The Method Of Direction Fields With Illustrative Examples

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Abstract. When the exact solution of a differential equation is impossible to find, the study of its direction field can provide valuable information. This article reviews the method and shows how it is applied to find the terminal velocities of falling objects subject to air resistance. Two different cases are discussed: air resistance proportional to the velocity and air resistance proportional to the velocity squared.

Consider a falling object, subject to air resistance, which is allowed to fall for a distance of infinite length and assume that the air resistance is proportional to either the velocity or the velocity squared. The differential equation that models the movement of the object is easily established by Newton's Second Law. Since the air resistance increases with velocity, the velocity of the falling object will not increase without bound. So what is the velocity as time approaches infinity? The standard way of obtaining the answer is to solve the differential equation, obtain an exact solution of the velocity in terms of the time, and then compute the limit as $t \to \infty$. There is an alternative approach: analyzing the direction field of the corresponding differential equation. Without solving the equation, information about the solution of a differential equation can often be readily obtained by studying its direction field. Even when the classical methods of solving differential equations can not be used to obtain an exact solution, the analysis of the direction field can still provide valuable information about the solution of the the equation. It is the purpose of this article to review the method of direction fields by illustrating its application to the problem of finding the terminal velocity of a falling object subject to air resistance.

Let us first recall a result regarding the existence and uniqueness of the solution of initial-value problems of the form

$$\frac{dy}{dx} = g(x, y) \tag{1.a}$$

$$y(x_0) = y_0, \tag{1.b}$$

as this result will be used on two separate occasions in our discussion. If there is a rectangle R, containing the point (x_0, y_0) in the xy-plane, over which g and $\frac{\partial g}{\partial y}$ are continuous, then there exists an interval I_0 containing x_0 and a unique function $y(x)$, defined on I_0, that is a solution of the initial-value problem.

It can be seen from eq. 1.a that the value of g at any point (x, y) in the xy-plane is equal to the slope of the tangent line of the solution curve $y(x)$ and hence determines the direction of $y(x)$ at that point. Knowledge of the directions of the solution curves is useful, especially when exact solutions are not available, because it allows us to trace the solution curves by following the directions prescribed by the values of g. More can be said for special forms of g. For our purpose, we now consider a special case of eq. 1.a and eq. 1.b in which $g(x, y) = f(y)$, that is, the directions of the solution are only determined by values of y and not x. We also assume that f and f' are continuous for $-\infty < y < \infty$. The initial-value problem then takes the form

$$\frac{dy}{dx} = f(y) \tag{2.a}$$

$$y(x_0) = y_0. \tag{2.b}$$

Since f and f' are continuous for all values of y, given any initial condition (x_0, y_0), there exists a unique solution $y(x)$ which passes through the point (x_0, y_0). Now, for the sake of discussion, suppose that f has two zeros c_1 and c_2 (i.e., $f(c_1) = f(c_2) = 0$), with $c_1 < c_2$. It is then apparent that $y = c_1$ and $y = c_2$ are two constant solutions of eq. 2.a, which we will call the *equilibrium solutions.* The lines $y = c_1$ and $y = c_2$ partition the xy-plane into three subregions $R_1 = \{(x, y) : y < c_1\}$, $R_2 = \{(x, y) : c_1 < y < c_2\}$, and $R_3 = \{(x, y) : y > c_2\}$. Without solving the initial-value problem, the following information can be concluded.

First, if (x_0, y_0) is in R_i (for $i = 1, 2, 3$), the solution $y(x)$ to eq. 2.a and eq. 2.b is confined to R_i for $-\infty < x < \infty$. Why? Otherwise, $y(x)$ intersects the boundary of R_i, let's say that $y(x)$ is in R_2 and intersects the boundary $y = c_1$ at $x = x_1$. Then the initial-value problem

$$\frac{dy}{dx} = f(y)$$
$$y(x_1) = c_1$$

will have two different solutions: $y(x)$ and the equilibrium solution $y = c_1$, violating the uniqueness of the solution to the problem.

Next, since $\frac{dy}{dx} = f(y)$ does not change sign in R_i, the solution $y(x)$ is strictly monotonic. Finally, if $y(x)$ is in R_i, then $y(x)$ will approach the boundary of R_i (the equilibrium solution(s)) as $x \to \infty$ and/or as $x \to -\infty$. For example, if $y(x)$ lies in R_2 and is increasing, then $\lim_{x\to-\infty} y(x) = c_1$ and $\lim_{x\to\infty} y(x) = c_2$. To see that the second limit is true, first note that $y(x)$ is increasing and bounded above by $y = c_2$. Therefore, $\lim_{x\to\infty} y(x) = c \le c_2$ (i.e., $y = c$ is a horizontal asymptote of $y(x)$). Hence, $\lim_{x\to\infty} y'(x) = 0$. Now on one hand, eq. 2.a implies that

$$\lim_{x\to\infty} f(y(x)) = \lim_{x\to\infty} y'(x) = 0.$$

On the other hand, by the property of limits of composite functions,

$$\lim_{x\to\infty} f(y(x)) = f\left(\lim_{x\to\infty} y(x)\right) = f(c).$$

Therefore $f(c) = 0$, and then $c = c_2$, by the continuity of f.

We are now ready to put the above analysis into use. In what follows, the original position of the object is taken as the origin and the positive direction is oriented downward. We study two cases.

1. Air Resistance Proportional to Velocity

By Newton's Second Law, the modeling differential equation is

$$m\frac{dv}{dt} = mg - kv, \tag{3}$$

where m denotes the the mass of the object, v velocity, t time, g the gravitational acceleration and k the constant of proportionality. The equation is subject to the initial condition $v(0) = v_0$, where v_0 is the initial velocity. Dividing both sides of the equation by m yields

$$\frac{dv}{dt} = g - \frac{kv}{m}. \tag{4}$$

Now setting the right hand side of eq. 4 equal to zero, we obtain the equilibrium solution

$$v_1 = \frac{mg}{k}$$

The line $v = v_1$ partitions tv-plane into two regions $R_1 = \{v < v_1\}$ and $R_2 = \{v > v_1\}$. From eq. 4 we see that if $(0, v_0)$ lies in R_2, $\frac{dv}{dt} < 0$ and v decreases; whereas if $(0, v_0)$ lies in R_1,

$\frac{dv}{dt} > 0$ and v increases. Since the solution will eventually approach the equilibrium solution, in both cases $\lim_{t\to\infty} v(t) = v_1$. Finally if $v_0 = v_1$, the equilibrium solution is obtained. (The physical interpretation in this case is that, since the air resistance is equal to the gravitational force, the total force equals zero and the object will move with a constant speed.) In all three cases: $(0, v_0) \in R_1$; $(0, v_0) \in R_2$; $v_0 = v_1$; the terminal velocity is $\frac{mg}{k}$.

2. Air Resistance Proportional to Velocity Squared

Let us assume the initial velocity $v_0 > 0$, i.e., the object moves in the downward direction at $t = 0$. The modeling differential equation now becomes:

$$m\frac{dv}{dt} = mg - kv^2, \tag{5}$$

which is equivalent to

$$\frac{dv}{dt} = g - \frac{kv^2}{m}. \tag{6}$$

(Note that $k > 0$.)

In this case, the equilibrium solutions are $v_1 = -\sqrt{\frac{mg}{k}}$, $v_2 = \sqrt{\frac{mg}{k}}$ and the tv-plane is now partitioned into three regions: $R_1 = \{v < v_1\}$, $R_2 = \{v_1 < v < v_2\}$, and $R_3 = \{v > v_2\}$. We are only interested in R_3 and the part of R_2 that lies above the t-axis, since eq. 5 can only be used as a model equation of the falling process (the last term in the equation needs a sign change if the object moves up initially.) If $(0, v_0)$ lies in R_3, $\frac{dv}{dt} < 0$, and v decreases; if $(0, v_0)$ lies above the t-axis in R_2, $\frac{dv}{dt} > 0$, and v increases. In both cases, $\lim_{t\to\infty} v(t) = v_2$. Finally if $v_0 = v_2$, we again obtain the equilibrium solution. Hence the terminal velocity in this case is $\sqrt{\frac{mg}{k}}$.

If $v_0 < 0$, i.e., the object moves up at $t = 0$, the air resistance will be oriented in the positive direction and the differential equation for the rising process becomes

$$m\frac{dv}{dt} = mg + kv^2,$$

(Again note that $k > 0$) which is different from eq. 5. Will the terminal velocity be different? Not really. Due to the positive acceleration caused by $mg + kv^2$, the velocity will increase to zero (when the object reaches the highest point) and then eq. 5 will apply and govern the falling process as soon as v is positive.

The exact solutions of eq. 3 and eq. 5 can be obtained by using the standard techniques of solving first-order linear differential equations and are included here for reference (It can be seen that the same terminal velocities are obtained by letting $t \to \infty$) :

$$v = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right) e^{-\frac{kt}{m}}$$

for eq. 3 and

$$v = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}} t + \tanh^{-1}\sqrt{\frac{k}{mg}} v_0\right)$$

for eq. 5. On the other hand, we see that such solutions are not necessary for our purpose, and, in general, the method of direction fields can provide us with useful information even in circumstances in which the exact solutions are impossible to find!

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