Viewing a Problem from Different Perspectives

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Abstract. The author examines a problem in probability from several different perspectives, with the objective of demonstrating how this can enhance a student's understanding of the underlying probability theory.

Six cards are drawn successively at random and without replacement from a deck of 52 playing cards. What is the probability that the third spade appears on the sixth draw?

Let A be the event of drawing (exactly) two spades in the first five cards drawn, and let B be the event of selecting a spade on the sixth draw. The probability that we wish to compute is $P(A \cap B)$. Out of C_5^{52} total number of ways of choosing 5 cards out of 52 cards, $C_2^{13} C_3^{39}$ of them consist of two spades. So

$$P(A) = \frac{C_2^{13} C_3^{39}}{C_5^{52}} = 0.274,$$

and $P(B|A) = \frac{11}{47} = 0.234$. Hence, the desired probability is:

$$P(A \cap B) = P(A)\, P(B|A) = (0.274)\,(0.234) = 0.064.$$

Note: Throughout this paper we use the symbol C_m^n to represent the number of combinations of n objects selected m at a time. This is sometimes denoted $\binom{n}{m}$. Similarly, we will denote the number of permutations of n objects selected m at a time by P_m^n.

Note Also: The preceding problem and the solution were taken from [1]. This is not a very easy probability problem and one might be content once the solution has been found. Although it is not obvious, other strategies for solving this problem exist. We now look at this problem from five alternate perspectives.

Approach #1

Let A and B be defined as above. For a change, we now write $P(A \cap B) = P(B) P(A|B)$. Since the sixth card is either a spade, a heart, a diamond or a club, $P(B) = \frac{1}{4}$. The probability that the first five cards contain two spades, given that the sixth card is a spade, is therefore:

$$P(A|B) = \frac{C_2^{12} C_3^{39}}{C_5^{51}} = 0.257.$$

So the desired probability is the product of the two found above,

$$P(B) P(A|B) = (.25)(0.257) = 0.064.$$

Approach #2

In both of the methods above, the event of our interest is expressed as the intersection of two events and the formula for conditional probability is used. We now use the same formula, but in a different way.

Let A be the event of our interest, i.e., the third spade appears on the sixth draw, and let B be the event that the first three cards are, in order, ace of clubs, ace of diamonds, ace of hearts, while the last three cards are spades. Out of the P_6^{52} number of permutations of six cards out of 52 cards, P_3^{13} yield the event B - the first three cards are fixed and the last three can be any permutations of three spades out of a total of 13 spades. Therefore,

$$P(B) = \frac{P_3^{13}}{P_6^{52}} = 1.1707 \times 10^{-7}.$$

We now compute $P(B|A)$. Given that the third spade is drawn on the sixth draw, we see that the number of ways to choose two positions out of the first five for the other two spades and then have three ordered non-spades on the remaining three positions is $C_2^5 P_3^{39}$, and that in only one of these ways do the two spades fall on the fourth and fifth position while the first three cards in order are ace of clubs, ace of diamonds and ace of hearts. Therefore,

$$P(B|A) = \frac{1}{C_2^5 \cdot P_3^{39}} = 1.8237 \times 10^{-6}.$$

Observing that, in this case, $A \cap B = B$, we now rewrite the equation $P(A \cap B) = P(A) P(B|A)$ as $P(B) = P(A) P(B|A)$, and hence:

$$P(A) = \frac{P(B)}{P(B|A)} = \frac{1.1707 \times 10^{-7}}{1.8237 \times 10^{-6}} = 0.064.$$

So the desired probability is the quotient of the two probabilities already found.

Approach #3

Let us define the following events:

A = a non-spade on the first draw
B = non-spade on the second draw
C = a non-spade on the third draw
D = a spade on the fourth draw
E = a spade on the fifth draw
F = a spade on the sixth draw.

The probability of three non-spades on the first three draws and three spades on the last three is therefore:

$$\begin{aligned} P(A \cap B \cap C \cap D \cap E \cap F) &= P(A) \cdot P(B|A) \cdot P(C|A \cap B) \\ &\cdot\ P(D|A \cap B \cap C) \\ &\cdot\ P(E|A \cap B \cap C \cap D) \\ &\cdot\ P(F|A \cap B \cap C \cap D \cap E) \\ &= \tfrac{39}{52} \cdot \tfrac{38}{51} \cdot \tfrac{37}{50} \cdot \tfrac{13}{49} \cdot \tfrac{12}{48} \cdot \tfrac{11}{47} = 0.0064. \end{aligned}$$

Notice that as long as the third spade appears on the six draw, the first two spades can be on *any* two of the first five draws, a total of $C_2^5 = 10$ possibilities. Also note that all ten events are equally likely to occur. So the desired probability is 10 times the number we obtained above. (i.e., $P(*) = 10 \cdot 0.0064 = 0.064$.)

Approach #4

If we are only interested in getting three spades in the first six draws with no regard to the order in which they are drawn, then the probability is:

$$\frac{C_3^{13} C_3^{39}}{C_6^{52}} = 0.128.$$

The probability of getting three spades, with the third spade being drawn on the sixth draw, is only a fraction of the above probability. To find the exact value of the fraction, we see that out of the possible C_3^6 ways of having three spades in the first six draws, C_2^5 of them have a spade on the sixth draw, so the desired probability is:

$$\frac{C_2^5}{C_3^6} \cdot \frac{C_3^{13} C_3^{39}}{C_6^{52}} = (0.5)(0.128) = 0.064.$$

Approach #5

We now look at the problem from the classical probability point of view. The total number of events in the sample space is P_6^{52}, the number of permutations of six cards taken out of 52 cards. To find out how many of them have the third spade on the sixth draw, we argue as follows. Start with the third spade on the sixth draw, there are C_1^{13}. Next, the other two spades can be on any two of the first five draws, a total of C_2^5 possibilities. Once the *positions* of the two spades are chosen, there are P_2^{12} different permutations of the remaining twelve spades, chosen two at a time. Finally, there are P_3^{39} possible permutations of 39 non-spades chosen three at a time, to fill the remaining three positions. So the number of outcomes in the sample space which yield the event of selecting the third spade on the sixth draw is:

$$C_1^{13} C_2^5 P_2^{12} P_3^{39} = 9.4095 \times 10^8,$$

and consequently the desired probability is:

$$\frac{C_1^{13} C_2^5 P_2^{12} P_3^{39}}{P_6^{52}} = \frac{9.4095 \times 10^8}{1.4658 \times 10^{10}} = 0.064.$$

References

[1] Hogg, Robert V., and Tanis, Elliot A., *Probability and Statistical Inference,* 6^{th} edition, Prentice-Hall, New Jersey, 2001.

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