Fuzzy Logic

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Abstract. The role of logic in mathematical activities is indisputable. Indeed, it has become a cornerstone for many of the important achievements in the field of mathematics. This paper is about a conversion of classic logic into the fascinating world of fuzzy logic. Fuzzy logic has put a new perspective on the once crisp ideas of classical logic. Multivalued logic, which has been in existence for some time, has opened new horizons and changed the way many think about logic. If you allow a set of multiple values, as used in multivalued logic, to become a set of infinitely many values contained between false and true, the underlying idea behind fuzzy set theory and fuzzy logic is exposed. This paper contains a short history of the origin of fuzzy logic, an accepted definition, and some of its uses in the field of mathematics. It also contains other noteworthy and interesting observations in the field of fuzzy logic.

Aristotle and many philosophers who preceded him deserve much credit for the current precision of mathematics. In their efforts to define a concise theory of logic, the present day idea of conventional (Boolean) logic was formed. The conclusion generated by their efforts led to a strict law: a statement is either true or not true. When placed in binary logic, their claim states that every false statement has a truth-value of 0 and every true statement has a truth-value of 1, and there can be no middle ground. This conclusion is entitled Aristotle's Law of Bivalence and was accepted as philosophically correct for over two thousand years. Notwithstanding its general acceptance, this law has not existed without objection. It can easily be conceived that things are simultaneously true and not true. For example, conventional (Boolean) logic states that a glass can either be full or not full of water. However, suppose a glass were filled only halfway. Then the glass can be half-full and not half-full at the same time. The statements half-full and not half-full simultaneously represent the same situation, and therefore, apparently violate Aristotle's Law of Bivalence. The given example gives way to the existence of values between false and true. This concept is the basis for multivalued logic. If allowed, this logic could have an infinitely vast number of values contained between false and true, and thus statements can take on truth values represented by real numbers x that satisfy the condition $0 \leq x \leq 1$. The concept of statements having certain degrees of truth is the fundamental principal that propelled Dr. Lotfi A. Zadeh of the University of California at Berkeley in 1964 to introduce fuzzy set theory, fuzzy logic, and many other extensions of fuzzy applications in mathematics. Fuzzy logic can be described as the construction of a membership function where the values false and true operate over the range of real numbers $[0.0, 1.0]$, where 0.0 and 1.0 are the extreme cases of truth. The essential characteristics of fuzzy logic as set by Zadeh, now called the Father of fuzzy logic, are:

- In fuzzy logic, exact reasoning is viewed as a limiting case of approximate reasoning,
- In fuzzy logic, everything is a matter of degrees,
- With fuzzy logic, any logical system can be fuzzified,
- In fuzzy logic, knowledge is interpreted as a collection of elastic or fuzzy constraints on a collection of variables,
- Inference is viewed as a process of propagation of fuzzy constraints [10].

Due to the third characteristic given above, it is now possible to take Boolean logic and generalize it to form a precise definition of fuzzy logic. By definition, fuzzy logic is a superset of conventional (Boolean) logic that has been extended to handle the concept of partial truth values between the extreme cases completely false and completely true, i.e., fuzzy logic describes modes of reasoning which are approximate rather than exact. The use of fuzzy logic holds precedence in that natural language and our human reasoning process are approximate in nature. These two human necessities, as well as most other activities in life are not easily translated into terms of 0's and 1's. It is even debatable if they can be put into these terms at all. Humans aggregate information and data to form a number of partial truths. We then take these and aggregate further to determine higher level of truths. Once a certain degree of acceptable truth is met or exceeded in the process, an action such as speech will occur. A similar kind of process occurs in computer technology in the creation of neural networks and expert systems to achieve certain tasks. In basic terms, fuzzy logic seems closer to the way our brains actually work than conventional logic.

In classical logic, statements are taken and examined in relationship to each other to derive a single conclusion from their connection. The common logical operations in classical logic include: negation, conjunction, disjunction, implication, and equivalence. The negation of a statement P is "not (P)," written as "$\tilde{}P$;" the conjunction of statements P and Q is "P and Q," or "$P \wedge Q$;" the disjunction is "P or Q," denoted by "$P \vee Q$;" the implication is "If P then Q," written "$P \rightarrow Q$;" lastly, the equivalence is "P if and only if Q," or "$P \leftrightarrow Q$.". The usual way of viewing changes in classical logic is through the use of truth tables. Truth tables allow the extreme points created in classical logic to be viewed in a true-false manner. In order to convert classical logic into fuzzy form, a few conversion standards had to be set into place. Let the notation $t(P)$ denote the truth value of the statement P. The standards for conversion are as follows:

- Negation: $t(\tilde{}P) = 1 - t(P)$
- Conjunction: $t(P \wedge Q) = \min\{t(P), t(Q)\}$
- Disjunction: $t(P \vee Q) = \max\{t(P), t(Q)\}$

To show the conversions by example, let P be a statement with the truth value 0.8, and let Q be a statement with the truth value 0.7. The negate or negation of the statement P in fuzzy form is represented $t(\tilde{}P) = 1 - 0.8$, and generates the truth value of 0.2. The conjunction of the statements P and Q in fuzzy form is $t(P \wedge Q) = \min\{0.8, 0.7\}$, and generates a truth value of 0.7. The disjunction, $t(P \vee Q) = \max\{0.8, 0.7\}$, and generates the truth value of 0.8. The rules for conversion create functions that can be used to generate values based on changing variables of truth. Implication by definition can be written in the form $\tilde{}(P \wedge \tilde{}Q)$; equivalence in the form $(P \rightarrow Q) \wedge (Q \rightarrow P)$. Now with simple substitution the fuzzy definitions of implication and equivalence can be stated. Fuzzy implication in formula form is $1 - \min\{t(P), 1 - t(Q)\}$. Fuzzy equivalence defined in functional form is $\min\{\,1 - \min\{t(P), t(Q)\},\ 1 - \min\{t(Q), 1 - t(P)\}\,\}$. It is easily seen that these definitions represent two-dimensional functions whose domain is the unit square in the first quadrant. The graphical representations of these two-dimensional functions are referred to as fuzzy truth surfaces. The fuzzy truth surface for implication is shown in Figure 1:

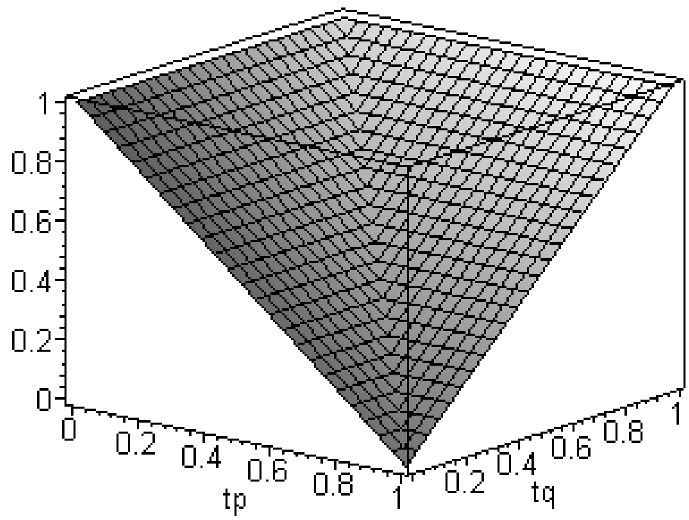

Figure 1 **Implication**

It is important to realize that in the conversion from classical logic to fuzzy, all the classical values of false and true remain the same. Indeed, the corner points of the cube in Figure 1 correspond to the classical logic values. Figures 2 – 5 illustrate additional examples of fuzzy truth surfaces.

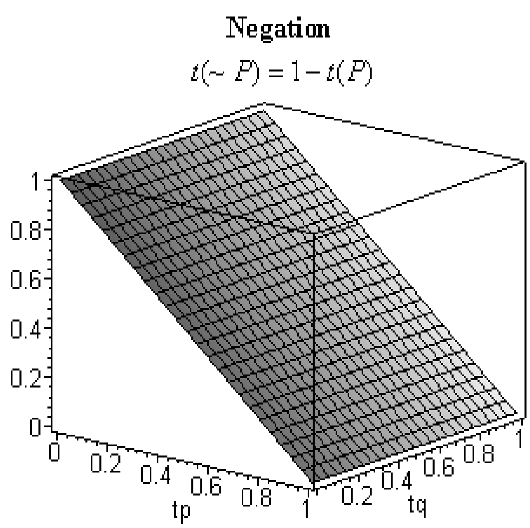

Figure 2

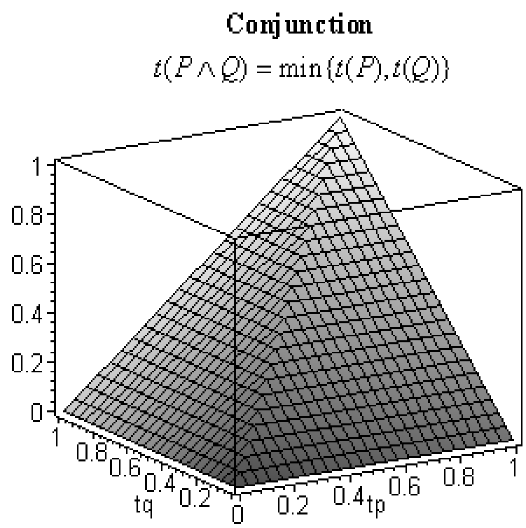

Figure 3

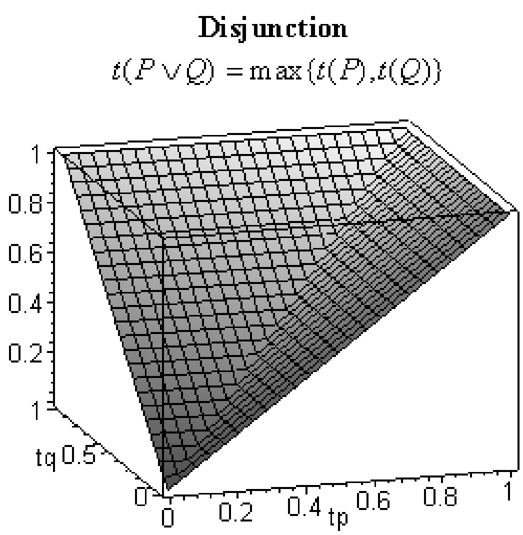

Figure 4

Equivalence

$t(P = Q) = \min\{[1 - \min(t(P), q - t(Q)), (1 - \min(T(Q), 1 - t(P))]\}$

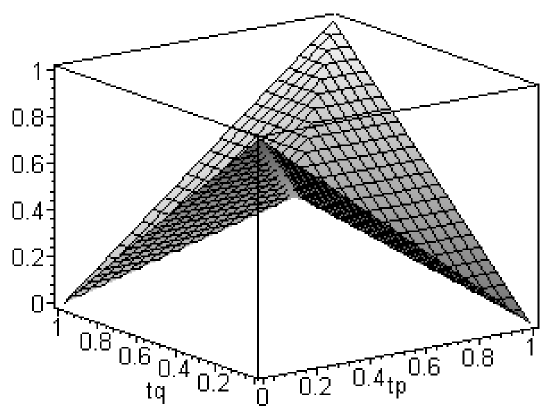

Figure 5

An important distinction between classical logic and fuzzy logic is the concept of a tautology. A tautology in classical logic is a statement or formula that is always true and has a truth-value of 1.0. Another defining term for a classical tautology is "strictly true theorems of logic." A tautology is evident in truth tables when, in each of all logical cases, the outcome is true. The notion of a fuzzy tautology is less clear. One interpretation of a fuzzy tautology is a statement or formula whose truth value is never 0.0, meaning that a fuzzy tautology can have varying degrees of truth. In classical logic, any statements, whether they are true or false when placed in a tautology, are such that the resulting compounded statement is still a tautology and has a truth-value of 1.0. In fuzzy logic, where statements are based on degrees of truth, the varying degrees, when substituted into converted fuzzy tautology formulas, generate outcomes higher than 0.0, but not always 1.0. For example, the classic tautology Modus Ponens is given logically as $((P \to Q) \wedge P) \to Q$. Translating into fuzzy form gives $1 - \min\{\, \min\{1 - \min(t(P), 1 - t(Q)),\ t(P)\},\quad 1 - t(Q)\,\}$ and is represented graphically in Figure 6.

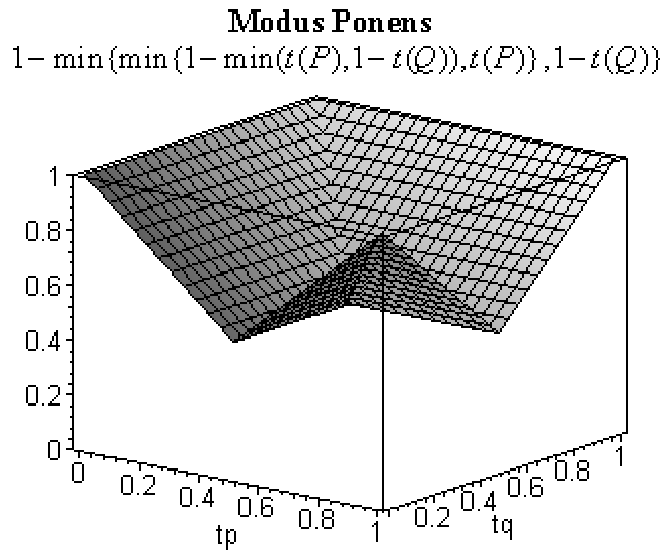

Figure 6

The introduction of fuzzy logic into the field of Mathematics has opened a profound array of new ideas. Fuzzy Mathematics is still in its academic infancy, but it has already proved its value in a number of areas. Almost every branch of applied mathematics has been influenced or directly affected by fuzzy logic. A simple definition for fuzzy mathematics would be the use of fuzzy quantities or numbers and other fuzzy variables introduced into the commonly strict field of mathematics in order to calculate somewhat crisp outcomes. Fuzzy quantities or numbers are fuzzy subsets of the universe of a numerical number, or simply a number whose precise value is somewhat uncertain. In other words, a fuzzy real number is a fuzzy subset of the domain of real numbers; a fuzzy integer is a fuzzy subset of the domain of integers, etc. An example of a fuzzy real number is "about 10" which is shown in Figure 7:

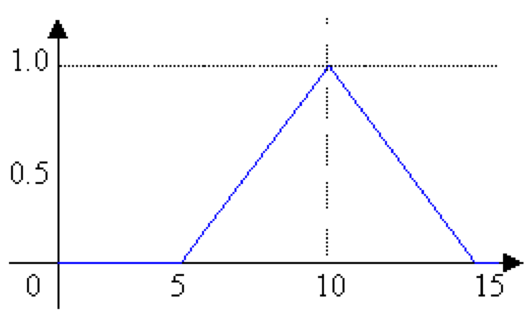

Figure 7

The example above represents a triangular fuzzy number. Figures 8 and 9 illustrate two other common ways to express fuzzy numbers graphically.

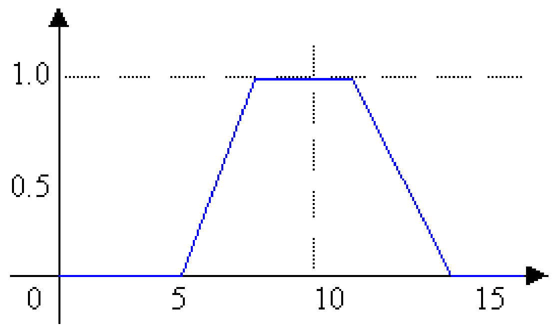

Figure 8

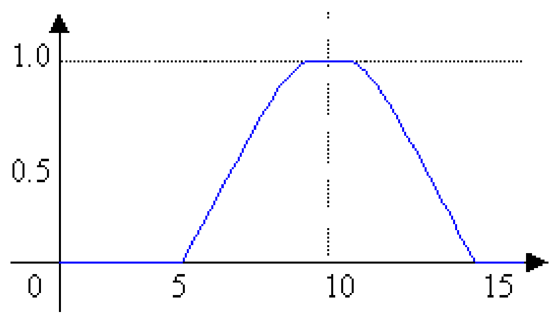

Figure 9

Figure 8 illustrates a trapezoidal fuzzy number whereas Figure 9 illustrates a bell-shaped fuzzy number. These types of fuzzy numbers are characterized as convex functions that start with a membership grade of 0, rise to a maximum of 1, signifying full membership, and then return to 0, or no membership again. There also exist alternative types of fuzzy numbers that begin or end in full membership such as "slightly less than or equal to 10." A graphical representation of this type of fuzzy number is a one-sided triangular fuzzy number also called a slope fuzzy number. We often use modifying words to express fuzzy numbers, such as "about 10," in the above example. These words are called hedges. A *hedge* is a linguistic term to qualify fuzzy variables by increasing, reducing, or restricting membership levels. Other hedges include "slightly," "somewhat," and "approximately." When working with fuzzy numbers, it is possible to make approximate comparisons. This is very useful when data is imprecise or when rigidity of input variables is not needed.

Fuzzy applications in mathematics include various other fields. Fuzzy Geometry deals with concepts of "somewhat straight lines" and "not quite round circles." Ovals are also studied in fuzzy geometry as a shape not fitting traditional geometry. Fuzzy Graph Theory is a generalization of conventional graph theory. Fuzzy graphs describe functional mappings between a set of input linguistic variables and an output linguistic variable. This process uses If-Then statements. Other fuzzy graphs are constructed, based on fuzzy points or incomplete data. Also, there exist Fuzzy Algebra, Fuzzy Topology, and Fuzzy Calculus. Presently, sets of fuzzy algorithms are being used to solve real world problems that are missing necessary variables. Through the introduction of fuzzy mathematics, many complexities that were once unfathomable are being examined. Classical mathematics can only deal with the simplest of models when compared to the complexities of human reckoning in the real world. Although these models enlighten us to many phenomena, they have the limitation of only allowing absolutes. Fuzzy mathematics permits not only absolutes, but also all possible degrees between the extremes. For this and other reasons, many mathematicians are using fuzzy methods and models in their current research and applications.

At this juncture it is important and necessary to point out the difference between probability and fuzzy systems. Both operate over the same numeric range, and at first glance have similar values: 0.0, representing false (or non-membership), and 1.0, representing true (or membership). However, there is a distinction to be made between the meaning of probability and the meaning of fuzzy truth values. In probability, the phrase, "there is an 80% chance that John is tall," takes the view that John is either tall or not tall, when compared to a set height for the term tall. This probability gives us a degree or likelihood to determine if John will belong to the set or not. The fuzzy statement sounds more like, "John's degree of membership within the set of tall people is 0.80." This statement gives us the view that John is in the set of tall people already and is "considerably" tall, corresponding to the value of 0.80.

Another distinction to be made between probability and fuzzy logic, is in the computing of operations. For independent events, the probabilistic operation for **and** is multiplication, whereas the fuzzy operation takes the minimum of the two separate values as the result; the probabilistic operation for **or** is $(x + y - x \cdot y)$, whereas the fuzzy operation takes the maximum. Although similar in some aspects, there is a clear distinction between probability and fuzzy logic both mathematically, as well as in interpretation and application.

It is not surprising that the far-reaching theories of fuzzy logic and fuzzy mathematics arouse several objections in, not only the mathematical community, but also many other professional communities. There have been many generic complaints about the "fuzziness" of the process of assigning values to linguistic terms. Many classical logicians argue that in every case where fuzzy logic is utilized, it can be shown that fuzzy logic is not necessary and could be done mathematically through the use of compounded classical logic or the combinations of classical logic statements. Many logicians also claim that the terms false and true are rigid, inflexible terms. Any fuzziness added into a statement arises from an imprecise definition of the terms within the statement, not in the nature of truth [4]. They also claim that no area of data manipulation is made easier with the introduction of fuzzy calculations; if anything, the calculations become more complex in nature. Other objections include ideas such as:

- There is no set standard for the "defuzzification" process (that is, how to convert fuzzy sets to usable data and apply the outcomes),
- The soundness of the whole concept (especially mathematically), and the use of fuzzy logic is too complex and could possibly be unstable (emphasizing that membership functions, their rules, and algorithmic steps used to create fuzzy systems are very complex, and because of their lack of subsistence, there have been no long term uses with fuzzy systems to show their stability and dependability) [1] [4].

In response to these arguments, proponents of fuzzy logic try to explain the notion that fuzzy logic and classical logic should not be viewed as competing against each other, but be seen as complements [4]. They claim that semantic clarity should not be the cause for objections, but that fuzzy statements should be translated into phrases which classical logicians would find acceptable. Fuzzy logic has been deemed acceptable in many various applications, notably consumer products, and has proved very successful there. The benefits created by fuzzy systems alone are reason for continued development and implementation to occur in this new horizon.

In conclusion, the bounds on fuzzy mathematics and fuzzy logic are almost limitless. New areas of their use are being found daily. Its applications in industry, business, healthcare, and education are successful and could potentially cause problems if removed. Although fuzzy logic was developed as a better method for handling and sorting data, it has proven itself to be much more useful in other applications, including the strict field of mathematics.

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