\ &\quad + \sum_{i=0}^{\lceil \frac{l}{2}\rceil -1} Q_d^{2i+1}(I + Q_d P)P^{2i}P^\pi,\end{aligned}$$

where $h = s + t \geq \text{Ind}(P)$, $k = 2p + 1 \geq \text{Ind}(Q)$ and $l = max(h, k)$(by Lemma 4).

Now we consider the matrices mentioned in the above equation. Clearly,

$$P_d = \begin{pmatrix} A_d & 0 \\ 0 & D_d \end{pmatrix}, \qquad P^\pi = \begin{pmatrix} A^\pi & 0 \\ 0 & D^\pi \end{pmatrix}.$$

By Lemma 4,

$$Q_d = \begin{pmatrix} 0 & (BC)_d B \\ C(BC)_d & 0 \end{pmatrix}$$

and then, by Lemma 2,

$$Q^\pi = \begin{pmatrix} (BC)^\pi & 0 \\ 0 & (CB)^\pi \end{pmatrix}.$$

Since

$$Q^2 = \begin{pmatrix} BC & 0 \\ 0 & CB \end{pmatrix},$$

for every integer $i \geq 1$,

$$Q^{2i} = \begin{pmatrix} (BC)^i & 0 \\ 0 & (CB)^i \end{pmatrix}$$

and then

$$Q_d^{2i} = \begin{pmatrix} (BC)_d^i & 0 \\ 0 & (CB)_d^i \end{pmatrix},$$

$$Q_d^{2i+1} = \begin{pmatrix} 0 & B(CB)_d^{i+1} \\ (CB)_d^{i+1}C & 0 \end{pmatrix}.$$

Hence

$$\begin{aligned}
&\sum_{i=0}^{\lceil \frac{l}{2} \rceil - 1} Q^\pi Q^{2i}(I + QP_d)P_d^{2i+1} \\
= &\sum_{i=0}^{\lceil \frac{l}{2} \rceil - 1} Q^\pi Q^{2i} \begin{bmatrix} I & BD_d \\ CA_d & I \end{bmatrix} P_d^{2i+1} \\
= &\sum_{i=0}^{\lceil \frac{l}{2} \rceil - 1} \begin{bmatrix} (BC)^\pi (BC)^i A_d^{2i+1} & (BC)^\pi (BC)^i BD_d^{2i+2} \\ (CB)^\pi (CB)^i CA_d^{2i+2} & (CB)^\pi (CB)^i D_d^{2i+1} \end{bmatrix} \\
= &\begin{bmatrix} \sum_{i=0}^{p-1} (BC)^\pi (BC)^i A_d^{2i+2} A & B\sum_{i=0}^{p-1} (CB)^\pi (CB)^i D_d^{2i+2} \\ C\sum_{i=0}^{q-1} (BC)^\pi (BC)^i A_d^{2i+2} & \sum_{i=0}^{q-1} (CB)^\pi (CB)^i D_d^{2i+2} D \end{bmatrix}
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&\sum_{i=0}^{\lceil \frac{l}{2} \rceil - 1} Q_d^{2i+1}(I + Q_d P)P^{2i}P^\pi \\
= &\sum_{i=0}^{\lceil \frac{l}{2} \rceil - 1} \begin{bmatrix} (BC)_d^{i+1} A^{2i+1} A^\pi & (BC)_d^{i+1} BD^{2i} D^\pi \\ (CB)_d^{i+1} CA^{2i} A^\pi & (CB)_d^{i+1} D^{2i+1} D^\pi \end{bmatrix} \\
= &\begin{bmatrix} \sum_{i=0}^{\lceil \frac{s}{2} \rceil - 1} (BC)_d^{i+1} A^{2i+1} A^\pi & B\sum_{i=0}^{\lceil \frac{s}{2} \rceil - 1} (CB)_d^{i+1} D^{2i} D^\pi \\ C\sum_{i=0}^{\lceil \frac{t}{2} \rceil - 1} (BC)_d^{i+1} A^{2i} A^\pi & \sum_{i=0}^{\lceil \frac{t}{2} \rceil - 1} (CB)_d^{i+1} D^{2i+1} D^\pi \end{bmatrix}.
\end{aligned}$$

Thus the desired result follows directly. □

Remark 2. By Lemma 3, (4) and (5), (3) is rewritten as

$$M_d = \begin{bmatrix} (A^2 + BC)_d A & B(CB + D^2)_d \\ C(A^2 + BC)_d & (CB + D^2)_d D \end{bmatrix}. \tag{7}$$

(ii) By Remark 1,

$$\max\{\text{Ind}(A), \text{Ind}(D), \text{Ind}(BC)\} \leq \text{Ind}(M)$$

$$\text{Ind}(M) \leq \text{Ind}(A) + \text{Ind}(D) + 2\text{Ind}(BC) + 1$$

if $AB = 0$ and $DC = 0$.

By Theorem 1 and Lemma 2, we have the following corollary.

Corollary 1. *Let M be a matrix of the form* (1) *with $A = 0$. If $DC = 0$, then*

$$M_d = \begin{bmatrix} 0 & BY \\ (CB)_d C & YD \end{bmatrix}, \tag{8}$$

where Y is defined in (5).

Using (7), we can easily see the following results.

Corollary 2. *Let M be a matrix of the form* (1).
(i) *If $AB = 0$, $DC = 0$, and $BC = 0$, then*

$$M_d = \begin{bmatrix} A_d & BD_d^2 \\ CA_d^2 & D_d + CBD_d^3 \end{bmatrix}. \tag{9}$$

(ii) *If $AB = 0$, $DC = 0$, and $CB = 0$, then*

$$M_d = \begin{bmatrix} A_d + BCA_d^3 & BD_d^2 \\ CA_d^2 & D_d \end{bmatrix}. \tag{10}$$

Proof. (i) When $BC = 0$, (7) becomes

$$M_d = \begin{bmatrix} A_d^2 A & B(CB + D^2)_d \\ CA_d^2 & (CB + D^2)_d D \end{bmatrix}, \tag{11}$$

and, by Lemma 2, $(CB)_d = 0$. Thus, since $D^2 CB = 0$, $(CB + D^2)_d = D_d^2 + CBD_d^4$ by Lemma 3. And then $B(CB + D^2)_d = BD_d^2$ and $(CB + D^2)_d D = D_d + CBD_d^3$. Consequently, (9) holds.

(ii) Similar as the proof of (i). □

The next result is an alternative generalization of (Deng & Wei, 2009, Theorem 3.1).

Theorem 2. *Let M be a matrix of the form* (1). *If $AB = 0$ and $BD = 0$, then*

$$M_d = \begin{bmatrix} XA & (BC)_d B \\ Z + T + R + D_d^2 CA^\pi - D_d^2 CBCX - D_d CXA & S + D_d(CB)^\pi \end{bmatrix}$$

where X is defined in (4),

$$
\begin{aligned}
Z &= \sum_{n=0}^{\lceil \frac{t}{2}\rceil-1} \left[D^\pi D^{2n+1}C(BC)_d^n X^2 A + D^\pi D^{2n} C(BC)_d^n X\right],\\
T &= \sum_{n=1}^{\lceil \frac{t}{2}\rceil-1} (L(n) + DL(n)A_d) + \sum_{n=0}^{\lceil \frac{k}{2}\rceil-1} (H(n) + D_d H(n) A),\\
S &= \sum_{n=0}^{\lceil \frac{t}{2}\rceil-1} D^\pi D^{2n+1}(CB)_d^{n+1} + \sum_{n=0}^{q-1} D_d^{2n+1}(CB)^\pi (CB)^n,\\
R &= \sum_{n=0}^{p-1} D_d^{2n+2} C(BC)^\pi (BC)^n,\\
L(n) &= D^\pi D^{2n} CX^2 A_d^{2n-1} A - D^\pi D^{2n} \sum_{i=1}^{n-1} C(BC)_d^{i+1} A_d^{2n-2i},\\
H(n) &= D_d^{2n+1} C(BC)^\pi \sum_{i=0}^{n-1} (BC)^i A^{2n-2i-1} - D_d^{2n+1} CXA^{2n+1},
\end{aligned} \tag{12}
$$

and $s = \mathrm{Ind}(A), t_1 = \mathrm{Ind}(D), t_2 = \mathrm{Ind}(P), t = max(t_1, t_2), p = \mathrm{Ind}(BC), q = \mathrm{Ind}(CB)$ *and* $k = s + 2p + 1$.

Proof. Let $M = P + Q$, where

$$P = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}.$$

Then

$$Q_d = \begin{bmatrix} 0 & 0 \\ 0 & D_d \end{bmatrix}, \qquad Q^\pi = \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix}.$$

Since $BD = 0$ implies $PQ = 0$, by Lemma 3,

$$M_d = (P+Q)_d = \sum_{n=0}^{\lceil \frac{t}{2}\rceil-1} Q^\pi Q^{2n}(I + QP_d)P_d^{2n+1} + \sum_{n=0}^{\lceil \frac{t}{2}\rceil-1} Q_d^{2n+1}(I + Q_d P)P^{2n}P^\pi,$$

.

Since $AB = 0$, by Theorem 1,

$$P_d = \begin{bmatrix} XA & (BC)_d B \\ CX & 0 \end{bmatrix}, \quad P^\pi = \begin{bmatrix} A^\pi - BCX & 0 \\ -CXA & (CB)^\pi \end{bmatrix},$$

where X is defined in (4), and, by Remark 2(ii),

$$\max\{\mathrm{Ind}(A), \mathrm{Ind}(BC)\} \le \mathrm{Ind}(P) \le \mathrm{Ind}(A) + 2\mathrm{Ind}(BC) + 1.$$

Thus, from $AB = 0$, we have

$$
\begin{aligned}
AX &= A_d,\\
(XA)^k &= X(AX)^{k-1}A = XA_d^{k-1}A, \quad k \ge 1,\\
XB &= (BC)_d B,\\
X^2 &= (BC)^\pi \sum_{i=0}^{p-1} (BC)^i A_d^{2i+4} + \sum_{i=0}^{\lceil \frac{s}{2}\rceil-1} (BC)_d^{i+2} A^{2i} A^\pi\\
&\quad -(BC)_d A_d^2,\\
(BC)_d X^2 A_d &= -(BC)_d^2 A_d^3
\end{aligned}
$$

and then, by mathematical induction, for every integer $n \ge 1$,

$$
\begin{aligned}
P_d^{2n+1} &= \begin{bmatrix} XA_d^{2n}A - \sum_{i=1}^{n-1}(BC)_d^{i+1} BCA_d^{2n+2-2i}A & \\ CX^2A_d^{2n-1}A - \sum_{i=1}^{n-1} C(BC)_d^{i+1}A_d^{2n+1-2i}A & \\ \quad +(BC)_d^n BCX^2A & (BC)_d^{n+1}B \\ \quad +C(BC)_d^n X & 0 \end{bmatrix},\\
P^{2n} &= \begin{bmatrix} \sum_{i=0}^{n}(BC)^i A^{2n-2i} & 0 \\ \sum_{i=0}^{n-1} C(BC)^i A^{2n-2i-1} & (CB)^n \end{bmatrix},
\end{aligned} \tag{13}
$$

where $\Sigma_i^j = 0$ if $i > j$.

Now consider the first sum in (13). Obviously,

$$
\begin{aligned}
Q^\pi(I + QP_d)P_d &= \begin{bmatrix} XA & (BC)_d B \\ D^\pi DCX^2A + D^\pi CX & D^\pi D(CB)_d \end{bmatrix},\\
Q^\pi Q^{2n}(I + QP_d) &= \begin{bmatrix} 0 & 0 \\ D^\pi D^{2n+1}CX & D^\pi D^{2n} \end{bmatrix}.
\end{aligned} \tag{14}
$$

By (13) and (14), for every integer $n \ge 1$,

$$
\begin{aligned}
&Q^\pi Q^{2n}(I + QP_d)P_d^{2n+1}\\
&= \begin{bmatrix} 0 & 0 \\ D^\pi D^{2n+1}CX^2A_d^{2n-1} + D^\pi D^{2n}CX^2A_d^{2n-1}A & D^\pi D^{2n+1}(CB)_d^{n+1} \end{bmatrix}\\
&\quad + \begin{bmatrix} 0 & \\ -D^\pi D^{2n+1}CX\sum_{i=1}^{n-1}(BC)_d^i A_d^{2n+1-2i} & \\ \quad 0 & \\ \quad -D^\pi D^{2n}\sum_{i=1}^{n-1} C(BC)_d^{i+1}A_d^{2n-2i} & 0 \end{bmatrix}\\
&\quad + \begin{bmatrix} 0 & 0 \\ D^\pi D^{2n+1}CX(BC)_d^n BCX^2A + D^\pi D^{2n}C(BC)_d^n X & 0 \end{bmatrix}\\
&= \begin{bmatrix} 0 & 0 \\ L(n) + DL(n)A_d & 0 \end{bmatrix}\\
&\quad + \begin{bmatrix} 0 & 0 \\ D^\pi D^{2n+1}C(BC)_d^n X^2A + D^\pi D^{2n}C(BC)_d^n X & D^\pi D^{2n+1}(CB)_d^{n+1} \end{bmatrix},
\end{aligned}
$$

where $L(n)$ is defined in (12).

Hence the first sum in (13) is

$$\begin{aligned}
&\begin{bmatrix} XA & (BC)_d B \\ D^\pi DCX^2A + D^\pi CX & D^\pi D(CB)_d \end{bmatrix} \\
&+ \sum_{n=1}^{\lceil \frac{\ell}{2}\rceil -1} \begin{bmatrix} 0 & 0 \\ L(n)+DL(n)A_d & 0 \end{bmatrix} \\
&+ \sum_{n=1}^{\lceil \frac{\ell}{2}\rceil -1} \begin{bmatrix} 0 & 0 \\ D^\pi D^{2n+1}C(BC)_d^n X^2A + D^\pi D^{2n}C(BC)_d^n X & D^\pi D^{2n+1}(CB)_d^{n+1} \end{bmatrix} \\
= &\sum_{n=0}^{\lceil \frac{\ell}{2}\rceil -1} \begin{bmatrix} 0 & 0 \\ D^\pi D^{2n+1}C(BC)_d^n X^2A + D^\pi D^{2n}C(BC)_d^n X & D^\pi D^{2n+1}(CB)_d^{n+1} \end{bmatrix} \\
&+ \sum_{n=1}^{\lceil \frac{\ell}{2}\rceil -1} \begin{bmatrix} 0 & 0 \\ L(n)+DL(n)A_d & 0 \end{bmatrix} \\
&+ \begin{bmatrix} XA & (BC)_d B \\ 0 & 0 \end{bmatrix}.
\end{aligned} \tag{18}$$

Next consider the second sum in (13). For every integer $n \geq 0$,

$$\begin{aligned}
&Q_d^{2n+1}(I+Q_dP)P^\pi \\
= &\begin{bmatrix} 0 & 0 \\ D_d^{2n+2}CA^\pi - D_d^{2n+2}CBCX - D_d^{2n+1}CXA & D_d^{2n+1}(CB)^\pi \end{bmatrix}
\end{aligned}$$

For every integer $n \geq 1$, since $PP_d = P_dP$ implies $(BC)^\pi - XA^2 = A^\pi - BCX$, we have

$$\begin{aligned}
&Q_d^{2n+1}(I+Q_dP)P^{2n}P^\pi \\
= &\begin{bmatrix} 0 & 0 \\ D_d^{2n+2}CA^\pi A^{2n} + D_d^{2n+2}C\sum_{i=1}^{n}(BC)^i A^{2n-2i} & 0 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ -D_d^{2n+2}CBCXA^{2n} - D_d^{2n+2}C(BC)_d\sum_{i=1}^{n}(BC)^{i+1}A^{2n-2i} & 0 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ D_d^{2n+1}(CB)^\pi \sum_{i=0}^{n-1} C(BC)^i A^{2n-2i-1} - D_d^{2n+1}CXA^{2n+1} & D_d^{2n+1}(CB)^\pi (CB)^n \end{bmatrix} \\
= &\begin{bmatrix} 0 & 0 \\ D_d^{2n+2}C(BC)^\pi A^{2n} - D_d^{2n+2}CXA^{2n+2} + D_d^{2n+2}C(BC)^\pi \sum_{i=1}^{n}(BC)^i A^{2n-2i} & 0 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ H(n) & D_d^{2n+1}(CB)^\pi (CB)^n \end{bmatrix} \\
= &\begin{bmatrix} 0 & 0 \\ H(n)+D_dH(n)A & 0 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ D_d^{2n+2}C(BC)^\pi (BC)^n & D_d^{2n+1}(CB)^\pi (CB)^n \end{bmatrix},
\end{aligned}$$

where $H(n)$ is defined in (12).

Hence the second sum in (13) is

$$\begin{aligned}
&\sum_{n=0}^{\lceil \frac{k}{2}\rceil -1} \begin{bmatrix} 0 & 0 \\ H(n)+D_dH(n)A & 0 \end{bmatrix} \\
&+ \sum_{n=0}^{\lceil \frac{k}{2}\rceil -1} \begin{bmatrix} 0 & 0 \\ D_d^{2n+2}C(BC)^\pi (BC)^n & D_d^{2n+1}(CB)^\pi (CB)^n \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ D_d^2CA^\pi - D_d^2CBCX - D_dCXA & D_d(CB)^\pi \end{bmatrix},
\end{aligned} \tag{19}$$

where $k = \mathrm{Ind}(A) + 2\mathrm{Ind}(BC) + 1$.

As a result, putting (18) and (19) into (13) yields M_d. The proof is complete. □

Since

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ I_m & 0 \end{bmatrix}\begin{bmatrix} D & C \\ B & A \end{bmatrix}\begin{bmatrix} 0 & I_m \\ I_n & 0 \end{bmatrix}, \tag{20}$$

we can obtain the following result, applying Theorem 2 to $\begin{bmatrix} D & C \\ B & A \end{bmatrix}$.

Remark 3. *Let M be a matrix of the form* (1)*. If $DC = 0$ and $CA = 0$, then we can get another representation for the Drazin inverse by Theorem 2 .*

In the rest of the paper we will exploit Theorem 1 or Corollary 2 to obtain some representation of M_d under some weaker conditions. Firstly we will present the following result.

Theorem 3. *Let M be a matrix of the form* (1). *If $AA^{\pi}B = 0$, $DD^{\pi}C = 0$, $CA_d = 0$ and $BD_d = 0$, then*

$$M_d = \begin{bmatrix} A_d + A^{\pi}XA + L & A^{\pi}BY + N \\ D^{\pi}CX + \widetilde{N} & D_d + D^{\pi}YD + \widetilde{L} \end{bmatrix}, \tag{21}$$

where $s = \mathrm{Ind}(A)$, $t = \mathrm{Ind}(D)$, $p = \mathrm{Ind}(BC)$, $k = s+t+2p+1$, and

$$\begin{aligned} L &= \sum_{n=0}^{k-1} A_d^{2n+1}[BC(S(n) - XA^{2n}) \\ &\quad + A_dBC(S(n) - XA^{2n})A], \\ \widetilde{L} &= \sum_{n=0}^{k-1} D_d^{2n+1}[CB(\widetilde{S}(n) - YD^{2n}) \\ &\quad + D_dCB(\widetilde{S}(n) - YD^{2n})D], \\ N &= \sum_{n=0}^{k-1} A_d^{2n+1}[B(\widetilde{S}(n) - YD^{2n})D \\ &\quad + A_dBCB(\widetilde{S}(n) - YD^{2n}) + A_dBD^{2n}], \\ \widetilde{N} &= \sum_{n=0}^{k-1} D_d^{2n+1}[C(Z - XA^{2n})A \\ &\quad + D_dCBC(Z - XA^{2n}) + D_dCA^{2n}], \qquad (22) \\ X &= \sum_{i=0}^{\lceil \frac{s}{2} \rceil - 1} (BC)_d^{i+1}A^{2i}, \quad Y = \sum_{i=0}^{\lceil \frac{t}{2} \rceil - 1} (CB)_d^{i+1}D^{2i}, \\ S(n) &= \sum_{i=0}^{n-1} (BC)^{\pi}(BC)^iA^{2n-2i-2}, \\ \widetilde{S}(n) &= \sum_{i=0}^{n-1} (CB)^{\pi}(CB)^iD^{2n-2i-2}. \end{aligned}$$

Proof. Let $M = P + Q$, where

$$P = \begin{bmatrix} AA^{\pi} & B \\ C & DD^{\pi} \end{bmatrix}, \qquad Q = \begin{bmatrix} A^2A_d & 0 \\ 0 & D^2D_d \end{bmatrix}.$$

Clearly, $PQ = 0$. By Lemma 5, we have

$$Q_d = \begin{bmatrix} A_d & 0 \\ 0 & D_d \end{bmatrix} \text{ and } Q^{\pi} = \begin{bmatrix} A^{\pi} & 0 \\ 0 & D^{\pi} \end{bmatrix}.$$

Apparently

$$Q_d^n = \begin{bmatrix} A_d^n & 0 \\ 0 & D_d^n \end{bmatrix}, \ for\ n \geq 1, \text{ and } QQ^{\pi} = 0.$$

Since $AA^{\pi}B = 0$ and $DD^{\pi}C = 0$, by Theorem 1,

$$P_d = \begin{bmatrix} XAA^{\pi} & BY \\ CX & YDD^{\pi} \end{bmatrix} = \begin{bmatrix} XA & BY \\ CX & YD \end{bmatrix},$$

where $X = \sum_{i=0}^{\lceil \frac{s}{2} \rceil - 1} (BC)_d^{i+1}A^{2i}$ and $Y = \sum_{i=0}^{\lceil \frac{t}{2} \rceil - 1} (CB)_d^{i+1}D^{2i}$ (by Lemma 5 and $CA_d = 0$ and $BD_d = 0$), and $k = s+t+2p+1$ by Remark 2. And therefore $AA^{\pi}X = 0$ and $DD^{\pi}Y = 0$. From this, we have

$$P^{\pi} = \begin{bmatrix} I - BCX & -BYD \\ -CXA & I - CBY \end{bmatrix}.$$

Moreover, we have, for $n \geq 1$,

$$P^{2n} = \begin{bmatrix} \sum_{i=0}^{n}(BC)^iA^{2n-2i}A^{\pi} & \sum_{i=0}^{n-1} B(CB)^iD^{2n-2i-1} \\ \sum_{i=0}^{n-1} C(BC)^iA^{2n-2i-1} & \sum_{i=0}^{n}(CB)^iD^{2n-2i}D^{\pi} \end{bmatrix},$$

Since $CA_d = 0$ and $BD_d = 0$, we obtain $PQ = 0$. Using Lemma 3, we get

$$\begin{aligned} M_d &= (P + Q)_d = Q^{\pi}P_d + Q_d(I + Q_dP)P^{\pi} \\ &+ \sum_{n=1}^{k-1} Q_d^{2n+1}(I + Q_dP)P^{2n}P^{\pi}. \end{aligned}$$

Clearly,

$$\begin{aligned} Q^{\pi}P_d &= \begin{bmatrix} A^{\pi}XA & A^{\pi}BY \\ D^{\pi}CX & D^{\pi}YD \end{bmatrix}. \qquad (23) \\ I + Q_dP &= \begin{bmatrix} I & A_dB \\ D_dC & I \end{bmatrix}. \end{aligned}$$

For $n \geq 0$,

$$\begin{aligned} & Q_d^{2n+1}(I + Q_dP)P^{\pi} \\ &= \begin{bmatrix} A_d^{2n+1} & A_d^{2n+2}B \\ D_d^{2n+2}C & D_d^{2n+1} \end{bmatrix} \begin{bmatrix} I - BCX & -BYD \\ -CXA & I - CBY \end{bmatrix} \\ &= \left[\begin{matrix} A_d^{2n+1} - A_d^{2n+1}BCX - A_d^{2n+2}BCXA \\ D_d^{2n+2}C - D_d^{2n+2}CBCX - D_d^{2n+1}CXA \end{matrix}\right. \\ &\qquad \left.\begin{matrix} A_d^{2n+2}B - A_d^{2n+2}BCBY - A_d^{2n+1}BYD \\ D_d^{2n+1} - D_d^{2n+1}CBY - D_d^{2n+2}CBYD \end{matrix}\right] \end{aligned} \tag{24}$$

Since $BD^kC = BD^{\pi}D^kC = 0$ and $CA^kB = CA^{\pi}A^kB = 0$ for

$k \geq 1$, we have, for $n \geq 1$,

$$\begin{aligned}
&Q_d^{2n+1}(I + Q_dP)P^{\pi}P^{2n} \\
=& \sum_{i=0}^{n-1}\begin{bmatrix} A_d^{2n+1}(BC)^{i+1}(BC)^{\pi}A^{2n-2i-2} & A_d^{2n+1}B(CB)^{i}(CB)^{\pi}D^{2n-2i-1} \\ D_d^{2n+2}C(BC)^{i+1}(BC)^{\pi}A^{2n-2i-2} & D_d^{2n+2}(CB)^{i+1}(CB)^{\pi}D^{2n-2i-1} \end{bmatrix} \\
&+ \sum_{i=0}^{n-1}\begin{bmatrix} A_d^{2n+2}(BC)^{i+1}(BC)^{\pi}A^{2n-2i-1} & A_d^{2n+2}B(CB)^{i+1}(CB)^{\pi}D^{2n-2i-2} \\ D_d^{2n+1}C(BC)^{i}(BC)^{\pi}A^{2n-2i-1} & D_d^{2n+1}(CB)^{i+1}(CB)^{\pi}D^{2n-2i-2} \end{bmatrix} \\
&- \begin{bmatrix} A_d^{2n+1}BCXA^{2n} & A_d^{2n+1}BYD^{2n+1} \\ D_d^{2n+2}(CBCX - C)A^{2n} & D_d^{2n+2}CBYD^{2n+1} \end{bmatrix} \\
&- \begin{bmatrix} A_d^{2n+2}BCXA^{2n+1} & A_d^{2n+2}(BCBY - B)D^{2n} \\ D_d^{2n+1}CXA^{2n+1} & D_d^{2n+1}CBYD^{2n} \end{bmatrix} \\
=& \begin{bmatrix} A_d^{2n+1}(BCS(n) + A_dBCS(n)A) & A_d^{2n+1}(A_dBCB\widetilde{S}(n) + B\widetilde{S}(n)D) \\ D_d^{2n+1}(D_dCBCS(n) + CS(n)A) & D_d^{2n+1}(CB\widetilde{S}(n) + D_dCB\widetilde{S}(n)D) \end{bmatrix} \\
&- \begin{bmatrix} A_d^{2n+1}(BCX + A_dBCXA)A^{2n} & A_d^{2n+1}\left[BYD + A_d(BCBY - B)\right]D^{2n} \\ D_d^{2n+1}\left[CXA + D_d(CBCX - C)\right]A^{2n} & D_d^{2n+1}(CBY + D_dCBYD)D^{2n} \end{bmatrix},
\end{aligned}$$

where $S(n)$ and $\widetilde{S}(n)$ are defined in (22).

By (24) ,

$$\begin{aligned}
&\sum_{n=0}^{k-1} Q_d^{2n+1}(I + Q_dP)P^{\pi}P^{2n} \\
=& \sum_{n=1}^{k-1}\begin{bmatrix} A_d^{2n+1}(BCS(n) + A_dBCS(n)A) & A_d^{2n+1}(A_dBCB\widetilde{S}(n) + B\widetilde{S}(n)D) \\ D_d^{2n+1}(D_dCBCS(n) + CS(n)A) & D_d^{2n+1}(CB\widetilde{S}(n) + D_dCB\widetilde{S}(n)D) \end{bmatrix} \\
=& \begin{bmatrix} A_d + L & N \\ \widetilde{N} & D_d + \widetilde{L} \end{bmatrix}, \qquad (25)
\end{aligned}$$

where N, $\widetilde{N}$, L and $\widetilde{L}$ are defined in (22).

From (23) and (25), (21) follows. □

The next result is a generalization of (Deng & Wei, 2009, Theorem 3.8).

Theorem 4. *Let M be matrix of a form* (1). *If $AA^{\pi}B = 0$, $BC(I - A^{\pi}) = 0$ and $DC = 0$, then*

$$M_d = \begin{bmatrix} T & \widetilde{T} \\ CA_dT + CA^{\pi}X - CA_dXA & CA_d\widetilde{T} + YD - CA_dBY \end{bmatrix}, \qquad (26)$$

where $k = s+t+2p+1$, $s = \operatorname{Ind}(A)$, $t = \operatorname{Ind}(D)$, $p = \operatorname{Ind}(BC)$, $q = \operatorname{Ind}(CA^{\pi}B)$, and

$$\begin{aligned}
T &= A^{\pi}XA + A_d + \sum_{n=0}^{\lceil \frac{k}{2} \rceil - 1}(G(n) - J(n)), \\
\widetilde{T} &= A^{\pi}BY + \sum_{n=0}^{\lceil \frac{k}{2} \rceil - 1}(H(n) - K(n)), \\
G(n) &= \sum_{i=0}^{n-1} A_d^{2n+1}[(BC)^{i+1} + A_d(BC)^{i+1}A]A^{2n-2i-2}, \\
H(n) &= A_d^{2n+2}B(CB)^{n}D^{\pi} \\
&\quad + \sum_{i=0}^{n-1} A_d^{2n+1}[B(CB)^{i} + A_dB(CB)^{i}D]D^{2n-2i-1}D^{\pi}, \\
J(n) &= A_d^{2n+1}(BC)^{n+1}X + A_d^{2n+2}(BC)^{n+1}XA, \\
K(n) &= A_d^{2n+1}B(CB)^{n}YD + A_d^{2n+2}B(CB)^{n+1}Y, \\
X &= \sum_{i=0}^{\lceil \frac{s}{2} \rceil - 1}(BC)_d^{i+1}A^{2i}, \\
Y &= (CA^{\pi}B)^{\pi}\sum_{i=0}^{q-1}(CA^{\pi}B)^{i}D_d^{2i+2} + \sum_{i=0}^{\lceil \frac{t}{2} \rceil - 1}(CA^{\pi}B)_d^{i+1}D^{2i}D^{\pi}.
\end{aligned}$$

Proof. Split matrix M as $M = P + Q$, where

$$P = \begin{bmatrix} AA^{\pi} & B \\ CA^{\pi} & D \end{bmatrix} \text{ and } Q = \begin{bmatrix} A^2A_d & 0 \\ CAA_d & 0 \end{bmatrix}.$$

From Lemma 1 and Lemma 5, we have, for every integer $n \geq 1$,

$$Q_d^n = \begin{bmatrix} A_d^n & 0 \\ CA_d^{n+1} & 0 \end{bmatrix}$$

and then

$$Q^{\pi} = \begin{bmatrix} A^{\pi} & 0 \\ -CA_d & I \end{bmatrix} \text{ and } Q^{\pi}Q = 0.$$

Since $PQ = 0$,

$$M_d = (P + Q)_d = Q^{\pi}P_d + \sum_{i=0}^{\lceil \frac{k}{2} \rceil - 1} Q_d^{2i+1}(I + Q_dP)P^{2i}P^{\pi}, \quad (27)$$

where $k \geq \operatorname{Ind}(P)$, by Lemma 3.

Since $AA^{\pi}B = 0$ and $DCA^{\pi} = 0$ in P, we get, by Theorem 1,

$$P_d = \begin{bmatrix} XAA^{\pi} & BY \\ CA^{\pi}X & YD \end{bmatrix},$$

where

$$X = (BCA^{\pi})^{\pi}\sum_{i=0}^{p-1}(BCA^{\pi})^{i}(AA^{\pi})_{d}^{2i+2} + \sum_{i=0}^{\lceil \frac{s}{2}\rceil-1}(BCA^{\pi})_{d}^{i+1}(AA^{\pi})^{2i}(AA^{\pi})^{\pi}$$
$$= \sum_{i=0}^{\lceil \frac{s}{2}\rceil-1}(BC)_{d}^{i+1}A^{2i} \text{ (by Lemma 5 and } BC(I-A^{\pi})=0),$$
$$Y = (CA^{\pi}B)^{\pi}\sum_{i=0}^{q-1}(CA^{\pi}B)^{i}D_{d}^{2i+2} + \sum_{i=0}^{\lceil \frac{t}{2}\rceil-1}(CA^{\pi}B)_{d}^{i+1}D^{2i}D^{\pi},$$

and $s = \mathrm{Ind}(AA^{\pi}) = \mathrm{Ind}(A), t = \mathrm{Ind}(D), p = \mathrm{Ind}(BCA^{\pi}) = \mathrm{Ind}(BC)$, $q = \mathrm{Ind}(CA^{\pi}B) \leq p+1$, and, by Remark 2(ii), $\mathrm{Ind}(P) \leq s+t+2p+1$.

Note that $XA^{\pi} = X$. Then

$$P_d = \begin{bmatrix} XA & BY \\ CA^{\pi}X & YD \end{bmatrix}.$$

Since $AA^{\pi}B = 0$ and $DC = 0$, $AA^{\pi}(BC)_d = 0$ and $D(CA^{\pi}B)_d = 0$ and therefore $AA^{\pi}X = 0$ and $DY = D_d$. Thus

$$P^{\pi} = I - \begin{bmatrix} AA^{\pi} & B \\ CA^{\pi} & D \end{bmatrix}\begin{bmatrix} XA & BY \\ CA^{\pi}X & YD \end{bmatrix} = \begin{bmatrix} I-BCX & -BYD \\ -CA^{\pi}XA & D^{\pi}-CA^{\pi}BY \end{bmatrix}.$$

For every integer $n \geq 0$,

$$Q_d^{2n+1}(I+Q_dP) = Q_d^{2n+1}\begin{bmatrix} I & A_dB \\ 0 & I+CA_d^2B \end{bmatrix} = \begin{bmatrix} A_d^{2n+1} & A_d^{2n+2}B \\ CA_d^{2n+2} & CA_d^{2n+3}B \end{bmatrix}. \quad (28)$$

Note that

$$P^2 = \begin{bmatrix} A^2A^{\pi}+BCA^{\pi} & BD \\ CAA^{\pi} & CA^{\pi}B+D^2 \end{bmatrix}.$$

We can prove, for every integer $n \geq 1$,

$$P^{2n} = \begin{bmatrix} \sum_{i=0}^{n}(BC)^iA^{2n-2i}A^{\pi} & \sum_{i=0}^{n-1}B(CB)^iD^{2n-2i-1} \\ \sum_{i=0}^{n-1}CA^{\pi}(BC)^iA^{2n-2i-1} & \sum_{i=0}^{n}(CA^{\pi}B)^iD^{2n-2i} \end{bmatrix}$$

by induction on n. By (28), for every integer $n \geq 1$,

$$Q_d^{2n+1}(I+Q_dP)P^{2n} = \sum_{i=0}^{n-1}\begin{bmatrix} A_d^{2n+1}[(BC)^{i+1}+A_d(BC)^{i+1}A]A^{2n-2i-2} & A_d^{2n+1}[B(CB)^i+A_dB(CB)^iD]D^{2n-2i-1} \\ CA_d^{2n+2}[(BC)^{i+1}+A_d(BC)^{i+1}A]A^{2n-2i-2} & CA_d^{2n+2}[B(CB)^i+A_dB(CB)^iD]D^{2n-2i-1} \end{bmatrix} + \begin{bmatrix} 0 & A_d^{2n+2}B(CB)^n \\ 0 & CA_d^{2n+3}B(CB)^n \end{bmatrix} = \begin{bmatrix} G(n) & \overline{H}(n) \\ CA_dG(n) & CA_d\overline{H}(n) \end{bmatrix},$$

where

$$\overline{H}(n) = A_d^{2n+2}B(CB)^n + \sum_{i=0}^{n-1}A_d^{2n+1}[B(CB)^i+A_dB(CB)^iD]D^{2n-2i-1}.$$

Note that $\overline{H}(n)C = A_d^{2n+2}(BC)^{n+1}$. Since $BCABC = BCA^{\pi}AB = 0$, $G(n)B = A_d^{2n+1}(BC)^nB$ for $n \geq 1$, we have

$$Q_d^{2n+1}(I+Q_dP)P^{2n}P^{\pi} = \begin{bmatrix} G(n)(I-BCX)-A_d^{2n+2}(BC)^{n+1}XA & \overline{H}(n)D^{\pi}-A_d^{2n+2}(BC)^{n+1}BY-G(n)BYD \\ CA_dG(n)(I-BCX)-CA_d^{2n+3}(BC)^{n+1}XA & CA_d\overline{H}(n)D^{\pi}-CA_d^{2n+3}(BC)^{n+1}BY-CA_dG(n)BYD \end{bmatrix}$$
$$= \begin{bmatrix} G(n) & \overline{H}(n)D^{\pi} \\ CA_dG(n) & CA_d\overline{H}(n)D^{\pi} \end{bmatrix} - \begin{bmatrix} A_d^{2n+1}(BC)^{n+1}X+A_d^{2n+2}(BC)^{n+1}XA & A_d^{2n+1}(BC)^nBYD+A_d^{2n+2}(BC)^{n+1}BY \\ CA_d^{2n+2}(BC)^{n+1}X+CA_d^{2n+3}(BC)^{n+1}XA & CA_d^{2n+2}(BC)^nBYD+CA_d^{2n+3}(BC)^{n+1}BY) \end{bmatrix}$$
$$= \begin{bmatrix} G(n) & H(n) \\ CA_dG(n) & CA_dH(n) \end{bmatrix} - \begin{bmatrix} J(n) & K(n) \\ CA_dJ(n) & CA_dK(n) \end{bmatrix}.$$

Also, by (28) and $G(0) = 0$ and $H(0) = A_d^2BD^{\pi}$,

$$Q_d(I+Q_dP)P^{\pi} = \begin{bmatrix} A_d(I-BCX)-A_d^2BCXA & A_d^2B(D^{\pi}-CBY)-A_dBYD \\ CA_d^2(I-BCX)-CA_d^3BCXA & CA_d^3B(D^{\pi}-CBY)-CA_d^2BYD \end{bmatrix}$$
$$= \begin{bmatrix} A_d & 0 \\ CA_d^2 & 0 \end{bmatrix} + \begin{bmatrix} G(0) & H(0) \\ CA_dG(0) & CA_dH(0) \end{bmatrix} - \begin{bmatrix} J(0) & K(0) \\ CA_dJ(0) & CA_dK(0) \end{bmatrix},$$

$$Q^{\pi}P_d = \begin{bmatrix} A^{\pi}XA & A^{\pi}BY \\ CA^{\pi}X-CA_dXA & YD-CA_dBY \end{bmatrix}.$$

The proof is complete. □

Using (20) and Theorem 4, we have the following result.

Remark 4. *Let M be matrix of a form* (1). *If $DD^{\pi}C = 0$, $CB(I - D^{\pi}) = 0$ and $AB = 0$, then then we can get another representation for the Drazin inverse by Theorem 4 .*

The last result is gained by utilizing Corollary 2.

Theorem 5. *Let M be matrix of a form* (1). *If $AA^{\pi}B = 0$, $D_dC = 0$, $CA_d = 0$ and $BD^{\pi} = 0$, then*

$$M_d = \begin{bmatrix} A_d & -A_dBD_d + A^{\pi}BD_d^2 \\ 0 & D_d + \sum_{n=0}^{t} D^nCBD_d^{n+3} \end{bmatrix}, \tag{29}$$

where $t = \operatorname{Ind}(D)$.

Proof. Let $M = P + Q$, where

$$P = \begin{bmatrix} AA^{\pi} & B \\ C & D^2D_d \end{bmatrix}, \qquad Q = \begin{bmatrix} A^2A_d & 0 \\ 0 & DD^{\pi} \end{bmatrix}.$$

Thus

$$\begin{aligned} P^n &= \begin{bmatrix} A^nA^{\pi} & BD^{n-1} \\ CA^{n-1}A^{\pi} & CBD^{n-2} + D^{n+1}D_d \end{bmatrix}, n \geq 2, \quad (30) \\ Q^n &= \begin{bmatrix} A^{n+1}A_d & 0 \\ 0 & D^nD^{\pi} \end{bmatrix}, n \geq 1, \end{aligned}$$

where $\operatorname{Ind}(Q) \leq \operatorname{Ind}(A^2A_d) + \operatorname{Ind}(DD^{\pi}) = 1 + t$ by Lemma 1 and Lemma 5.

By Lemma 2, we have

$$Q_d = \begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix}, \quad Q^{\pi} = \begin{bmatrix} A^{\pi} & 0 \\ 0 & I \end{bmatrix},$$

$$Q^{\pi}Q^n = \begin{bmatrix} 0 & 0 \\ 0 & D^nD^{\pi} \end{bmatrix},$$

$n \geq 1$.

Since $BD^{\pi} = 0$ and $DD_dC = 0$ imply $BD^jC = BD^{j+1}D_dC = 0$ for $j \geq 0$, P satisfies the conditions of Corollary 2(i). Then, by Corollary 2(i) and Lemma 5,

$$P_d = \begin{bmatrix} 0 & BD_d^2 \\ 0 & D_d + CBD_d^3 \end{bmatrix}, P_d^n = \begin{bmatrix} 0 & BD_d^{n+1} \\ 0 & D_d^n + CBD_d^{n+2} \end{bmatrix},$$

$n \geq 1$,

$$P^{\pi} = \begin{bmatrix} I & -BD_d \\ 0 & D^{\pi} - CBD_d^2 \end{bmatrix}$$

and

$$P^nP^{\pi} = \begin{cases} \begin{bmatrix} AA^{\pi} & 0 \\ C & -CBD_d \end{bmatrix}, & n = 1, \\ \begin{bmatrix} A^nA^{\pi} & 0 \\ CA^{n-1}A^{\pi} & 0 \end{bmatrix}, & n \geq 2. \end{cases}$$

Thus $Q_dP^nP^{\pi} = 0$ for $n \geq 1$.

Since $AA^{\pi}B = 0$, $CA_d = 0$ and $BD^{\pi} = 0$, we obtain $PQ = 0$. Therefore, by Lemma 3, we get

$$\begin{aligned} M_d &= \sum_{n=0}^{(t+1)-1} Q^{\pi}Q^nP_d^{n+1} + Q_dP^{\pi} \\ &= \begin{bmatrix} A^{\pi} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 & BD_d^2 \\ 0 & D_d + CBD_d^3 \end{bmatrix} \\ &\quad + \sum_{n=1}^{t}\begin{bmatrix} 0 & 0 \\ 0 & D^nD^{\pi} \end{bmatrix}\begin{bmatrix} 0 & BD_d^{n+2} \\ 0 & D_d^{n+1} + CBD_d^{n+3} \end{bmatrix} \\ &\quad + \begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & -BD_d \\ 0 & D^{\pi} - CBD_d^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & A^{\pi}BD_d^2 \\ 0 & D_d + CBD_d^3 \end{bmatrix} + \sum_{n=1}^{t}\begin{bmatrix} 0 & 0 \\ 0 & D^nCBD_d^{n+3} \end{bmatrix} \\ &\quad + \begin{bmatrix} A_d & -A_dBD_d \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Hence we reach (29). □

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