Solutions and Discussions

Problem 4 — Vol. 24 , No. 2, Fall, 2000

Consider a gambling game where first you pay an entrance fee of X dollars, then you flip a coin repeatedly for as long as you only toss heads. If you toss Y heads before the first tail, then you will receive a payoff of Y dollars. What value of X makes this a fair game? Extra Credit: Suppose instead that the payoff is $2Y$ dollars. What value of X makes this a fair game?

Solution

Richard Harem, Senior, Troy State University, Troy, AL

To start off with, we define the probability of flipping a sequence of n heads and then a tail (a tail must be flipped, in order to create a sequence of n heads) as $P(n) = \frac{n}{2^{n+1}}$.

In order to find the expected return, the summation of all possible outcomes must be computed, namely, $\sum_{n=1}^{\infty} \frac{n}{2^{n+1}}$.

The first five terms of the corresponding sequence of partial sums is given below:

$$\begin{aligned}
&S_1 = \tfrac{1}{4}\\
&S_2 = \tfrac{1}{4} + \tfrac{2}{8} = \tfrac{4}{8}\\
&S_3 = \tfrac{4}{8} + \tfrac{3}{16} = \tfrac{11}{16}\\
&S_4 = \tfrac{11}{16} + \tfrac{4}{32} = \tfrac{26}{32}\\
&S_5 = \tfrac{26}{32} + \tfrac{5}{64} = \tfrac{57}{64}
\end{aligned}$$

A pattern, easily verified by induction, emerges; the general term is given by:

$$S_n = \frac{2^{n+1} - (n+2)}{2^{n+1}} = 1 - \frac{n+2}{2^{n+1}}.$$

Thus, $\sum_{n=1}^{\infty} \frac{n}{2^{n+1}} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{n+2}{2^{n+1}}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\left(\frac{n+2}{2^{n+1}}\right) = 1 - 0 = 1$, the latter limit being computed using a simple application of L'Hôpital's Rule. So we are left with \$1 as the expected payoff.

For a payoff of $2n$ dollars, the expected return would just be multiplied by 2, so \$2 is the expected return.

Also solved by Sheena Richards, Sophomore, Troy State University.

Problem 1 — Vol. 25 , No. 2, Fall, 2001

Four cars travel along a road at constant speeds. Cars A and B travel north, and cars C and D travel south. Car A meets car C at 1:00. Car A meets car D at 1:20. Car B meets car C at 2:00. Car B meets car D at 2:10. If cars C and D travel at identical speeds, what time will car B overtake car A? Alternatively, if cars A and B travel at identical speeds, what time will car D overtake car C?

Solution

Richard Harem, Senior, Troy State University, Troy, AL

Let a, b, c, d represent the rates of cars A, B, C, and D, respectively. Relative rates are obtained by adding or subtracting individual rates when the cars travel in the opposite or same direction, respectively. We arbitrarily chose noon as a reference time before any cars meet, and measure time in minutes from here. We define three distances, x, y, z, as shown in Figure 1.

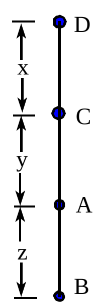


Figure 1

Repeated applications of time $= \frac{\text{distance}}{\text{rate}}$, the assumption that $c = d$, and the given information yield:

$$\frac{y}{a+c} = 60 \quad or \quad y = 60a + 60c \tag{1}$$

$$\frac{y+x}{a+c} = 80 \quad or \quad y + x = 80a + 80c \tag{2}$$

$$\frac{y+z}{c+b} = 120 \quad or \quad y + z = 120c + 120b \tag{3}$$

$$\frac{x+y+z}{c+b} = 130 \quad or \quad x + y + z = 130c + 130b \tag{4}$$

Combining equations 1 and 3, yields:

$$60a + z = 60c + 120b \tag{5}$$

Combining equations 2 and 4, yields:

$$80a + z = 50c + 130b \tag{6}$$

Finally, eliminating c from equations 5 and 6 yields:

$$z = 180b - 180a = 180\,(b - a)$$

We seek the value for $\frac{z}{b-a}$. From the previous equation, we have $\frac{z}{b-a} = 180$ minutes. So car B overtakes car A at 3 o'clock.

The alternative problem is solved in a similar manner and has the same solution.

Also solved by Sheena Richards, Sophomore, Troy State University; and Bruce Myers, Kankakee Community College, Kankakee, IL.

Problem 2 — Vol. 25 , No. 2, Fall, 2001

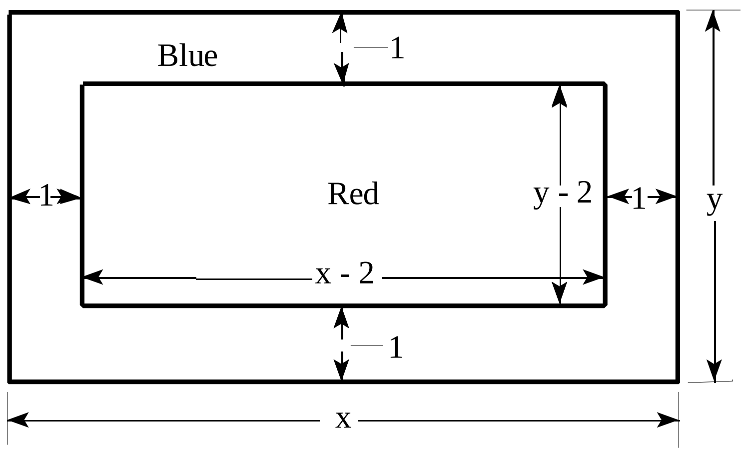
Mr. Jones wishes to add one-foot square tiles to his porch floor. He selects a pattern that consists of a central rectangle of blue tiles surrounded by red tiles along the outer perimeter. Mrs. Smith also wishes to add one-foot square tiles to her porch floor. She selects a pattern that consists of a central rectangle of red tiles surrounded by blue tiles along the outer perimeter. Mr. Jones' porch floor is 2 feet wider and 6 feet shorter than Mrs. Smith's porch floor.

Coincidentally, both porches require identical numbers of blue tiles, and also identical numbers of red tiles. Determine the dimensions of both porch floors.

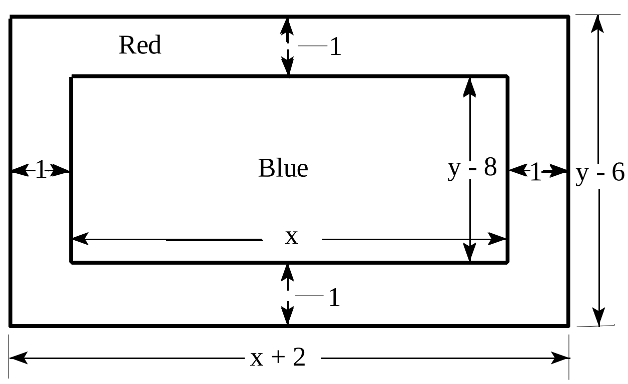
Solution

Sheena Richards, Sophomore, Troy State University, Troy, AL

Let x be the width of Mrs. Smith's porch, and let y be its length. The the side of Mr. Jones' porch is $(x+2)$, and its length is $(y-6)$. The situation is illustrated below.



Mrs. Smith's Porch



Mr. Jones' Porch

It follows that the dimensions of the inner rectangle of red tiles on Mrs. Smith's porch are $(x-2)$ and $(y-2)$, and the dimensions of the inner rectangle of blue tiles on Mr. Jones" porch are x and $(y-8)$.

Since the number of blue tiles on Mrs. Smith's porch equals the number of blue tiles on Mr. Jones' porch, it follows that

$$xy - (x-2)(y-2) = x(y-8)$$

$$\Rightarrow \qquad x = \frac{-2y+4}{10-y} \tag{1}$$

Since the number of red tiles on Mrs. Smith's porch equals the number of red tiles on Mr. Jones' porch, it follows that

$$(x-2)(y-2) = (x+2)(y-6) - x(y-8)$$

$$\Rightarrow \qquad xy - 4x = 4y - 16 \tag{2}$$

$$\Rightarrow \qquad x = \frac{4y-16}{y-4} \Rightarrow x = 4$$

We mention, parenthetically, that $y = 4$ is also a solution of equation 2. We discard this solution, however, since it yields a negative value for x.

Substituting $x = 4$ into equation 1 yields

$$4 = \frac{-2y+4}{10-y} \qquad \Rightarrow y = 18$$

Thus, the dimensions of Mrs. Smith's porch are 4×18, and the dimensions of Mr. Jones porch are 6×12.

Also solved by Richard Harem, Senior, Troy State University; Taoufiq Bellamine, Freshman, Troy State University; and Bruce Myers, Kankakee Community College, Kankakee, IL.

Problem 5 — Vol. 25 , No. 2, Fall, 2001

Compute the value of $G(4,6)$ *for the function* $G(x,y)$ *that is defined recursively below. For extra credit, also find a closed formula for* $G(x,y)$*; that is, provide a non-recursive definition of* $G(x,y)$*.*

$$G(x,y) = \begin{cases} 1 & x=0,\ y=0 \\ 2G(x,y-1) & x=0,\ y>0 \\ 3G(x-1,y) & x>0,\ y=0 \\ G(x,y-1)+G(x-1,y)-G(x-1,y-1) & x>0,\ y>0 \end{cases}$$

Solution

Taoufiq Bellamine, Freshman, Troy State University, Troy, AL

$G(0,0) = 1$ by definition. A straightforward induction argument yields $G(0,y) = 2^y$ for all points $(0,y)$ on the y-axis, where $y \in \mathbf{N}$.

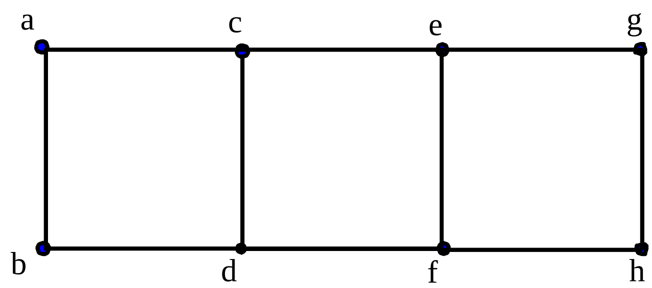


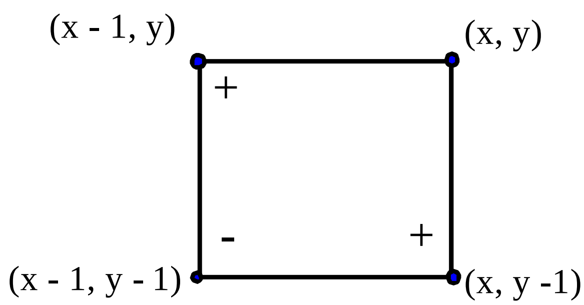
FIGURE 1

Similarly, $G(x,0) = 3^x$ for all points $(x,0)$ on the x-axis, where $x \in \mathbf{N}$.

For $x, y \in \mathbf{N}$, the recursive definition,

$$G(x,y) = G(x,y-1) + G(x-1,y) - G(x-1,y-1)$$

has the following geometric interpretation:



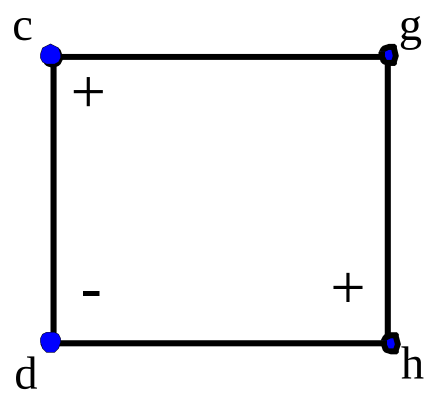
"The G-value of the upper right corner is the sum of the G-values of the lower right and upper left corners, minus the G-value of the lower left corner."

A telescoping cancellation argument shows that this geometric interpretation holds for any horizontal concatenation of squares. For example, letting letters represent G-functional values, $g = h + e - f$ (see below).

Substituting $e = f + c - d$ yields:

$$g = h + (f + c - d) - f = h + c - d$$

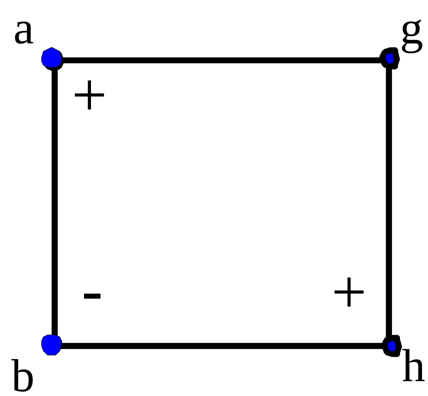
which is the geometric interpretation of the rectangle:



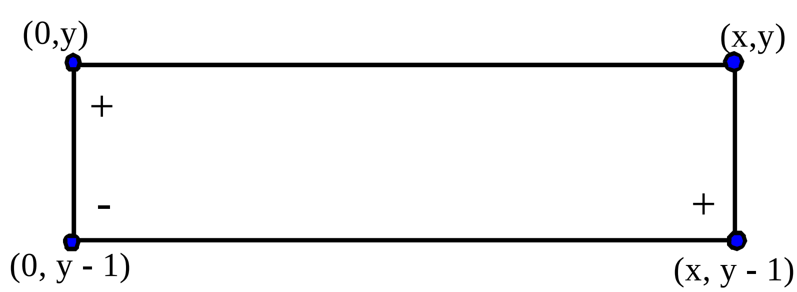
Continuing with the substitution $c = d + a - b$, we have:

$$g = h + (d + a - b) - d = h + a - b$$

which corresponds to the geometric interpretation for the rectangle:



Thus, $G(x, y)$ may be computed via the rectangle:



Thus, $G(x, y) = G(x, y-1) + G(0, y) - G(0, y-1)$.

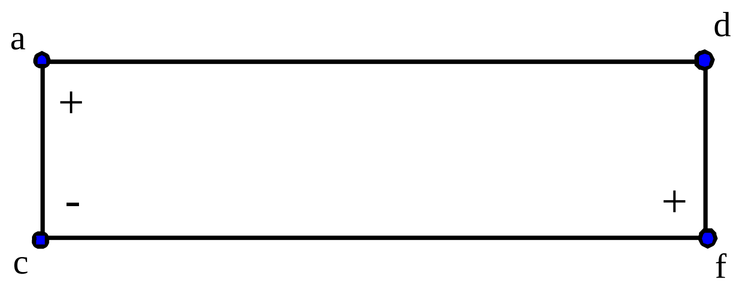
The previous result combined with a similar telescoping cancellation argument can be used to show that this geometric interpretation is valid for any vertical concatenation of **rectangles** (not just squares). Thus, referring to the diagram below, and letting lower case letters represent G-values, as before, we have:

$$\begin{array}{ll} d = e + a - b & \text{(by a previous argument)} \\ e = f + b - c & \text{(by a previous argument).} \end{array}$$

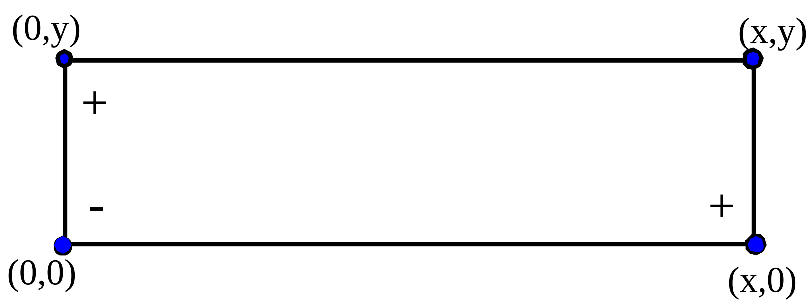
Substituting the latter into the former, we have:

$$\begin{array}{rl} & d = (f + b - c) + a - b \\ \Rightarrow & d = f + a - c, \end{array}$$

which corresponds to the geometric interpretation for the rectangle:



Thus, for $x, y \in \mathbf{N}$, $G(x, y)$ is determined by the rectangle:



So $G(x, y) = G(x, 0) + G(0, y) - G(0, 0) = 3^x + 2^y - 1$. (Note that this formula also holds on the x and y axes, as well.)

In answer to the first part of the problem, $G(4, 6) = 3^4 + 2^6 - 1 = 144$.