Solutions and Discussions

Problem 2 — Volume 26, No. 2, Fall, 2002

Buildings A and B are separated by a 12 foot wide alley. One 15 foot ladder rests at the base of building A and leans against the wall of building B. Another 20 foot ladder rests at the base of building B and leans against the wall of building A. What is the height where the two ladders cross?

Editor's Comment: The solution to this problem, given by Laura Steil of Samford University, was published in the Spring, 2003 issue of the *Journal.* Here, we present a generalization of the problem and solution.

Solution

Eric Cleckler, Junior, Troy State University, Troy, AL.

We generalize the problem as follows: Buildings A and B are separated by an alley of width w. A ladder of length L_1 rests at the base of building A and leans against the wall of building B. Another ladder of length L_2 rests at the base of building B and leans against the wall of building A. We wish to derive a formula for the height h where the two ladders cross. Note that our problem will have a solution, provided that $L_1, L_2 > w$. Hence, we will assume that this is the case. The situation is depicted in Figure 1.

From the triangle with sides of length L_1, w, and the wall of building B, we have $\cos(\alpha) = \frac{w}{L_1}$, or:

(Eq. 1) $$\alpha = \cos^{-1}\left(\frac{w}{L_1}\right).$$

Similarly, from the triangle with sides of length L_2, w, and the wall of building A, we have $\cos(\beta) = \frac{w}{L_2}$, or:

$$\beta = \cos^{-1}\left(\frac{w}{L_2}\right). \quad \text{(Eq. 2)}$$

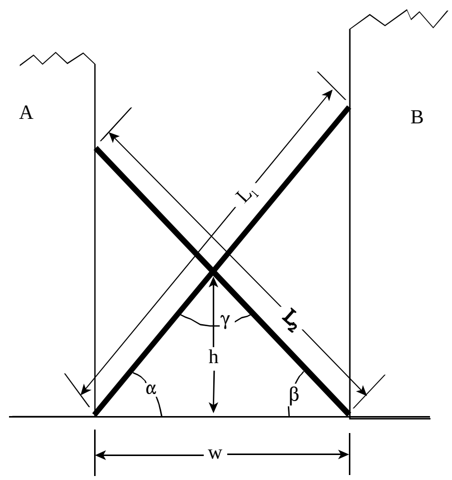

Figure 1

Next, consider the triangle with base w, height h, and angles α, β, and γ shown in Figure 2.

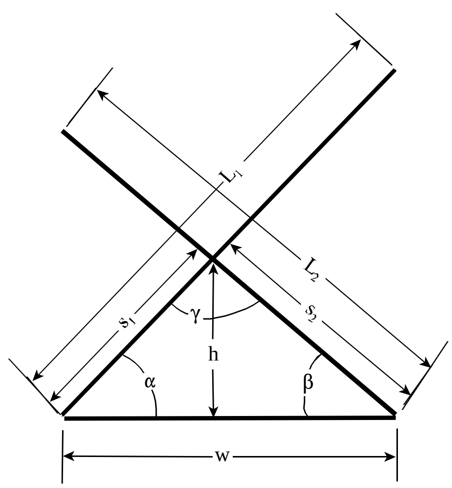

Figure 2

The Law of Sines yields $\frac{\sin(\alpha)}{s_2} = \frac{\sin(\gamma)}{w}$. From this we can see that

(Eq. 3) $$s_2 = \frac{w\sin(\alpha)}{\sin(\gamma)}.$$

Also, since $m\angle\alpha + m\angle\beta + m\angle\gamma = 180°$, we have $m\angle\gamma = 180° - (m\angle\alpha + m\angle\beta)$. Again, using Figure 2, we have $\sin(\beta) = \frac{h}{s_2}$, and hence, $h = s_2 \sin(\beta)$. Substituting the values of a, b, and s_2, from Eq. 1, 2, and 3, we get our answer:

$$\begin{aligned} h &= \frac{w\sin\left(\cos^{-1}\left(\frac{w}{L_1}\right)\right)\sin\left(\cos^{-1}\left(\frac{w}{L_2}\right)\right)}{\sin\left(180° - \left(\cos^{-1}\left(\frac{w}{L_1}\right) + \cos^{-1}\left(\frac{w}{L_2}\right)\right)\right)} \\ &= \frac{w\sin\left(\cos^{-1}\left(\frac{w}{L_1}\right)\right)\sin\left(\cos^{-1}\left(\frac{w}{L_2}\right)\right)}{\sin\left(\cos^{-1}\left(\frac{w}{L_1}\right) + \cos^{-1}\left(\frac{w}{L_2}\right)\right)} \\ &= \frac{w\sqrt{1-\left(\frac{w}{L_1}\right)^2}\sqrt{1-\left(\frac{w}{L_2}\right)^2}}{\frac{w}{L_2}\sqrt{1-\left(\frac{w}{L_1}\right)^2} + \frac{w}{L_1}\sqrt{1-\left(\frac{w}{L_2}\right)^2}} \\ &= \frac{\sqrt{L_1^2-w^2}\sqrt{L_2^2-w^2}}{\sqrt{L_1^2-w^2}+\sqrt{L_2^2-w^2}} \end{aligned}$$

i.e., the height h where the two ladders cross is given by

$$h = \frac{\sqrt{L_1^2 - w^2}\sqrt{L_2^2 - w^2}}{\sqrt{L_1^2 - w^2} + \sqrt{L_2^2 - w^2}}.$$