# Tomographic Reconstruction of 2D Images

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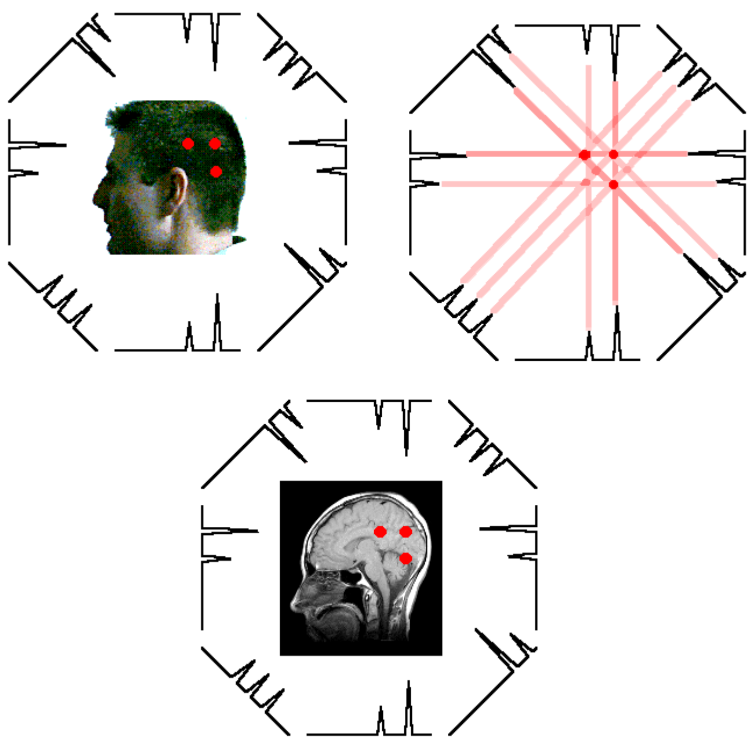
## Introduction:

We are all familiar with the idea of a doctor sending someone to get a "catscan" or CT Scan.

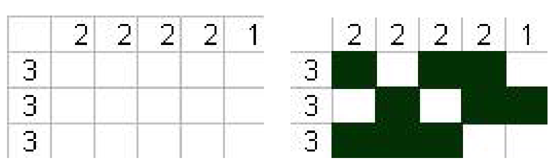


Figures 1a, 1b, 1c

The figures seen above (Figures 1a, 1b, 1c) found in [9] summarize the process and, for purposes of simplicity, we will restrict ourselves to 2D situations where only a "slice" of the object is analyzed in any one scanning "run." As intimated above, a CT (computed tomography) scanner "surrounds" an object to be reconstructed in the form of a useful image. Rays are emitted from the surrounding periphery, sent through the object, and then detected by sensors on the opposite side as Figure 1b suggests. The data received by the sensors is then "integrated," in ways which we will describe, to reconstruct the object (Figure 1c) based on the absorption levels undergone by the rays as they travel through the object. Notice the spikes (sketched on the periphery of the figures) which are a graphical visualization of the absorption levels in each direction. The intensity of a particular ray as it is emitted is known and its intensity upon arrival, after having traveled through the object, is recorded by the sensor directly across from the original emitter. The difference in the intensities is an indicator of the level of attenuation experienced by the ray in question, and in our situation, larger values correspond to greater attenuation due to the presence of higher density material in the path of the ray. (In reality, ratios rather than differences are considered, but our approach is sufficient for our level of explanation). Emitter/Sensor pairs face each other and form emitting/recording banks surrounding the object.

*Our purpose here is to describe, in a very simple way, the mathematical process used to "reconstruct" a visual representation of the 2D object from the absorption measurements determined by the CT scanner.* The original idea for these remarks was found in [1] and our goal is to generate a Maple worksheet that will interactively implement a 2D "catscan" simulation in simple situations.

Consider Figure 2, descriptive of the simplest geometric situation possible:

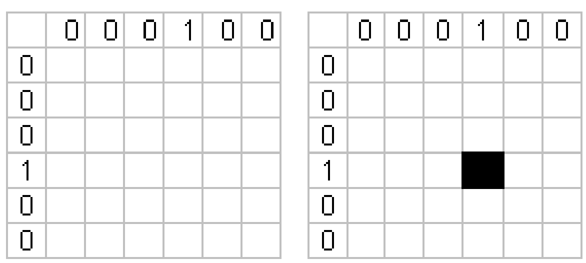



Figures 2a Numerical data, and 2b Object reconstructed

On the left is the "result" of a 2D catscan. Rays emanating from sensors below (southern border of the box) as well as on the right (eastern border) are sent respectively upward and leftward through an object (whose shape we are to deduce) and the numerical values respectively on the northern and western borders of the box (Figure 2a) are the measurements recorded (the higher the value, the higher the attenuation). In Figure 2b, with the (dark) object reconstructed from the data, we are able to understand what is taking place. Very simply, the numerical readings seen in Figure 2a indicate the number of black cells traveled through upward and leftward respectively. The goal therefore is to devise an algorithm which reconstructs the sequence of black cells seen at the right from the numerical data seen on the left. The larger the number of cells available, the finer the resolution and, therefore, the more informative the final image is in terms of the internal constituents of the object (Figure 1c). Notice that the values added along the horizontal border give $N = 2+2+2+2+1 = 9$ and that we obtain the same value of $N = 9$ when adding the values along the vertical border. The number $N$ represents the number of cells that contribute to the attenuation of the rays and, more specifically, indicates the number of cells to be shaded.

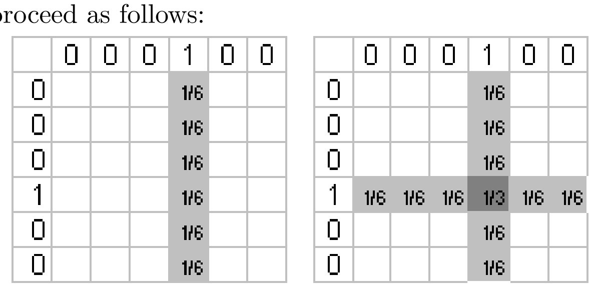
## Simple Introduction to the Reconstruction Algorithm ($N = 1$ case):

The overall idea behind our scheme is rather simple and, in fact, suggested itself as a slight variant of the "cumulative" idea presented in [2, 3]. Let us consider Figure 3 for purposes of illustration:




Figures 3a Data, 3b Solution

Figure 3a shows the absorption readings on the northern and western borders for a CT scan session, and Figure 3b shows the reconstructed image of the object while Figure 4 indicates the reconstruction process. To "reconstruct" the object from the data, let us proceed as follows:




Figures 4a Vertical smear and 4b Horizontal smear

**Step 1:**

In the column with the numerical value 1 in the heading (Figure 4a), we evenly apportion that value through the entire column which is made up of 6 rows. This means that we will "write in" a 1/6 in every relevant box.

**Step 2:**

We then do the same in the horizontal direction (Figure 4b), the key idea being that, as we proceed row-wise, we add the new values to the pre-existing ones. This explains why the cell corresponding to the center of the "cross" seen in Figure 4b receives a value of $1/6+1/6 = 1/3$. That cell was visited twice (as we recorded 1/6 each time), once vertically and once horizontally. We are also using a shading process (with the intensity of the shading proportional to the numerical value in the cell) for greater visual impact. Each of the remaining gray areas has a lesser value of 1/6.

**Step 3:**

This last step in the reconstruction process is an overexposure/underexposure process where we first scan the matrix of fractions, ranking the values in descending order. One piece of information that we do have is the number $N$ of cells that are supposed to be darkened. In our simple example, we have only one, so we let $N = 1$. It is the cell that stood in the way of the horizontal and vertical rays giving a reading of 1 in the relevant row and column. Note that $N$ is simply obtained by adding all the values along the northern border OR the eastern border but not both. Since there is ultimately only one cell to be darkened, we locate within the grid the cell with the maximal fractional value and change its density level to 1. We then "zero out" all remaining cells, obtaining the final result in Figure 3b).

**Details of the Algorithm in the General Case** ($N > 1$):

We have decided, for the sake of convenience, to implement the process using the Maple environment [4]. Let the data presented in Figure 5a be the numerical results of a CT scan session. Using the simple idea presented in the previous paragraph, we take each of the values on the top row and partition ("smear") them down, distributing equally the apportioned value in each cell of a given column (Figure 5b). For instance, in the column with heading 3, a value of 3/8 is entered in each of the cells in that particular column, since there are 8 rows.

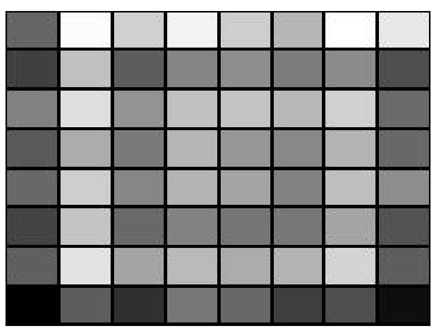
Next, we duplicate the process described above, this time horizontally, adding to the pre-existing values. So, for instance, in the first row (below the header row) we consistently add a value of 1/8 across each cell since the absorption value of 1 seen in the left-most column is to be evenly apportioned horizontally across each of the 8 cells. We did not simplify the final values (Figure 6a) so one can see the process at work.

| | 8 | 1 | 6 | 3 | 4 | 5 | 2 | 7 |
|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | |
| 7 | | | | | | | | |
| 3 | | | | | | | | |
| 5 | | | | | | | | |
| 4 | | | | | | | | |
| 6 | | | | | | | | |
| 2 | | | | | | | | |
| 8 | | | | | | | | |

| | 8 | 1 | 6 | 3 | 4 | 5 | 2 | 7 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1/8 | 3/4 | 3/8 | 1/2 | 5/8 | 1/4 | 7/8 |
| 7 | 1 | 1/8 | 3/4 | 3/8 | 1/2 | 5/8 | 1/4 | 7/8 |
| 3 | 1 | 1/8 | 3/4 | 3/8 | 1/2 | 5/8 | 1/4 | 7/8 |
| 5 | 1 | 1/8 | 3/4 | 3/8 | 1/2 | 5/8 | 1/4 | 7/8 |
| 4 | 1 | 1/8 | 3/4 | 3/8 | 1/2 | 5/8 | 1/4 | 7/8 |
| 6 | 1 | 1/8 | 3/4 | 3/8 | 1/2 | 5/8 | 1/4 | 7/8 |
| 2 | 1 | 1/8 | 3/4 | 3/8 | 1/2 | 5/8 | 1/4 | 7/8 |
| 8 | 1 | 1/8 | 3/4 | 3/8 | 1/2 | 5/8 | 1/4 | 7/8 |

Figures 5a Data, and 5b Reconstruction First Approximation

| | 8 | 1 | 6 | 3 | 4 | 5 | 2 | 7 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1+1/8 | 1/8+1/8 | 3/4+1/8 | 3/8+1/8 | 1/2+1/8 | 5/8+1/8 | 1/4+1/8 | 7/8+1/8 |
| 7 | 1+7/8 | 1/8+7/8 | 3/4+7/8 | 3/8+7/8 | 1/2+7/8 | 5/8+7/8 | 1/4+7/8 | 7/8+7/8 |
| 3 | 1+3/8 | 1/8+3/8 | 3/4+3/8 | 3/8+3/8 | 1/2+3/8 | 5/8+3/8 | 1/4+3/8 | 7/8+3/8 |
| 5 | 1+5/8 | 1/8+5/8 | 3/4+5/8 | 3/8+5/8 | 1/2+5/8 | 5/8+5/8 | 1/4+5/8 | 7/8+5/8 |
| 4 | 1+1/2 | 1/8+1/2 | 3/4+1/2 | 3/8+1/2 | 1/2 +1/2 | 5/8+1/2 | 1/4+1/2 | 7/8+1/2 |
| 6 | 1+3/4 | 1/8+3/4 | 3/4+3/4 | 3/8+3/4 | 1/2+3/4 | 5/8+3/4 | 1/4+3/4 | 7/8+3/4 |
| 2 | 1+1/4 | 1/8+1/4 | 3/4+1/4 | 3/8+1/4 | 1/2+1/4 | 5/8+1/4 | 1/4+1/4 | 7/8+1/4 |
| 8 | 1+1 | 1/8+1 | 3/4+1 | 3/8+1 | 1/2+1 | 5/8+1 | 1/4+1 | 7/8+1 |



Figures 6 a Numerical distribution, and 6b Density plot

For better visual impact we shade each cell in proportion to its numerical value. One may wish to stare at the density plot (Figure 6b), perhaps "squinting" a bit to see if a recognizable pattern emerges. This ad-hoc approach (squinting) probably lowers our eye resolution, bringing forth the main features while deleting the nuances. To simulate this approach via the density plot, we proceed as follows: recall that numerical values are present inside each cell. Let us start with the higher number in sight, change its value to a 1, go to the next lower one, also change it to a 1, and keep this "descending" process going, entering a value of 1 each time. For how long? Until we have shaded the correct number of cells! As mentioned earlier, this is a number that we can obtain by summing up the initial data listed either along a row or a column (Figure 5a). These two values will be identical. So, in our example, the number of cells to be "filled" is $N = 8+1+6+3+4+5+2+7 = 36 = 1+7+3+5+4+6+2+8$. Once the proper 36 cells have been identified, the remaining ones are "zeroed out." The density plot (black for a 1 and white for a 0) for our example is as follows (Figure 7):

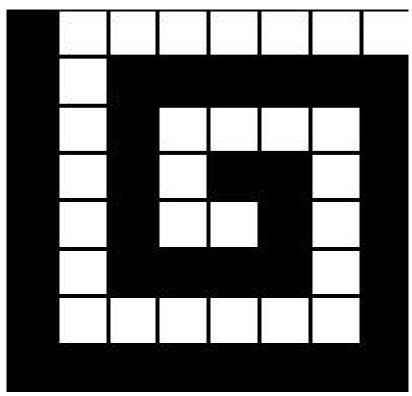


Figure 7: Reconstruction Final Approximation

For "double-checking" purposes, we can add the number of black cells row-wise and column-wise and compare these figures with the original numerical data.

## Problems Encountered and Solutions:

Perhaps the following question has come to mind: "Is it possible for two different 2D designs to generate the same numerical data set?" The answer is "yes." Figure 8 shows a very simple example of a *catastrophic* case (a multi-valued situation in which a single set of data values results from two different object configurations) to use the terminology introduced in [1]:

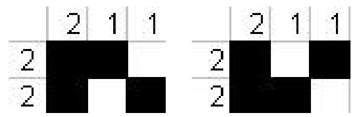


Figure 8: Catastrophic Case

It was expected that the simple method described above would, at times, fail since its very deterministic nature does not have the means to distinguish between competing alternatives. Consider implementing the above algorithm on a piece of paper for the example of Figure 8 and see your quandary when trying to determine which 4 boxes to fully shade. To alleviate this problem, we may decide to collect data from four pairs of opposing sides rather than just two as shown above. The reconstruction process will involve 4 consecutive "smearing" steps (North-South, South-North, East-West, West-East). This is the approach that we took in our Maple application. Using this approach, one may consider increasing the number of cells in both directions as an attempt to improve the resolution. However, doing so increases the likelihood of the occurrence of catastrophic cases. This is understandable from a common-sense point of view, since it is extremely unrealistic to expect a set of only four readings (from four different directions) to enable us to reconstruct complex objects.

Figures 1a, b, c, seen at the very beginning of this article, suggest a hexagonal improvement as a further generalization. Such generalizations point us in a direction that may have been obvious from the start, namely an eventual circular arrangement of emitter/sensor banks surrounding the object to be scanned. One notion that we should address that did not fully emerge in our simple presentation is that the internal grid of cells has its own *Cartesian* geometry (rows and columns) while the sensor banks seem to evolve into a *polar* geometry. Lengths of paths traveled through each cell, because they differ depending on the direction of approach, need to be taken into account in a circular model. It is also probably obvious that better resolution of the final image calls for a finer mesh for our internal grid while maintaining its Cartesian character of rows and columns. For those who are interested, Modules [5, 6], in which the different rays' angle of approach to the grid is figured into the analysis, provide a good place to continue investigating this fascinating topic. You will see where modifying our simplified technique gives rise to the use of linear algebra with systems of equations recording the level of absorption at each cell cumulated along the path of each ray. [7] is the next level of generalization and brings the power of calculus into the fray for a more powerful handling of the situation. Finally [8,9] present results that open the door to further and more sophisticated improvements introducing the reader to Fourier and Radon Transforms, mathematical operations very prevalent in computed tomography.

## References

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