Undergraduate Research

A Trigonometric Simpson's Rule

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Biographical Sketch

Catherine Cusimano Kirby is the daughter of Donna and Sam Cusimano. Originally from Vestavia Hills, Alabama, she is a 1999 graduate of John Carroll Catholic High School. She graduated with honors from Samford University in 2003 with majors in mathematics and Spanish. She credits Dr. Sonya Stanley, who directed her research for this paper, with having considerable influence on her mathematical development. Catherine now resides in Chattanooga, Tennessee.

Introduction

There are some integrable functions f for which the definite integral $\int_a^b f(x)\,dx$ cannot be calculated exactly. Because of this, numerous methods of approximating definite integrals exist, including Simpson's rule and the trapezoidal rule. These well-known numerical integration methods are based on polynomial interpolation and are well-suited for computer implementation. In this paper, we develop similar methods using trigonometric polynomials and trigonometric splines and present comparisons with existing numerical integration methods.

Simpson's Rule

The scheme for approximating an integral using Simpson's rule is based on the fact that if $P(x)$ is any polynomial of degree three or less, then $\int_a^b P(x)\,dx = \frac{b-a}{6}\left(P(a) + 4P\left(\frac{a+b}{2}\right) + P(b)\right)$. Since there exists a unique quadratic polynomial $Ax^2 + Bx + C$ that interpolates the function f at the points, $(a, f(a))$, $\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right)$, and $(b, f(b))$, the integral of this quadratic polynomial serves as an approximation to $\int_a^b f(x)\,dx$. In other words,

$$\int_a^b f(x)\,dx \approx \int_a^b \left(Ax^2 + Bx + C\right) dx.$$

Hence,

$$\text{(Eq. 1)} \qquad \int_a^b f(x)\,dx \approx \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right).$$

The error term associated with Simpson's rule is

$$-\frac{1}{90}\left[\frac{b-a}{2}\right]^5 f^{(4)}(\xi), \qquad \xi \in (a,b).$$

As a motivating example, we will derive Simpson's rule using the method of undetermined coefficients. This method, illustrated here with $a = 0$ and $b = 1$, yields the formula

$$\int_0^1 f(x)\,dx \approx A_0 f(0) + A_1 f\left(\frac{1}{2}\right) + A_2 f(1).$$

which is exact if $f(x)$ is a polynomial of degree less than or equal to 3. To obtain a formula that calculates integrals of quadratic polynomials exactly, we simply force the integrals of the basis functions,

$f_1(x) = 1$, $f_2(x) = x$, and $f_3(x) = x^2$ to be represented exactly. Doing this we obtain the following:

$$1 = \int_0^1 dx = A_0 + A_1 + A_2$$

$$\tfrac{1}{2} = \int_0^1 x dx = \tfrac{1}{2} A_1 + A_2$$

$$\tfrac{1}{3} = \int_0^1 x^2 dx = \tfrac{1}{4} A_1 + A_2.$$

The solution of the system is $A_0 = \frac{1}{6}$, $A_1 = \frac{2}{3}$, $A_2 = \frac{1}{6}$, thereby yielding

$$\int_0^1 f(x)\, dx \approx \frac{1}{6} f(0) + \frac{2}{3} f\left(\frac{1}{2}\right) + \frac{1}{6} f(1).$$

A more general application of the foregoing procedure on the interval $[a, b]$ can be used to obtain the form of Simpson's rule given in Eq. 1.

Integration Using Trigonometric Polynomials.

It is natural to ask whether a numerical integration formula based on trigonometric polynomials would be better suited to approximate the integral of a trigonometric function than an approach based on quadratic polynomials. In the case of trigonometric polynomials, we have developed an approximation for the integral $\int_a^b f(x)\, dx$ by imitating the method of undetermined coefficients described above in the derivation of Simpson's rule. In our case we use the following basis functions, first introduced in [1]

$$f_1(x) = \left(\tfrac{\sin(t_2 - t)}{\sin(t_2 - t_1)}\right)^2,$$

$$f_2(x) = 2 \tfrac{\sin(t_2 - t)\sin(t - t_1)}{(\sin(t_2 - t_1))^2},$$

$$f_3(x) = \left(\tfrac{\sin(t - t_1)}{\sin(t_2 - t_1)}\right)^2.$$

If we let $h = \frac{b-a}{2}$, the formula that results is:

$$\begin{aligned}
\int_a^b f(x)\, dx \quad &\approx \\
&\csc(2h)\left(-\tfrac{1}{2} + h\cot(2h) + \tfrac{1}{2}\csc(2h)\left(h - \tfrac{1}{4}\sin(4h)\right)\right) f(a) \\
+ \quad &\left(\csc(2h)\left(-\tfrac{1}{2} + \cot(2h) + \csc(2h)\right)\right) f\left(\tfrac{a+b}{2}\right) \\
+ \quad &\csc(2h)\left(-\tfrac{1}{2} + h\cot(2h) + \csc(2h)\left(h - \tfrac{1}{4}\sin(4h)\right)\right) f(b).
\end{aligned}$$

This formula is exact for functions of the form $af_1(x)+bf_2(x)+cf_3(x)$. We demonstrate this in examples in Table 1. Also, even though this formula was derived on the interval $[-h,h]$, it is valid on any interval of length $2h$. For example, if $h=1$ and $b-a=2$, the result is $\int_a^b f(x)\,dx \approx 0.385095 f(a)+1.22981 f\left(\frac{a+b}{2}\right)+0.385095 f(b)$.

Trigonometric-Spline-Based Integration Approximation Method

Now we construct an approximation method based on trigonometric splines. The first step in this process is to choose an appropriate representation for the spline used in the construction. We define a second-degree trigonometric spline in the following way:

Definition 1. *Let $\tilde{t}\in[t_1,t_2]$, $0<t_2-t_1<\pi$ and let A_1, B_1, C_1, A_2, B_2, and C_2 be real numbers. Then, the function $p(t)$ given by*

$$p(t) = A_1\left(\frac{\sin(\tilde{t}-t)}{\sin(\tilde{t}-t_1)}\right)^2 + 2B_1\left(\frac{\sin(\tilde{t}-t)}{\sin(\tilde{t}-t_1)}\right)\left(\frac{\sin(t-t_1)}{\sin(\tilde{t}-t_1)}\right) + C_1\left(\frac{\sin(t-t_1)}{\sin(\tilde{t}-t_1)}\right)^2, \text{ for } t_1\le t<\tilde{t}$$

and

$$p(t) = A_2\left(\frac{\sin(t_2-t)}{\sin(t_2-\tilde{t})}\right)^2 + 2B_2\left(\frac{\sin(t_2-t)}{\sin(t_2-\tilde{t})}\right)\left(\frac{\sin(t-\tilde{t})}{\sin(t_2-\tilde{t})}\right) + C_2\left(\frac{\sin(t-\tilde{t})}{\sin(t_2-\tilde{t})}\right)^2, \text{ for } \tilde{t}\le t\le t_2$$

is a second-degree trigonometric spline.

It is our intention to use this trigonometric spline to develop a numerical integration formula similar to Simpson's rule and to compare the performance of this new formula in approximating integrals of several families of functions with the performance of Simpson's rule. In particular, we will investigate whether our trigonometric version of Simpson's rule approximates trigonometric integrals better than the polynomial-based Simpson's rule.

Using the definition of spline in definition 1, we will develop a trigonometric-spline-based integration method to approximate $\int_a^b f(x)\,dx$. The value of the spline will be calculated at $t=-h$, $t=0$, and $t=h$ and set equal to $y_0=f(-h)$, $y_1=f(0)$, and $y_2=f(h)$ respectively. We set the value of the left-hand side of the spline equal to the value of the right-hand side at $t=0$, and we set the "slope value" at $t = 0$ equal from both the left and the right. Also, we let $a = t_1 = -h$ and $b = t_2 = h$. Using this information, the following system of equations is set up and solved for A_1, B_1, C_1, A_2, B_2, and C_2 :

$$\begin{aligned}
p(t_1) &= A_1\left(\tfrac{\sin(-t_1)}{\sin(-t_1)}\right)^2 + 2B_1\left(\tfrac{\sin(-t_1)}{\sin(-t_1)}\right)\left(\tfrac{\sin(0)}{\sin(-t_1)}\right) \\
&+ C_1\left(\tfrac{\sin(0)}{\sin(-t_1)}\right)^2 = A_1 = y_0 \\
p'(0) &= -2\csc(h)\,B_1 + 2\cot(h)\,C_1 = s
\end{aligned}$$

$$\begin{aligned}
p(0) &= A_1\left(\tfrac{\sin(0)}{\sin(-t_1)}\right)^2 + 2B_1\left(\tfrac{\sin(0)}{\sin(-t_1)}\right)\left(\tfrac{\sin(-t_1)}{\sin(-t_1)}\right) \\
&+ C_1\left(\tfrac{\sin(-t_1)}{\sin(-t_1)}\right)^2 = C_1 = y_1 \\
p(0) &= A_2\left(\tfrac{\sin(t_2)}{\sin(t_2)}\right)^2 + 2B_2\left(\tfrac{\sin(t_2)}{\sin(t_2)}\right)\left(\tfrac{\sin(0)}{\sin(t_2)}\right) \\
&+ C_2\left(\tfrac{\sin(0)}{\sin(t_2)}\right)^2 = A_2 = y_1 \\
p'(0) &= -2\cot(h)\,A_2 + 2\csc(h)\,B_2 = s \\
p(t_2) &= A_2\left(\tfrac{\sin(0)}{\sin(t_2)}\right)^2 + 2B_2\left(\tfrac{\sin(0)}{\sin(t_2)}\right)\left(\tfrac{\sin(t_2)}{\sin(t_2)}\right) \\
&+ C_2\left(\tfrac{\sin(t_2)}{\sin(t_2)}\right)^2 = C_2 = y_2
\end{aligned}$$

where $y_0 = f(-h)$, $y_1 = f(0)$, $y_2 = f(h)$, and $s = \frac{2\delta_1\delta_2}{\delta_1+\delta_2}$ with $\delta_1 = \frac{y_1-y_0}{h}$ and $\delta_2 = \frac{y_2-y_1}{h}$.This system can be represented in matrix form as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\csc(h) & 2\cot(h) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2\cot(h) & 2\csc(h) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ A_2 \\ B_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ s \\ y_1 \\ y_1 \\ s \\ y_2 \end{bmatrix}$$

From this it follows that

$$\int_{-h}^{h} f(x)\,dx = \int_{-h}^{0}\left[A_1\left(\frac{\sin\left(\tilde{t}-t\right)}{\sin\left(\tilde{t}-t_1\right)}\right)^2 + 2B_1\left(\frac{\sin\left(\tilde{t}-t\right)}{\sin\left(\tilde{t}-t_1\right)}\right)\left(\frac{\sin(t-t_1)}{\sin\left(\tilde{t}-t_1\right)}\right)\right.$$

$$\left. + \quad C_1\left(\frac{\sin(t-t_1)}{\sin\left(\tilde{t}-t_1\right)}\right)^2\right] dt$$

$$+\int_{0}^{h}\left[A_2\left(\frac{\sin(t_2-t)}{\sin\left(t_2-\tilde{t}\right)}\right)^2 + 2B_2\left(\frac{\sin(t_2-t)}{\sin\left(t_2-\tilde{t}\right)}\right)\left(\frac{\sin\left(t-\tilde{t}\right)}{\sin\left(t_2-\tilde{t}\right)}\right)\right.$$

$$\left. + \quad C_2\left(\frac{\sin\left(t-\tilde{t}\right)}{\sin\left(t_2-\tilde{t}\right)}\right)^2\right] dt$$

which results in

$$\int_a^b f(x)\,dx \approx$$

$$-\frac{1}{4}\csc^2(h)\left[-2h\left(y_0+y_2\right)+4hy_1\cos(2h)+\left(y_0-2y_1+y_2\right)\sin(2h)\right].$$

Comparison of Simpson's Rule and Trigonometric Integration Methods

We now present a comparison of exact integral values with approximations calculated through the use of Simpson's Rule, the trigonometric polynomial-based method, and the trigonometric spline-based method. These comparisons are shown for various functions on the same interval and for several functions on different intervals.

Function $f(x) =$	Exact Integral Value	Simpson's Rule Approximation	Trigonometric Polynomial-based Approximation	Trigonometric Spline-based Approximation
5	10	10	10	10
$5x + 7$	14	14	14	14
x^2	$0.\bar{6}$	$0.\bar{6}$	0.77019	0.77019
$3x^2 + 2$	6	6	6.31057	6.31057
x^3	0	0	0	0
$3x^3 + 2$	4	4	4	4
x^4	0.4	$0.\bar{6}$	0.77019	0.77019
$x^4 + 5$	10.4	$10.\bar{6}$	10.7702	10.7702
x^5	0	0	0	0
$(x+1)^5$	$10.\bar{6}$	12	13.5529	13.5529
$\sin(x)$	0	0	0	0
$\cos(x)$	1.68294	1.69353	1.64595	1.64595
$\sin^2(x)$	0.54535	0.472049	0.54535	0.54535
$\cos^2(x)$	1.45465	1.52795	1.45465	1.45465
$\sin^3(x)$	0	0	0	0
$\cos^3(x)$	1.28573	1.43849	1.35129	1.35129
$\cos^3(x) + \sin^2(x)$	1.83108	1.91053	1.89664	1.89664

Table 1. Comparisons of exact and approximate integral values for $\int_{-1}^{1} f(x)\,dx$.

Integral	Exact Integral Value	Simpson's Rule Approximation	Trigonometric Polynomial-based Approximation
$\int_{-1}^{1} x^2 dx$	0.66667	0.66667	0.77019
$\int_{1}^{3} x^2 dx$	18.6667	18.6667	18.7702
$\int_{2}^{4} x^2 dx$	18.6667	18.6667	18.7702
$\int_{-1}^{1} \left(\cos^2(x) + \sin^3(x)\right) dx$	1.60497	1.61679	1.5626
$\int_{1}^{3} \left(\cos^2(x) + \sin^3(x)\right) dx$	1.85711	1.85689	1.85796
$\int_{2}^{6} \left(\cos^2(x) + \sin^3(x)\right) dx$	0.997832	1.20019	0.695208

Table 2. Comparisons of exact and approximate integral values over various intervals.

From these examples, it can be seen that there was no observable difference in the trigonometric polynomial-based approximation method and the trigonometric-spline approximation method. The reason for this has not been investigated at this time. Also, it can be seen in these examples that in both the linear and constant functions, the trigonometric approximations agree with the exact integral values. Whereas the trigonometric based methods give exact integral values for odd polynomial functions, they give only close approximations for the even polynomial functions. Also, in the cases of the trigonometric functions, the trigonometric-based methods often give closer integral approximations than Simpson's rule.

References

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