Heronian Proofs Of The Pythagorean Theorem

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In his delightful book, *Journey Through Genius,* William Dunham strains no one's credulity in recognizing Pythagorean's theorem and Heron's formula as two of "the great theorems of mathematics" [4]. What may be far less obvious, however, is the uncanny reciprocity that can be demonstrated between these two ostensibly unrelated ideas.

Modern derivations of Heron's formula, which are typically algebraic [2] or trigonometric ([1], [3], [8]), either employ or presuppose the Pythagorean theorem; and so, the latter has been used often enough in the service of the former. But, how does Heron's formula return the favor? Fortunately, in his *Metrica* Heron derived his formula entirely by *geometric* means and *without any reliance* on the Pythagorean theorem [5] (see [6] and [7] for later geometric derivations). What this means is that, without fear of circularity, Heron's formula may itself be used to prove the Pythagorean theorem; thus, an opportunity for reciprocity arises. Interestingly, however, very few Heronian proofs have been published, only three to be exact, and all of these, quite recently.

In 1990 Dunham published the first of these in *Journey Through Genius.* On the construction of any right triangle Δ with sides a, b, and hypotenuse c, he set Heron's formula (area of $\Delta = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$, where s, the semiperimeter, is equal to $\frac{a+b+c}{2}$) equal to the standard area formula for a triangle (area of $\Delta = \text{base} \cdot \text{height}/2$).

$$\text{(Eq. 1)} \quad \frac{\sqrt{(a+b+c)\,(-a+b+c)\,(a-b+c)\,(a+b-c)}}{4} = \frac{ab}{2}.$$

The proof, which follows in his book, occupies more than a page of text and has been aptly described by Dunham himself as "rather like traveling from Boston to New York by way of Spokane" (p. 129).

A decade later, I discovered there is no need for a layover in Spokane! A commuter flight from Boston to New York can be easily arranged. In 2000 [9], I found a way of using Heron's formula to produce a much simpler proof of the Pythagorean theorem, albeit one requiring four right triangles instead of Dunham's one. While the construction I employed was more complex (a parallelogram replaces his right triangle), the flight itself was brief, as the algebra required occupied only a few lines. Accordingly, I joined four right triangles (AEB, CEB, AED, CED), each with sides a, b, and c, to form parallelogram $ABCD$, where $AC = 2b$ and $BD = 2a$ (see Fig. 1).

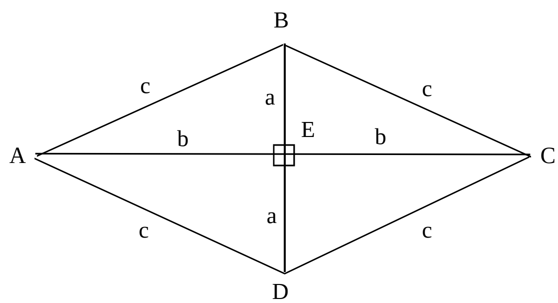

Figure 1. Parallelogram ABCD

The semiperimeter for each of the triangles ABC and ADC is $c+b$; and for each of the triangles DAB and DCB, the semiperimeter is $c + a$. So, by Heron's formula,

(Eq. 2) $$\text{area of } (\Delta ABC + \Delta ADC) = 2\sqrt{(c+b)\,(c-b)\,b^2}$$

and

(Eq. 3) $$\text{area of } (\Delta DAB + \Delta DCB) = 2\sqrt{(c+a)\,(c-a)\,a^2}.$$

Also, by the standard formula,

(Eq. 4) $$\text{area of } (\Delta AEB + \Delta CEB + \Delta AED + \Delta CED) = 2ab.$$

Since area of $(\Delta AEB + \Delta CEB + \Delta AED + \Delta CED)$ = area of $(\Delta ABC + \Delta ADC)$, we have (from Eq. 2 and Eq. 4):

$$\begin{aligned} 2ab &= 2\sqrt{(c+b)(c-b)\,b^2} \\ \Rightarrow \quad a^2b^2 &= (c+b)(c-b)\,b^2 \\ \Rightarrow \quad a^2 &= (c+b)(c-b) \\ \Rightarrow \quad a^2 &= c^2 - b^2 \\ \Rightarrow \quad a^2 + b^2 &= c^2. \end{aligned}$$

Similarly, the reader will note that Eq. 3 and Eq. 4 can be used to yield the same result.

The simplicity of this proof led me to a further question. Dunham and I had both made use of the standard area formula to achieve our respective results. Would it be possible to prove the Pythagorean theorem relying *only* on Heron's formula? I first posed this question in 2000 [9]. In 2001 [10], I showed how it could be done, again relying on Fig. 1. A two-step proof is necessary, one for the general case $(a \neq b)$, and another, for the special case of an isosceles right triangle $(a = b)$. Interestingly, a special case is required because, if the general case is used for an isosceles right triangle, division by zero results!

General Case: $(a \neq b)$

Since area of $(\Delta ABC + \Delta ADC)$ = area of $(\Delta DAB + \Delta DCB)$, it follows that

$$\begin{aligned} 2\sqrt{(c+b)(c-b)\,b^2} &= 2\sqrt{(c+a)(c-a)\,a^2} \\ \Rightarrow \quad \left(c^2 - b^2\right) b^2 &= \left(c^2 - a^2\right) a^2 \\ \Rightarrow \quad c^2b^2 - b^4 &= c^2a^2 - a^4 \\ \Rightarrow \quad a^4 - b^4 &= c^2a^2 - c^2b^2 \\ \Rightarrow \quad \left(a^2 + b^2\right)\left(a^2 - b^2\right) &= c^2\left(a^2 - b^2\right) \\ \Rightarrow \quad \left(a^2 + b^2\right) &= c^2. \end{aligned}$$

Special Case: $(a = b)$

Here, it suffices to prove $2a^2 = c^2$. Observe that the semiperimeter for each of triangles AEB, CEB, AED, and CED is $a + \frac{c}{2}$; so, by Heron's formula,

$$\text{area of } (\Delta AEB + \Delta CEB + \Delta AED + \Delta CED) = 4\sqrt{\left(a + \tfrac{c}{2}\right)\left(a - \tfrac{c}{2}\right)\tfrac{c^2}{4}}.$$

Since area of $(\Delta AEB + \Delta CEB + \Delta AED + \Delta CED)$ = area of $(\Delta DAB + \Delta DCB)$, it follows that

$$\begin{aligned} 4\sqrt{\left(a + \tfrac{c}{2}\right)\left(a - \tfrac{c}{2}\right)\tfrac{c^2}{4}} &= 2\sqrt{(c+a)(c-a)\,a^2} \\ \Rightarrow \quad 2\sqrt{\left(a^2 - \tfrac{c^2}{4}\right)\tfrac{c^2}{4}} &= \sqrt{(c+a)(c-a)\,a^2} \end{aligned}$$

$$
\begin{aligned}
\Rightarrow \quad 4\left(a^2-\tfrac{c^2}{4}\right)\tfrac{c^2}{4} &= \left(c^2-a^2\right)a^2 \\
\Rightarrow \quad \left(a^2-\tfrac{c^2}{4}\right)c^2 &= \left(c^2-a^2\right)a^2 \\
\Rightarrow \quad a^2c^2-\tfrac{c^4}{4} &= a^2c^2-a^4 \\
\Rightarrow \quad a^4 &= \tfrac{c^4}{4} \\
\Rightarrow \quad 4a^4 &= c^4 \\
\Rightarrow \quad 2a^2 &= c^2
\end{aligned}
$$

It is reflective, perhaps, of some small measure of mathematical elegance, that each of these two great formulae of antiquity can be used in the service of the other! Whether or not further reciprocal connections can be found, and whether or not additional Heronian proofs of the Pythagorean theorem may be discovered, loom today as open questions.

References

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