An Example in Elementary Probability

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Abstract. The authors investigate whether independence of stages in a multi-stage experiment is sufficient to guarantee that $P\left(\cap_{i=1}^{m} A_i\right) = \prod_{i=1}^{m} P(A_i)$ for $m \geq 3$; where A_i is an event of the i^{th} stage.

Although some of our discussion applies in more general circumstances, let us suppose that S is a finite set of outcomes of some probabilistic experiment, and that P is a presumably correct assignment of probabilities to the elements of S. All events admitted to discussion will be subsets of S. The probability $P(A)$ of an event A is the sum of the probabilities assigned by P to the elements of A. The disjunction and conjunction of events A and B will be denoted $A \cup B$ and $A \cap B$, respectively.

By the standard definitions, events A and B are *independent* if and only if

$$P(A \cap B) = P(A)\, P(B),$$

and are *mutually exclusive* if and only if

$$P(A \cap B) = 0.$$

However, because it is well known that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

the condition for mutual exclusivity is equivalent to:

$$P(A \cup B) = P(A) + P(B).$$

Thus, there is an analog (some would say symmetry) between the two definitions.

The analogy breaks down immediately when we consider more than two events. By all sorts of elementary considerations it can be shown that if $A_i, \ldots, A_k$ are pairwise mutually exclusive events then

$$P\left(\cup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P\left(A_i\right),$$

whereas examples abound of sequences $A_1, \ldots, A_k$, $k \geq 3$, which are pairwise independent and yet $P\left(\cap_{i=1}^{k} A_i\right) \neq \prod_{i=1}^{k} P\left(A_i\right)$. In the simplest example of this phenomenon that we know of, $S = \{a, b, c, d\}$, each outcome has probability $\frac{1}{4}$ (for instance, S could be the set of outcomes of the experiment of flipping a fair coin twice), and we let $A = \{a, b\}$, $B = \{b, c\}$, and $C = \{a, c\}$. Then A. B, C are pairwise independent and yet

$$0 = P\left(A \cap B \cap C\right) \neq P\left(A\right) P\left(B\right) P\left(C\right) = \frac{1}{8}.$$

But what if the underlying experiment takes place in "stages," or "components," and the pairwise independent events $A_1, \ldots, A_m$ "belong" to different, pairwise independent components? Would such super-duper pairwise independence imply that $P\left(\cap_{i=1}^{m} A_i\right) = \prod_{i=1}^{m} P\left(A_i\right)$? Let us make the question precise: we will say that S has k components if and only if $S = S_1 \times S_2 \times \ldots \times S_k$ for some $S_1, S_2, \ldots, S_k$; an event A belongs to the j^{th} component if and only if $A = A_1 \times A_2 \times \ldots \times A_k$ where $A_i = S_i$ for $i \neq j$, and $A_j \subseteq S_j$. Components i and j are *independent* if and only if whenever A is an event belonging to the i^{th} component, and B is an event belonging to the j^{th} component, then A and B are independent.

For instance, consider the experiment of flipping a (not necessarily fair) coin twice. The experiment involves two apparently "independent" actions, or stages, and this separation of the stages of the experiment can be, and usually is, taken into account by the choice of a set of outcomes with two components: $S = \{H, T\} \times \{H, T\}$. To belabor the obvious, the components of S are the sets of outcomes of the different stages of the experiment. It is elementary to verify that the components of S are independent, by the definition above. The example above of three events A, B, C which are pairwise independent yet not "jointly" independent can be instantiated with this S (if the coin is fair), but we cannot arrange for A, B, C to belong to different independent components, because there are only two components to choose from.

Here is the example promised in the title. We describe a 3-stage experiment involving a fair coin and two urns, Same and Different. Same contains one red ball, Different contains one green ball. The fair coin is flipped twice. If the results of the two flips are the same, the red ball is drawn from Same; otherwise, the green ball is drawn from Different. We take $S = \{H, T\} \times \{H, T\} \times \{r, g\} = \{HHr, HHg, HTr, HTg, THr, THg, TTr, TTg\}$, where, for instance, HHr means "heads came up on the first flip, heads came up on the second flip, and then a red ball was drawn." The correct probability assignment is elementary, and it is elementary to verify that the components of S are pairwise independent. Let $A =$ "heads came up on the first flip" $= \{H\} \times \{H, T\} \times \{r, g\}$; $B =$ "heads came up on the second flip" $= \{H, T\} \times \{H\} \times \{r, g\}$; and $C =$ "the ball drawn was green" $= \{H, T\} \times \{H, T\} \times \{g\}$. Then A, B, and C belong to the first, second, and third components of S, respectively, but

$$P(A \cap B \cap C) = P(HHg) = 0 \neq \frac{1}{8} = P(A)\,P(B)\,P(C)\,.$$

(For those who feel that the third stage of this experiment is somehow bogus: the example can be gussied up by letting Same contain x red and y green balls, and Different contain y red and x green balls, with $o < x < y$. Then the outcomes of the first two stages do not determine the outcome of the third, but the example still works.)

Editor's Note: Shuangchi He, Zhigang Li, and Weidong Tang were students in Dr. Peter Johnson's Information Theory course at Auburn University. In the class, Dr. Johnson posed the problem of whether or not events associated with different pairwise independent stages of an experiment must necessarily be jointly independent. His three students submitted roughly the same counter-example.

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