Undergraduate Research

A Remark on the Concavity of a C^2 Function

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Biographical Sketch

Aaron Benson, the son of Bruce Benson and Chanel Wilson, entered Tuskegee University in 2001, after graduating from Lake Forest High School in Felton, DE. He graduated in 2006 and is currently a software engineer for the Department of Defense. This paper was written under the direction and supervision of Dr. Hussain Talibi, his senior seminar advisor at Tuskegee University.

ABSTRACT: We prove that if a function $f : [0,1] \to \mathbb{R}$ has a continuous second derivative for all x in $[0,1]$, and the tangent line to the graph of f intersects the graph only at $(x, f(x))$ for all x in $[0,1]$, then f is always concave up or always concave down on $[0,1]$.

Introduction

When students take a first semester Calculus course, and learn about second derivatives and their use in graphing, it is only natural to ask the question: "If the tangent line to the graph of a function intersects the graph only at the point of tangency for all points in some interval, could the function change concavity on that interval?" By looking at graphs, it seems obvious that the answer is "no." However, students at this stage of mathematical development do not have the prerequisite knowledge to prove it. We give a proof here that requires knowledge of Advanced Calculus.

THEOREM 1. *Let $f : [0,1] \to \mathbb{R}$ be a function with a continuous second derivative, and suppose that for every $x \in [0,1]$, the tangent line to the graph of f intersects the graph of f only at $(x, f(x))$. Then either $f''(x) > 0$ for all $x \in [0,1]$ or $f''(x) < 0$ for all $x \in [0,1]$.*

Proof: The tangent line at $(x, f(x))$ intersects the graph of f at $(y, f(y))$ if and only if

$$f'(x) = \frac{f(y) - f(x)}{y - x},$$

that is

$$f(y) - f(x) = f'(x)(y - x).$$

This equation can be rewritten as

$$\int_x^y f'(t)dt = \int_x^y f'(x)dt,$$

and in turn as

$$\int_x^y (f'(t) - f'(x))dt = 0,$$

and finally as

$$\int_x^y \int_x^t f''(s)dsdt = 0.$$

Let us define the function $F : [0,1] \times [0,1] \to \mathbb{R}$ by

$$F(x,y) = \int_x^y \int_x^t f''(s)dsdt.$$

The statement of the theorem is then equivalent to the statement:

If F never has value zero at any (x, y) with $x \neq y$, then either $f'' > 0$ everywhere on $[0, 1]$, or $f'' < 0$ everywhere on $[0, 1]$.

We will prove the contrapositive of this statement.

Suppose that $f''(v) > 0$ for some $v \in [0, 1]$, and $f''(w) < 0$ for some $w \in [0, 1]$, with $v \neq w$. Without loss of generality, we may assume that $v < w$. Since $f''(v) > 0$ and f'' is continuous, we can find an interval (a, b) containing v, with $0 \leq a < b \leq 1$, such that $f^{''} > 0$ everywhere on (a, b). Similarly we can find an interval (c, d) containing w with $0 \leq c < d \leq 1$, such that $f'' < 0$ everywhere on (c, d). We can clearly choose those intervals so that $(a, b) \cap (c, d) = \emptyset$. So $F(a, b) > 0$, and $F(c, d) < 0$, and it follows by the (Generalized) Intermediate Value Theorem (see [1], p. 154) that there is a point $(x, y) \in [0, 1]^2$, with $x \neq y$, such that $F(x, y) = 0$, which finishes the proof.

References

[1] Munkres, James, *Topology - A First Course,* Prentice-Hall, 1975.

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