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Certain problems on the approximation of functions by Fourier-Jacobi sums in the space $\mathbb{L}_2^{(\alpha,\beta)}$

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In this paper, two useful estimates are proved concerning the approximation of one-variable functions from the space $\mathbb{L}_2^{(\alpha,\beta)}$ by partial sums of Fourier-Jacobi series.

Introduction and Preliminaries

It is well known that many problems for partial differential equations are reduced to a power series expansion of the desired solution in terms of special functions or orthogonal polynomials (such as Laguerre, Hermite, Jacobi, etc., polynomials). In particular, this is associated with the separation of variables as applied to problems in mathematical physics (see, e.g.,Tikhonov and Samarskii (1953)-A. Sveshnikov and Kravtsov (2004)).
In V. A. Abilov and Kerimov (2008), Abilov et al. proved two useful estimates for the Fourier transform in the space of square integrable functions on certain classes of functions characterized by the generalized continuity modulus, using a translation operator.
In this paper, we also discuss this subject. More specifically, two useful estimates are proved concerning the approximation of functions form the space $\mathbb{L}_2^{(\alpha,\beta)}$ by partial sums of Fourier-Jacobi series, analogous of the statements proved in V. A. Abilov and Kerimov (2008). For this purpose, we use a generalized translation operator which was defined by Flensted-Jensen an Koornwinder (see Flensted-Jensen and Koornwinder (1973)).

Throughout the paper, α and β are arbitrary real numbers with $\alpha \geq \beta \geq -1/2$ and $\alpha \neq -1/2$. We put $w(x) = (1-x)^\alpha(1+x)^\beta$ and consider problems of the approximation of functions in the Hilbert spaces $L_2([-1,1], w(x)dx)$. Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi orthogonal polynomials, $n \in \mathbb{N}_0 := \{0,1,2,.....\}$ (see Szegö (1959) or A. Erdélyi and Tricomi (1953)). The polynomials $P_n^{(\alpha,\beta)}(x)$, $n \in \mathbb{N}_0$, form a complete orthogonal system in the Hilbert space $L_2([-1,1], w(x)dx)$.

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It is known (see Szegö (1959), Ch. IV) that

$$\max_{-1\leq x\leq 1} |P_n^{(\alpha,\beta)}(x)| = P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{\alpha} = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}.$$

The polynomials

$$R_n^{(\alpha,\beta)}(x) := \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}$$

are called normalized Jacobi polynomials.
In what follows it is convenient to change the variable by the formula $x = \cos t, t \in I := [0,\pi]$. We use the notation

$$\rho(t) = w(\cos t)\sin t = 2^{\alpha+\beta+1}\left(\sin\frac{t}{2}\right)^{2\alpha+1}\left(\cos\frac{t}{2}\right)^{2\beta+1},$$

$$\varphi_n(t) = \varphi_n^{(\alpha,\beta)}(t) := R_n^{(\alpha,\beta)}(\cos t), n \in \mathbb{N}_0.$$

Let $\mathbb{L}_2^{(\alpha,\beta)}$ denote the space of square integrable functions $f(t)$ on the closed interval I with the weight function $\rho(t)$ and the norm

$$\|f\| = \sqrt{\int_0^\pi |f(t)|^2\rho(t)dt}.$$

The Jacobi differential operator is defined as

$$\mathcal{B} := \frac{d^2}{dt^2} + \left(\left(\alpha+\frac{1}{2}\right)\cot\frac{t}{2} - \left(\beta+\frac{1}{2}\right)\tan\frac{t}{2}\right)\frac{d}{dt}.$$

The function $\varphi_n(t)$ satisfies the differential equation

$$\mathcal{B}\varphi_n = -\lambda_n\varphi_n, \qquad \lambda_n = n(n+\alpha+\beta+1), n \in \mathbb{N}_0,$$

with the initial conditions $\varphi_n(0) = 1$ and $\varphi_n'(0) = 0$.

Lemma 1 (Platonov (2009), Proposition 3.2, 3.3.)**.** *The following inequalities are valid for Jacobi functions $\varphi_n(t)$*
1. *For $t \in (0,\pi/2]$ we have*

$$|\varphi_n(t)| < 1.$$

2. *For $t \in [0,\pi/2]$ we have*

$$1 - \varphi_n(t) \leq c_1\lambda_n t^2.$$

3. *For every γ there is a number $c_2 = c_2(\gamma, \alpha, \beta) > 0$ such that for all n and t with $\gamma \le nt \le \frac{\pi n}{2}$ we have*

$$|\varphi_n(t)| \le c_2(nt)^{-\alpha-1/2}.$$

Recall from Platonov (2014), the Fourier-Jacobi series of a function $f \in \mathbb{L}_2^{(\alpha,\beta)}$ is defined by

$$f(t) = \sum_{n=1}^{\infty} a_n(f)\tilde{\varphi}_n(t), \qquad (1)$$

where

$$\tilde{\varphi}_n = \frac{\varphi_n}{\|\varphi_n\|} \quad , \quad a_n(f) = \langle f, \tilde{\varphi}_n \rangle = \int_0^{\pi} f(t)\tilde{\varphi}_n(t)\rho(t)dt.$$

Let

$$S_m f(t) = \sum_{n=1}^{m-1} a_n(f)\tilde{\varphi}_n(t),$$

be a partial sums of series (1), and let

$$E_m(f) = \inf_{P_m} \|f - P_m\|,$$

denote the best approximation of $f \in \mathbb{L}_2^{(\alpha,\beta)}$ by polynomials of the form

$$P_m(t) = \sum_{n=1}^{m-1} c_n\tilde{\varphi}_n(t), c_n \in \mathbb{R}.$$

It is well known that

$$\|f\| = \sqrt{\sum_{n=1}^{\infty} |a_n(f)|^2},$$

$$E_m(f) = \|f - S_m f\| = \sqrt{\sum_{n=m}^{\infty} |a_n(f)|^2}.$$

The Jacobi generalized translation is defined by the formula

$$T_h f(t) = \int_0^{\pi} f(\theta)K(t, h, \theta)\rho(\theta)d\theta, \quad 0 < t, h, \theta < \pi,$$

where $K(t, h, \theta) \ge 0$ is a symmetric function and

$$\int_0^{\pi} K(t, h, \theta)\rho(\theta)d\theta = 1,$$

(see Askey and Wainger (1969) for details).
Below are some properties of the operator T_h (see Platonov (2014)):
(i) $T_h : \mathbb{L}_2^{(\alpha,\beta)} \to \mathbb{L}_2^{(\alpha,\beta)}$ is a continuous linear operator,
(ii) $\|T_h f\| \le \|f\|$,
(iii) $T_h(\varphi_n(t)) = \varphi_n(h)\varphi_n(t)$,
(iv) $a_n(T_h f) = \varphi_n(h)a_n(f)$,
(v) $\|T_h f - f\| \to 0, \quad h \to 0$,
(vi) $\mathcal{B}(T_h f) = T_h(\mathcal{B} f)$.
For every function $f \in \mathbb{L}_2^{(\alpha,\beta)}$ we define the differences $\Delta_h^k f$ of order, $k \in \mathbb{N} = \{1, 2, 3, ...\}$, with step h, $0 < h < \pi$, and the modulus of smoothness $\Omega_k(f, \delta)$ by the formulae

$$\Delta_h^1 f(t) = \Delta_h f(t) = (T_h - I)f(t),$$

where I is the identity operator in $\mathbb{L}_2^{(\alpha,\beta)}$.

$$\Delta_h^k f(t) = \Delta_h(\Delta_h^{k-1} f(t)) = (T_h - I)^k f(t) = \sum_{i=0}^{k} (-1)^{k-1} \binom{k}{i} T_h^i f(t),$$

$k > 1$,

$$\Omega_k(f, \delta) = \sup_{0<h\le\delta} \|\Delta_h^k f\|, \quad \delta > 0,$$

where

$$T_h^0 f(t) = f(t), \quad T_h^i f(t) = T_h(T_h^{i-1} f(t)), \quad i = 1, 2, ..., k.$$

Let $W_{2,\phi}^{r,k}(\mathcal{B})$,$r \in \mathbb{N}_0$, denote the class of functions $f \in \mathbb{L}_2^{(\alpha,\beta)}$ that have generalized derivatives satisfying the estimate

$$\Omega_k(\mathcal{B}^r f, \delta) = O(\phi(\delta^k)), \quad \delta \to 0,$$

where $\phi(x)$ is any nonnegative function given on $[0, \infty)$, and $\mathcal{B}^0 f = f$, $\mathcal{B}^r f = \mathcal{B}(\mathcal{B}^{r-1} f)$, $r = 1, 2, ...$
i.e.,

$$\begin{aligned} W_{2,\phi}^{r,k}(\mathcal{B}) \quad &= \quad \{f \in \mathbb{L}_2^{(\alpha,\beta)}, \mathcal{B}^r f \in \mathbb{L}_2^{(\alpha,\beta)} \\ &\quad \text{and } \Omega_k(\mathcal{B}^r f, \delta) = O(\phi(\delta^k)), \delta \to 0\}. \end{aligned}$$

Lemma 2. *If $f \in W_{2,\phi}^{r,k}(\mathcal{B})$, then*

$$a_n(f) = (-1)^r \frac{1}{\lambda_n^r} a_n(\mathcal{B}^r f), r \in \mathbb{N}_0.$$

Proof. Since $\mathcal{B}$ is self-adjoint (see Platonov (2014)), we have

$$\begin{aligned} a_n(f) \quad &= \quad \langle f, \tilde{\varphi}_n \rangle = -\frac{1}{\lambda_n} \langle f, \mathcal{B}\tilde{\varphi}_n \rangle \\ &= \quad -\frac{1}{\lambda_n} \langle \mathcal{B} f, \tilde{\varphi}_n \rangle = -\frac{1}{\lambda_n} a_n(\mathcal{B} f). \end{aligned}$$

This completes the proof of Lemma 2. □

Lemma 3. *If*

$$f(t) = \sum_{n=1}^{\infty} a_n(f)\tilde{\varphi}_n(t),$$

then

$$T_h f(t) = \sum_{n=1}^{\infty} \varphi_n(h)a_n(f)\tilde{\varphi}_n(t).$$

Here, the convergence of the series on the right-hand side is understood in the sense of $\mathbb{L}_2^{(\alpha,\beta)}$.

Proof. By the definition of the operator T_h,

$$T_h(\tilde{\varphi}_n(t)) = \varphi_n(h)\tilde{\varphi}_n(t).$$

Therefore, for any polynomial

$$Q_N(t) = \sum_{n=1}^{N} a_n(f)\tilde{\varphi}_n(t),$$

we have

$$T_h Q_N(t) = \sum_{n=1}^{N} a_n(f) T_h(\tilde{\varphi}_n(t)) = \sum_{n=1}^{N} \varphi_n(h) a_n(f)\tilde{\varphi}_n(t). \quad (2)$$

Since T_h is a linear bounded operator in $\mathbb{L}_2^{(\alpha,\beta)}$ and the set of all polynomials $Q_N(t)$ is everywhere dense in $\mathbb{L}_2^{(\alpha,\beta)}$, passage to the limit in (2) gives the required equality. □

Remark. Since

$$T_h f(t) - f(t) = \sum_{n=1}^{\infty} (\varphi_n(h) - 1) a_n(f)\tilde{\varphi}_n(t),$$

the Parseval's identity gives

$$\|T_h f - f\|^2 = \sum_{n=1}^{\infty} (1 - \varphi_n(h))^2 |a_n(f)|^2.$$

If $f \in W^{r,k}_{2,\phi}(\mathcal{B})$, from Lemma 2, we have

$$\|\Delta_h^k(\mathcal{B}^r f)\|^2 = \sum_{n=1}^{\infty} (1 - \varphi_n(h))^{2k} \lambda_n^{2r} |a_n(f)|^2. \quad (3)$$

Estimate of Best Approximation

The goal of this work is to prove several estimates for $E_m(f)$ in certain classes of functions in $\mathbb{L}_2^{(\alpha,\beta)}$.

Theorem 1. *Let $r \in \mathbb{N}_0$, $k \in \mathbb{N}$. Then there is a constant $c > 0$ such that, for every $f \in W^{r,k}_{2,\phi}(\mathcal{B})$,*

$$E_m(f) = O(\lambda_m^{-r} \phi((c/m)^k)),$$

when $m \to +\infty$.

Proof. Let $f \in W^{r,k}_{2,\phi}(\mathcal{B})$. By the Hölder inequality, we have

$$\begin{aligned} & E_m^2(f) - \sum_{n=m}^{\infty} \varphi_n(h)|a_n(f)|^2 \\ = & \sum_{n=m}^{\infty} (1 - \varphi_n(h))|a_n(f)|^2 \\ = & \sum_{n=m}^{\infty} |a_n(f)|^{2-\frac{1}{k}} (1 - \varphi_n(h))|a_n(f)|^{\frac{1}{k}} \\ \leq & \left(\sum_{n=m}^{\infty} |a_n(f)|^2\right)^{\frac{2k-1}{2k}} \left(\sum_{n=m}^{\infty} |a_n(f)|^2 (1 - \varphi_n(h))^{2k}\right)^{\frac{1}{2k}} \\ \leq & \left(E_m^2(f)\right)^{\frac{2k-1}{2k}} \left(\lambda_m^{-2r} \sum_{n=m}^{\infty} \lambda_n^{2r} |a_n(f)|^2 (1 - \varphi_n(h))^{2k}\right)^{\frac{1}{2k}}. \end{aligned}$$

From (3), we have

$$\sum_{n=m}^{\infty} (1 - \varphi_n(h))^{2k} \lambda_n^{2r} |a_n(f)|^2 \leq \|\Delta_h^k(\mathcal{B}^r f)\|^2.$$

Therefore

$$E_m^2(f) \leq \sum_{n=m}^{\infty} \varphi_n(h)|a_n(f)|^2 + \left(E_m^2(f)\right)^{\frac{2k-1}{2k}} \lambda_m^{-\frac{r}{k}} \|\Delta_h^k(\mathcal{B}^r f)\|^{\frac{1}{k}}. \quad (4)$$

From Lemma 1, we have

$$\sum_{n=m}^{\infty} \varphi_n(h)|a_n(f)|^2 \leq c_2 (mh)^{-\alpha-1/2} E_m^2(f).$$

For $f \in W^{r,k}_{2,\phi}(\mathcal{B})$ there exist a constant $C > 0$ such that

$$\|\Delta_h^k(\mathcal{B}^r f)\| \leq C\phi(h^k).$$

Choose a constant c_3 such that the number $c_4 = 1 - c_2 c_3^{-\alpha-1/2}$ is positive.
Setting $h = c_3/m$ in the inequality (4), we have

$$c_4 E_m^2(f) \leq (E_m(f))^{2-\frac{1}{k}} \lambda_m^{-\frac{r}{k}} C^{\frac{1}{k}} (\phi((c_3/m)^k))^{\frac{1}{k}}.$$

By raising both sides to the power k and simplifying by $(E_m(f))^{2k-1}$ we finally obtain

$$c_4^k E_m(f) \leq C\lambda_m^{-r} \phi((c_3/m)^k).$$

for all $m > 0$. The theorem is proved with $c = c_3$. □

Theorem 2. *Let $\phi(t) = t^\nu$, then*

$$E_m(f) = O(m^{-2r-k\nu}) \Leftrightarrow f \in W^{r,k}_{2,\phi}(\mathcal{B}),$$

where, $r = 0, 1, ...; k = 1, 2, ...; 0 < \nu < 2$.

Proof. We prove sufficiency by using Theorem 1. Let $f \in W^{r,k}_{2,\phi}(\mathcal{B})$. Then

$$E_m(f) = O(m^{-2r-k\nu}).$$

To prove necessity let

$$E_m(f) = O(m^{-2r-k\nu}).$$

It is easy to show, that there exists a function $f \in \mathbb{L}_2^{(\alpha,\beta)}$ such that $\mathcal{B}^r f \in \mathbb{L}_2^{(\alpha,\beta)}$ and

$$\mathcal{B}^r f(t) = (-1)^r \sum_{n=1}^{\infty} \lambda_n^r a_n(f)\tilde{\varphi}_n(t). \quad (5)$$

From formula (5) and Parseval's identity, we have

$$\|\Delta_h^k(\mathcal{B}^r f)\|^2 = \sum_{n=1}^{\infty} (1 - \varphi_n(h))^{2k} \lambda_n^{2r} |a_n(f)|^2.$$

This sums is divided into two

$$\sum_{n=1}^{\infty} = \sum_{n=1}^{m-1} + \sum_{n=m}^{\infty} = I_1 + I_2,$$

where $m = [h^{-1}]$, We estimate them separately.
From Lemma 1, we have the estimate

$$\begin{aligned} I_2 &\leq c_5 \sum_{n=m}^{\infty} n^{4r}|a_n(f)|^2 = c_5 \sum_{l=0}^{\infty} \sum_{n=m+l}^{m+l+1} n^{4r}|a_n(f)|^2 \\ &\leq c_5 \sum_{l=0}^{\infty} a_l(u_l - u_{l+1}), \end{aligned}$$

with $a_l = (m + l + 1)^{4r}$ and $u_l = \sum_{n=m+l}^{\infty} |a_n(f)|^2$.
For all integers $n \geq 1$, the Abel transformation shows

$$\begin{aligned} \sum_{l=0}^{n} a_l(u_l - u_{l+1}) &= a_0u_0 + \sum_{l=1}^{n}(a_l - a_{l-1})u_l - a_nu_{n+1} \\ &\leq a_0u_0 + \sum_{l=1}^{n} (a_l - a_{l-1})\, u_l, \end{aligned}$$

because $a_nu_{n+1} \geq 0$. Moreover by the finite increments theorem, we have

$$a_l - a_{l-1} \leq 4r(m + l + 1)^{4r-1}$$

Furthermore, by the hypothesis on f there exists $c_6 > 0$ such that, for all $m > 0$

$$E_m^2(f) \leq c_6 m^{-4r-2k\nu}.$$

For $N \geq 1$, we have

$$\begin{aligned} \sum_{l=1}^{n} (a_l - a_{l-1})\, u_l &\leq c_6 \left(1 + \frac{1}{m}\right)^{4r} m^{-2k\nu} \\ &\quad +4rc_6 \sum_{l=1}^{n} \left(1 + \frac{1}{m+l}\right)^{4r-1} (m + l)^{-1-2k\nu} \\ &\leq 2^{4r}c_6m^{-2k\nu} + 4r2^{4r-1}c_6 \sum_{l=1}^{n}(m + l)^{-1-2k\nu}. \end{aligned}$$

Finally, by the integral comparison test we have

$$\sum_{l=1}^{n}(m + l)^{-1-2k\nu} \leq \int_m^{\infty} x^{-1-2k\nu}dx = \frac{1}{2k\nu}m^{-2k\nu}.$$

Letting $n \to \infty$ we see that, for $r \geq 0$ and $k, \nu > 0$, there exists a constant c_7 such that, for all $m \geq 1$ and for $h > 0$,

$$I_2 \leq c_7m^{-2k\nu}.$$

Now, we estimate I_1. From Lemma 1, we have

$$\begin{aligned} I_1 &\leq c_8h^{4k} \sum_{n=1}^{m} n^{4k+4r}|a_n(f)|^2 = c_8h^{4k} \sum_{l=1}^{m-1} \sum_{n=l}^{l+1} n^{4k+4r}|a_n(f)|^2 \\ &\leq c_8h^{4k} \sum_{l=1}^{m-1}(l + 1)^{4k+4r}(v_l - v_{l+1}), \end{aligned}$$

with $v_l = \sum_{n=l}^{\infty} |a_n(f)|^2$.
Using an Abel transformation and proceeding as with I_2 we obtain

$$\begin{aligned} I_1 &\leq c_8h^{4k} \left(v_0 + \sum_{l=1}^{m-1}((l + 1)^{4k+4r} - l^{4k+4r})v_l\right) \\ &\leq c_8h^{4k} \left(v_0 + (4k + 4r)c_6 \sum_{l=1}^{m-1}(l + 1)^{4k+4r-1}l^{-4r-2k\nu}\right), \end{aligned}$$

since $v_l \leq c_6l^{-4r-2k\nu}$ by hypothesis. From the inequality $l + 1 \leq 2l$ we conclude

$$I_1 \leq c_8h^{4k} \left(v_0 + c_9 \sum_{l=1}^{m-1} l^{4k-2k\nu-1}\right).$$

As a consequence of a series comparison for $\mu \geq 1$ and $\mu < 1$ we have the inequality,

$$\mu \sum_{l=1}^{m-1} l^{\mu-1} < m^{\mu}, \text{for} \quad \mu > 0 \quad \text{and} \quad m \geq 2.$$

If $\mu = 4k - 2k\nu > 0$ for $\nu < 2$ then we obtain

$$I_1 \leq c_8h^{4k} \left(v_0 + c_{10}m^{4k-2k\nu}\right) \leq c_8h^{4k} \left(v_0 + c_{10}h^{-4k+2k\nu}\right),$$

since $m \leq 1/h$. If h is sufficiently small then $v_0 \leq c_{10}h^{-4k+2k\nu}$. Then we have

$$I_1 \leq c_{11}h^{2k\nu}.$$

Combining the estimates for I_1 and I_2 gives

$$\|\Delta_h^k(\mathcal{B}^r f)\| = O(h^{k\nu}).$$

The necessity is proved. □

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