Division I Comprehensive Test 2001 ACTM — AACTM Alabama Statewide Mathematics Contest

CONSTRUCTION OF THIS TEST DIRECTED BY LAURIE EDLER, JACKSONVILLE STATE UNIVERSITY

INSTRUCTIONS.. All variables and constants represent real numbers, except when a particular problem indicates otherwise. We use the following geometric notation: If A and B are points, then $\overline{AB}$ is the segment between A and B, $\overleftrightarrow{AB}$ is the line containing A and B, and AB is the distance between A and B. If A is an angle, then $m\angle A$ is the measure of angle A in degrees. Diagrams are *not* drawn to scale.

The correct answer for each problem is followed by a star $(\star)$. Under each possible answer is the percentage of contestants who chose that answer.

(1) What is the distance between the vertex of the parabola $y = x^2 - 4x + 3$ and the center of the circle $x^2 = 9 - (y-3)^2$?

(A)	4 3%	**(B)**	$2\sqrt{2}$ 9%	**(C)**	$3\sqrt{2}$ 3%	**(D)**	$2\sqrt{3}$ 4%
(E)	$2\sqrt{5}\,\star$ 53%	Omit 28%					

(2) Let $i = \sqrt{-1}$. Find the ninth term of the geometric sequence $i, -2, -4i, \ldots$.

(A)	-512 4%	**(B)**	$-256i$ 12%	**(C)**	$16i$ 3%	**(D)**	$128i$ 3%
(E)	$256i \star$ 69%	Omit 9%					

(3) Suppose s varies directly with t and inversely with r^2. If $s = 5$ when $r = 1$ and $t = 3$, what is the value of s when $r = \sqrt{3t}$?

(A)	$\frac{1}{5}$ 2%	**(B)**	$\frac{5}{9} \star$ 57%	**(C)**	5 3%	**(D)**	$5t^2$ 1%
(E)	$\frac{\sqrt{3}}{t}$ 2%	Omit 35%					

(4) Find the sum of all solutions $x \in [-90^\circ, 90^\circ]$ for the equation $2\tan x \sin x + 2\sin x = \tan x + 1$.

(A)	$-15^\circ \star$ 26%	**(B)**	0° 10%	**(C)**	15° 4%	**(D)**	75° 3%
(E)	180° 4%	Omit 53%					

(5) The binary operation $*$ is defined by $a * b = \frac{a-2b}{2ab}$ for $a, b \neq 0$. If $x * y = -1$ and $y * x = 5/4$, then $y * y = (?)$

(A)	-1 4%	**(B)**	$\frac{1}{2} \star$ 45%	**(C)**	$\frac{7}{6}$ 3%	**(D)**	$\frac{3}{2}$ 5%
(E)	2 1%	Omit 42%					

(6) Let $i = \sqrt{-1}$. $\frac{6\cos\left(\frac{\pi}{6}\right) + 6i\sin\left(\frac{\pi}{6}\right)}{2\cos\left(\frac{2\pi}{3}\right) + 2i\sin\left(\frac{2\pi}{3}\right)} = (?)$

(A)	3 5%	**(B)**	$3i$ 2%	**(C)**	$-3i \star$ 45%
(D)	$6 - 3\sqrt{3}$ 4%	**(E)**	$3\left(\sqrt{3} + 2\right)$ 2%	Omit 42%	

(7) A natural number is *abundant* if it is less than the sum of its positive proper divisors (including 1, but not including the number itself). How many abundant numbers are less than 20?

(A)	1 6%	**(B)**	2 ⋆ 52%	**(C)**	3 10%	**(D)**	4 5%
(E)	5 4%	Omit 23%					

(8) $\sqrt{\sin^2 x + \csc^2 x + \cos^2 x + \sec^2 x - \left(\tan^2 x + \cot^2 x\right)} = (?)$

(A)	$\frac{1}{2}$ 1%	**(B)**	$\frac{\sqrt{3}}{2}$ 1%	**(C)**	1 18%	**(D)**	$\frac{2\sqrt{3}}{3}$ 1%
(E)	$\sqrt{3}$ ⋆ 41%	Omit 38%					

(9) If the five digit number $5dddd$ is divisible by 6, what is the digit d?

(A)	2 2%	**(B)**	4 ⋆ 82%	**(C)**	6 8%	**(D)**	7 1%
(E)	8 2%	Omit 5%					

(10) What is the cosine of the largest angle in a 3-4-6 triangle?

(A)	$-\frac{3}{4}$ 2%	**(B)**	$-\frac{11}{24}$ ⋆ 46%	**(C)**	0 4%	**(D)**	$\frac{2}{3}$ 19%
(E)	$\frac{29}{36}$ 3%	Omit 27%					

(11) What is the diameter in feet of a pulley which is driven at 6 revolutions per second by a belt moving at 40 feet per second?

(A) $\frac{10}{3\pi}$ 8% **(B)** $\frac{20}{3\pi}$ ⋆ 57% **(C)** $\frac{3\pi}{10}$ 1% **(D)** $\frac{20\pi}{3}$ 5%

(E) $\frac{40\pi}{3}$ 1% Omit 27%

(12) Let $i = \sqrt{-1}$. $(1-i)^8 = (?)$

(A) 8 3% **(B)** 16 ⋆ 56% **(C)** $8e^{i\frac{\pi}{2}}$ 4%

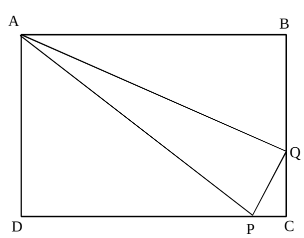
(D) $4\sqrt{2}e^{i\frac{\pi}{2}}$ 3% **(E)** $16e^{i\frac{\pi}{2}}$ 2% Omit 33%

(13) $\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right) = (?)$

(A) $\frac{\pi}{6}$ 3% **(B)** $\frac{\pi}{4}$ ⋆ 24% **(C)** $\frac{\pi}{2}$ 3% **(D)** $\frac{\pi}{3}$ 4%

(E) 1 3% Omit 62%

(14) In the diagram, $ABCD$ is a rectangle. If $AQ = 8$, $PQ = 6$, $AP = 10$, and $m\angle QAB = 45\,^\circ$, then $\sin(\angle DAP) = (?)$



(A) $\frac{\sqrt{2}}{10}$ ⋆ 56% **(B)** $\frac{\sqrt{2}}{6}$ 2% **(C)** $\frac{\sqrt{2}}{2}$ 4% **(D)** $\frac{4}{5}$ 2%

(E) $\frac{3\sqrt{2}}{5}$ 3% Omit 33%

(15) A \$100 bicycle was marked down by 12% for an end of month close-out sale. The bike did not sell and was marked up by 12%. For the next sale, the same bike was marked down by 12% and subsequently marked up by 12% when it did not sell. If this process continues indefinitely, what will the price of the bike approach?

(A)	\$112 1%	**(B)**	\$100 6%	**(C)**	\$98.56 7%	**(D)**	\$88 12%
(E)	\$0 ⋆ 56%	Omit 19%					

(16) Find the solution of $\sec x - 1 = \tan x$ with $x \in [0, 2\pi)$.

(A)	$0\star$ 42%	**(B)**	$\frac{\pi}{2}$ 10%	**(C)**	π 5%	**(D)**	$\frac{4\pi}{3}$ 2%
(E)	$\frac{3\pi}{2}$ 1%	Omit 40%					

(17) What is the range of $f(x) = \frac{2x-1}{x+4}$?

(A)	$(-\infty, \infty)$	6%
(B)	$(-\infty, -4) \cup (-4, \infty)$	23%
(C)	$(-\infty, 2) \cup (2, \infty)\ \star$	40%
(D)	$\left[-4, \frac{1}{2}\right]$	2%
(E)	$\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$	3%
Omit		26%

(18) The prime factorization of a positive integer is unique except for the order of the factors. How many distinct orderings of factors are possible for the prime factorization of 504 ?

(A)	3 2%	**(B)**	6 15%	**(C)**	56 7%	**(D)**	60 ⋆ 38%
(E)	720 7%	Omit 31%					

(19) What is the sum of the *squares* of the real and complex solutions of $x^4 + x^3 + 4x^2 + 4x = 0$?

(A) $-7\star$ 33% **(B)** -3 7% **(C)** 0 4% **(D)** 1 14%

(E) 9 7% Omit 36%

(20) How many integers satisfy the inequality $\left|x^2 - 8\right| < 4$?

(A) 0 3% **(B)** 1 20% **(C)** $2\star$ 64% **(D)** 3 4%

(E) 3 3% Omit 5%

(21) If n is an integer greater than 3, then $\frac{n!(n-3)!}{(n-2)!(n-1)!} = (?)$

(A) $\frac{n}{n-2}\star$ 75% **(B)** $\frac{n-3}{n-1}$ 1% **(C)** $n!$ 0%

(D) $\frac{n}{(n-2)(n-1)!}$ 2% **(E)** $\frac{(n^2-3n)!}{(n^2-3n+2)!}$ 6% Omit 16%

(22) If $4^{\frac{x}{y}} = 256$ and $3^x = \frac{1}{27}$, then $y = (?)$

(A) $-\frac{3}{2}$ 1% **(B)** -1 3% **(C)** $-\frac{3}{4}\star$ 82% **(D)** $\frac{2}{3}$ 0%

(E) $\frac{4}{3}$ 3% Omit 12%

(23) A circle has radius of length 5, is tangent to the line with equation $4x - 3y = 18$ at the point $(3, -2)$, and lies above the line. What is the equation of the circle?

(A)	$x^2 - 14x + y^2 + 10y = -49$	8%
(B)	$x^2 - 6x + y^2 + 4y = 12$	8%
(C)	$x^2 + 2x + y^2 - 2y = 3$	3%
(D)	$x^2 + 2x + y^2 - 2y = 23\star$	35%
(E)	$x^2 + 6x + y^2 + 4y = 12$	1%
Omit		46%

(24) $32^{-(\sqrt{5})^{-2}} = (?)$

(A)	32^6 0%	**(B)**	$\sqrt[3]{32}$ 1%	**(C)**	4 1%	**(D)**	$32^{2\sqrt{3}}$ 2%		
(E)	$\frac{1}{2}\star$ 76%	Omit 21%							

(25) Find the sum of the coefficients in the expansion of

$$\left(\tfrac{1}{2} - 4x + 4x^3\right)^{274}\left(6x^4 - 6x + 2\right)^{275}.$$

(A)	-1 2%	**(B)**	0 2%	**(C)**	$2\star$ 23%	**(D)**	549 4%
(E)	2746 1%	Omit 68%					

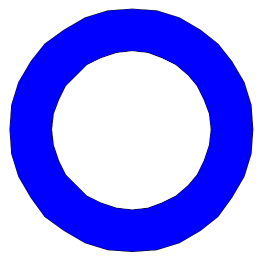
(26) Three circles, each with radius of length 6, intersect so that each circle passes through the centers of the other two circles. Find the area of the region that is the intersection of the interiors of the three circles.

(A)	$9\sqrt{3}$ 2%	**(B)**	6π 8%	**(C)**	9π 6%
(D)	$18\left(\pi - \sqrt{3}\right)\star$ 30%	**(E)**	$48\pi - \sqrt{3}$ 1%	Omit 52%	

(27) If 17! is written as a base eight numeral, how many zeros will appear at the end of the numeral?

(A)	0 2%	**(B)**	3 5%	**(C)**	$5\star$ 18%	**(D)**	9 2%
(E)	17 4%	Omit 68%					

(28) The diagram shows a circular pool surrounded by a garden of uniform width. The area of the garden is the same as the area of the surface of the pool. What is the ratio of the length of fencing needed to surround the pool to the length of fencing needed to enclose the entire region?



(A) $\frac{1}{4}$ 2% **(B)** $\frac{\sqrt{2}}{4}$ 3% **(C)** $\frac{1}{2}$ 10% **(D)** $\frac{\sqrt{3}}{3}$ 1%

(E) $\frac{\sqrt{2}}{2}\star$ 39% Omit 45%

(29) An investment with annual interest rate r yields an annual simple interest of \$1500. If \$500 more is invested and the rate is 2 percentage points less, the annual simple interest is \$1300. Find r.

(A) 7% 1% **(B)** 8% 4% **(C)** 9% 4% **(D)** 11% 4%

(E) 12% $\star$ 20% Omit 66%

(30) What is the probability that a 4-digit number consisting only of sixes and twos is divisible by 11 ?

(A) 0 3% **(B)** $\frac{1}{16}$ 3% **(C)** $\frac{1}{8}$ 6% **(D)** $\frac{3}{8}\star$ 32%

(E) $\frac{5}{16}$ 8% Omit 48%

(31) Find the final digit of $9^{412} \cdot 16^8$.

(A) 1 3% **(B)** 4 9% **(C)** 6 ⋆ 57% **(D)** 8 2%

(E) 9 1% Omit 29%

(32) In a class of 100, 40 students study Spanish, 35 study German, and 27 study French. Two study all three subjects, while 3 study Spanish and German only, and 40 do not study either Spanish or French. Finally, 20 students study no language at all. How many students who study German do not study Spanish?

(A) 25 5% **(B)** 30 ⋆ 58% **(C)** 31 1% **(D)** 33 2%

(E) 34 2% Omit 32%

(33) If $\left(1+\frac{1}{x^2}\right)\left(1+\frac{1}{x^4}\right)\left(1+\frac{1}{x^8}\right)\left(1+\frac{1}{x^{16}}\right)\left(1+\frac{1}{x^{32}}\right) = \frac{x^m-1}{x^m-x^{m-2}}$, then $m = (?)$

(A) 32 1% **(B)** 64 ⋆ 8% **(C)** 70 2% **(D)** 72 3%

(E) 128 1% Omit 85%

(34) The length of rectangle $ABCD$ is twice its width w. P is a point such that the area of $\triangle PBD$ is equal to the area of the rectangle. What is the length of the altitude of $\triangle PBD$ to the base BD ?

(A) $\frac{w}{2}$ 2% **(B)** $\frac{2w\sqrt{5}}{5}$ 7% **(C)** $\frac{2w\sqrt{3}}{3}$ 5%

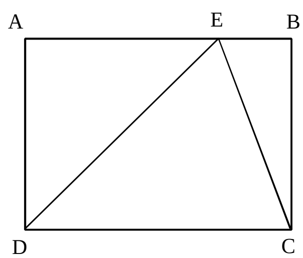
(D) $\frac{4w\sqrt{5}}{5}$ ⋆ 31% **(E)** $\frac{4w\sqrt{3}}{3}$ 2% Omit 54%

(35) What is the product of the solutions to the equation $ax^2 + bx + c = 0$, where $a \neq 0$?

(A) $4ac$ 1% **(B)** $\frac{c}{2a}$ 2% **(C)** $\frac{c}{a}$ ⋆ 62%

(D) $\frac{b^2-2ac}{a}$ 7% **(E)** $\frac{b^2+2ac}{a}$ 1% Omit 27%

(36) In the diagram, $ABCD$ is a rectangle, AB and BC are integers, $AE = 4$, $EC = \sqrt{13}$, and the area of $\triangle DEC$ is 9. What is the perimeter of the rectangle?



(A) 6 0% **(B)** 9 1% **(C)** 14 4% **(D)** 18 ⋆ 55%

(E) 22 2% Omit 37%

(37) A person has a hours at his disposal. How many miles may he ride in a car traveling b miles per hour and yet have time to return on foot walking c miles per hour?

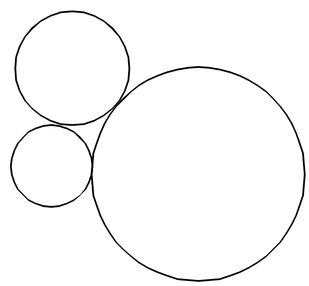
(A) $\frac{ab^2}{a+b}$ 2% **(B)** $\frac{ab-ac}{ab+c}$ 5% **(C)** $\frac{abc}{b+c}$ ⋆ 28%

(D) $\frac{ac}{b+c}$ 2% **(E)** $\frac{b+c}{ac}$ 2% Omit 62%

(38) The difference between the roots of $x^2 + px + q = 0$ is the same as the difference between the roots of $x^2 + qx + p$, and $p \neq q$. Then $p + q =$ (?)

(A)	$-4\star$ 14%	**(B)**	-1 3%	**(C)**	-2 4%	**(D)**	0 7%
(E)	2 1%	Omit 71%					

(39) The diagram shows three mutually tangent circles. If the circles have radii 2, 3, and 7, respectively, what is the area of the triangle formed by the segments joining their centers?



(A)	$\frac{45}{2}$ 20%	**(B)**	$\sqrt{42}$ 2%	**(C)**	$4\sqrt{3}$ 3%	**(D)**	$6\sqrt{14}\star$ 36%
(E)	25 3%	Omit 36%					

(40) If the 3-digit number $4x3$ is added to 134, the result is the 3-digit number $5y7$, which is divisible by 7. Then $x + y =$ (?)

(A)	5 1%	**(B)**	7 2%	**(C)**	$9\star$ 77%	**(D)**	11 4%
(E)	13 3%	Omit 13%					

(41) How many solutions of the equation $\cos x = \tan^{-1} 2x$ are in the interval $[-2\pi, 2\pi]$?

(A)	0 4%	**(B)**	1 $\star$ 7%	**(C)**	2 8%	**(D)**	3 3%
(E)	4 9%	Omit 69%					

(42) Exactly two integers between 75 and 85 are divisors of $3^{32} - 1$.What is the product of those integers?

(A)	5852 2%	**(B)**	6560 $\star$ 26%	**(C)**	6804 3%	**(D)**	6888 3%
(E)	6972 1%	Omit 65%					

(43) What is the domain of the function $f(x) = \log_2(\log_2 x)$?

(A)	$(-\infty, \infty)$ 4%	**(B)**	$(0, \infty)$ 12%	**(C)**	$(1, \infty)$ $\star$ 27%
(D)	$(2, \infty)$ 5%	**(E)**	$[2, \infty)$ 14%	Omit 38%	

(44) The lengths of the sides of $\triangle ABC$ are $AB = 7.5$, $BC = 10$, and $AC = 5$. Segment $\overline{BC}$ is extended through C to point P so that $\triangle PAB$ is similar to $\triangle PCA$. Then $CP = (?)$

(A)	7.5 7%	**(B)**	8 $\star$ 12%	**(C)**	10 12%	**(D)**	12 2%
(E)	15 3%	Omit 66%					

(45) What is the product of the solutions of $x^{\log_{10} x} = \frac{10000}{x^3}$?

(A)	$\frac{1}{1000}$ $\star$ 26%	**(B)**	$\frac{1}{10}$ 4%	**(C)**	100 4%
(D)	1000 2%	**(E)**	$1,000,000$ 1%	Omit 62%	

(46) The coefficient of x^2 in the expansion of the product $(x+a)(x+b)(x+c)$ is 0. The coefficient of x in the expansion of the product $(x-a)(x+b)(x+c)$ is 0. The coefficient of x in the first expansion is equal to the coefficient of x^2 in the second expansion. What must the value(s) of a be?

(A)	0 or 1 $\star$ 8%	**(B)**	0 only 7%	**(C)**	-1 or 2 4%
(D)	1 only 2%	**(E)**	-1 only 1%	Omit 78%	

(47) A path consisting of 100 square stones is to be painted. The length of a side of the first stone is 1 foot; the length of a side of the second stone is 2 feet; and so on, until the length of a side of the final stone is 100 feet. If one gallon of paint covers 101 square feet, how many gallons of paint will be necessary to paint the entire path?

(A)	50 13%	**(B)**	99 2%	**(C)**	100 2%	**(D)**	2875 14%
(E)	3350 $\star$ 17%	Omit 52%					

(48) If a committee of 6 members is to be chosen from among 5 Democrats and 3 Republicans so that at least two members of each party serve on the committee, how many committees are possible?

(A)	15 2%	**(B)**	16 4%	**(C)**	20 8%	**(D)**	25 $\star$ 26%
(E)	28 5%	Omit 55%					

(49) A container in the shape of a right circular cone of height 2 feet and radius r feet is full of water. A hole is punched in the vertex at the bottom of the cone, and the water drips into a second container in the shape of a right circular cylinder of the same radius. What is the depth (in feet) of the water in the cylinder when the volume of water in the cone is half of the original amount?

(A)	$\frac{1}{3}\star$ 37%	**(B)**	$\frac{2}{3}$ 6%	**(C)**	1 1%	**(D)**	$\frac{r}{3}$ 5%
(E)	r 1%	Omit 49%					

(50) H, A, and L are positive integers such that $HAL = 2001$. If five *different* digits are required to write the numerals for H, A, and L, what is the largest of these integers?

(A)	29 4%	**(B)**	69 3%	**(C)**	79 2%	**(D)**	87 $\star$ 30%
(E)	667 9%	Omit 52%					